



General equilibrium real and nominal interest rates [☆]

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Abstract

We derive the general equilibrium short-term real and nominal interest rates in a monetary economy affected by technological and monetary shocks and where the price level dynamics is endogenous. Assuming fairly general processes for technology and money supply, we show that an inherent feature of our equilibrium is that any real variable dynamics, in particular that of the short-term real interest rate, is driven by both monetary and real factors. This money non-neutrality is generic, as it does not stem from any friction such as price stickiness, or from a particular utility function. Non-neutrality obtains because the ex ante cost of real money holdings is random due to inflation uncertainty. We then analyze in depth a specialized version of this economy in which the state variables follow square root processes, and the representative investor has a log separable utility function. The short-term nominal rate dynamics we obtain encompasses most of the dynamics present in the literature, from Vasicek and CIR to recent quadratic and, more generally, non-linear interest rate models. Moreover, our results pave the way to several new nominal term structures.

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1. Introduction

We propose a general equilibrium of a frictionless monetary economy in which money is an argument of the representative individual's utility function. In a fairly general framework set in continuous time, we first derive and analyze the behavior of macroeconomic aggregates such as consumption, investment and real wealth and devote special attention to the inflation rate and the real and nominal interest rates. In a specialized version of the economy, where the representative agent has a log separable utility function and the state variables follow square root processes, we then provide explicit solutions to our model and derive in particular the implied dynamics for the real and short-term nominal rates. The main characteristic of our economy is that generically money is neither neutral nor superneutral, as monetary policy always affects the level and the dynamics of all real variables.¹ The transmission mechanism works as follows. An individual holding real balances faces an opportunity cost that *ex ante* is the nominal interest rate. However, the effective cost of money holding is not the nominal rate but the sum of the real rate and the inflation rate realized *ex post*. Under uncertain inflation, the two costs are distinct since, at the beginning of each period, the first one is known while the second is random. Investors' real wealth, and thereby all other endogenous real variables, are affected by this uncertainty. To further investigate the consequences of money non-neutrality, we provide a closed form solution for a specialized economy that can be viewed as the monetary extension of the real economy developed by Cox et al. (1985a,b), hereafter CIR. Our monetary economy turns out to possess original properties as compared to pure real economies or monetary economies in which the real and the nominal sectors are linked artificially or not at all.

The abundant and ever growing literature on term structure modeling and interest rate derivatives pricing witnesses the sizeable progress that has been accomplished in recent years both at the theoretical and the empirical levels. The adoption of new parametric and non-parametric techniques to estimate the term structure enhanced our understanding of the behavior of bond market prices and of the shortcomings of standard models. By comparison, relatively little effort has been devoted to providing these new models a sound economic background. The most widely used approach simply consists in assuming *a priori* grounds a given dynamics for the short-term nominal rate and then deriving the dynamics of bond prices and/or the price of derivatives. Although the CIR model of the term structure was set in a gen-

¹ According to the standard definition, money is superneutral with respect to a given variable if a change in its growth rate does not affect the level of the variable (see Walsh, 1998, p. 56). This paper deals essentially with superneutrality, as it considers changes in the money growth rate, although it discusses neutrality occasionally. Since however we generally examine the impact of changes in the money growth rate on the dynamics of economic variables, not their levels, we will mostly make use of the following definition. Money is superneutral with respect to a given variable if a change in its growth rate does not affect the *dynamics* (expectation and/or volatility of the growth rate) of the variable. It will be obvious from the context whether this definition or the standard one is used. Finally, the phrase "non-neutrality" will be used throughout to mean "non-superneutrality".

eral equilibrium framework, this was the case for almost none of the extensions proposed thereafter. In addition, the CIR model has been derived for a real, non-monetary, economy and nonetheless was used as if it were applicable to a nominal term structure. Others have followed this tradition. For example, Longstaff and Schwartz (1992) and Ahn et al. (2002) derive their nominal term structure dynamics within a CIR-like purely real economy. More generally, Jin and Glasserman (2001) show that every Heath et al. (1992) arbitrage free model of the term structure can be supported by a real-economy equilibrium a la CIR. This can be true only if interest rates are interpreted as real ones, or, equivalently, if the inflation rate is deterministic.

This paper, however, is not the first one to attempt to build a truly monetary economy, and to derive the equilibrium real and nominal term structures. The standard approach, followed by CIR themselves and others, merely consists in adding to CIR's framework an *exogenous* process for the price level or the inflation rate and then deriving the relevant variables, assuming along the way that money has no real effect. Other authors such as Pennacchi (1991) added artificially some non-neutrality, for instance by assuming on a priori grounds that the drift of the technological process depends on inflation in an otherwise CIR economy, the dynamics of the inflation rate being exogenous. Important progress has however been accomplished by Bakshi and Chen (1996) for a domestic economy and Basak and Galemeyer (1999) for an international economy. These authors built monetary economies in which the price level is found endogenously within a money-in-utility framework with a representative agent. However, both papers proposed a partial equilibrium framework in which output and consumption are in fact exogenous. Not surprisingly then, these models exhibit money superneutrality in equilibrium. Therefore, the present model of a truly monetary economy that leads to money non-neutrality without "ad hoc" assumptions fills an obvious gap. In particular, our dynamics for the real and nominal interest rates are dramatically different from each other.²

The first main contribution of this paper thus is to introduce a consistent framework of a monetary economy in which money is held because it provides utility and cannot in general be neutral in equilibrium, regardless of the *shape* of the utility function. Moreover, both *expected* and *non-anticipated* changes in the money supply rate of growth affect the level and the growth rate of all relevant variables. Such non-neutrality is achieved without introducing the imperfections (such as price stickiness and/or wage rigidities) characteristic of today standard models. The key is the correct modeling of the representative investor's wealth dynamics. This has important bearings on monetary theory and policy, term structure modeling and asset pricing.

The second main contribution concerns the behavior of the real interest rate in equilibrium. The short-term (in fact, continuous) real rate is equal to the expected return on real investment adjusted by a risk premium. The latter has two components, one related to consumption risk and the other to real balances risk. Since

² Buraschi and Jiltsov (2002) extend Bakshi and Chen's (1996) analysis to a production economy with taxes. Introducing a capital depreciation by firms that is imperfectly indexed on inflation does yield money non-neutrality. However, once this (possibly important) friction is removed, money does not affect the dynamics of physical capital any more in this model and recovers its neutrality.

consumption and real money holdings are affected by monetary factors, so is the risk premium. This is the first transmission mechanism of monetary impulses to the real rate. Consumption risk has three components, technology risk, monetary risk, and the risk associated with changes in the proportion of total wealth devoted to real investment. Since the latter risk is itself related to monetary risk, this compounds the impact of monetary policy on the real interest rate and makes its relationship to monetary factors highly non-linear. In the particular case of log separable utility functions, we solve completely for both the real rate level and its dynamics. They are shown to be very different from both what CIR obtained in their purely real economy and what is obtained in frictionless monetary economies in which money is in fact neutral (for a representative example, see Bakshi and Chen, 1996). To the best of our knowledge, such results for the short-term real rate are novel.

Another contribution concerns the behavior of the equilibrium nominal rate of interest. We recover the well-known result that this rate is equal to the marginal rate of substitution between real money balances and consumption. Although it is in general affected by both technological and monetary parameters, it is solely influenced by monetary factors in the log utility case. Its dynamics does nevertheless encompass most interest rate models offered in the literature, which therefore obtain as special cases of ours. First and foremost, we can recover CIR's square root interest rate model with the crucial provision that the latter was derived for the *real* rate, not the *nominal* rate as here. This provides a sound theoretical background to the numerous papers that used CIR's model as if it was obtained in a monetary economy and vindicates its adoption as the nominal interest rate model of a truly monetary economy in general equilibrium. Second, not only well known affine models of the term structure but also more complicated ones such as the non-linear models of Ahn and Gao (1999) or the log normal model of Miltersen et al. (1997) can be derived as particular cases. Third, our model also embeds the quadratic term structure model recently developed by Leippold and Wu (2002) and Ahn et al. (2002).

Finally, our model has obvious implications as to the potential factors explaining time series and cross-sectional features of nominal bond prices. In general, the factors found in the literature are related to the properties of the term structure itself such as the general level, steepness and convexity of the curve, or the volatility of the interest rates. This is similar to explaining the cross-section of asset returns by the return on the market portfolio, its volatility, skewness and kurtosis, and/or by the returns of particular "ad hoc" portfolios deemed to reflect common exposures to (generally) non-specified risks. Thus, bond returns are not related to fundamental economic risks. Our paper identifies exactly the factors affecting the short-term real and nominal rates. While three factors are needed to explain the level and dynamics of the real rate, namely the technology, the money supply and the investment/wealth ratio, one factor only, the money stock, plays a role for the determination of the nominal rate. Our identification thus is parsimonious and provides theoretical support to the many papers that showed that one factor in general explains about 90% of nominal bond price fluctuations (see for instance Chapman and Pearson, 2001). Moreover, estimating the level and dynamics of the short-term nominal rate does not require the use of consumption data.

The remainder of the article is organized as follows. Section 2 presents the monetary economy under investigation. Section 3 derives the equilibrium in the general economy and characterizes all the relevant variables, either real or nominal. Section 4 derives explicit solutions for all endogenous processes and variables in a specialized version of our economy in which the state variables follow square root processes, and the representative investor has a log separable utility function. Section 5 concludes. A mathematical appendix, which is not provided to save space but is available upon request to the authors, gathers all proofs and technical derivations.

2. The economic framework

The structure of the real sector of the economy is similar to that of CIR. We add a monetary sector by introducing money supply on the part of the Central Bank and money demand on the part of investors. The representative individual's wealth dynamics then is derived. Notationwise, real economic variables will be denoted by lower case letters (e.g. q , w , m or r) and nominal variables by capital letters (e.g. P , M or R).

2.1. Real sector, traded assets and money supply process

In the considered economy, there is a single physical good that may be allocated to either consumption or investment. When variables are said to be expressed in real (respectively, nominal) terms, it is understood that the implicit numeraire used is this physical good (respectively, money). The good is produced by a single technology (firm). The amount (real value) of the good invested at date t in the technology is denoted by $q(t)$. Production over time through the technology is governed by the following stochastic differential equation (SDE):

$$dq(t) = q(t)\mu_q(t, Y(t))dt + q(t)\sigma_q(t, Y(t))'dZ(t), \quad (1)$$

where $Z(t)$ is an $(N + K) \times 1$ dimensional Wiener process in R^{N+K} , $Y(t)$ is a $K \times 1$ dimensional vector of state variables, $\mu_q(t, Y(t))$ is a bounded function of t and Y , and $\sigma_q(t, Y(t))$ is a bounded $(N + K) \times 1$ vector valued function of t and Y . The Wiener process is defined on the usual complete probability space (Ω, F, P) where P is the true (historical) probability. We impose the normalization $q(0) = 1$. Since we make the classic simplifying assumption that the consumption/investment good is a non-durable good and that investment in the technological process is continuously destroyed, there is strictly speaking no capital accumulation.

The dynamics of the K state variables are determined by the following system of SDEs:

$$dY(t) = \mu_Y(t, Y(t))dt + \Sigma_Y(t, Y(t))dZ(t), \quad (2)$$

where $\mu_Y(t, Y(t))$ is a bounded $K \times 1$ vector valued function of t and Y and $\Sigma_Y(t, Y(t))$ is a bounded $K \times (N + K)$ matrix valued function of t and Y .

Consider an investor in this economy. At time t , he or she holds a number $a(t)$ of units of the technology, whose real value is $a(t)q(t)$. The proportion of wealth invested in the technology is denoted by $\alpha(t) \equiv a(t)q(t)/w(t)$, where $w(t)$ is the investor's real wealth. In the remainder of the paper, the terms “technology”, “equity”, and “real investment” will be used interchangeably.

Investors have also access to various markets for contingent (financial) claims to units of the consumption good. There are H non-redundant contingent claims, all of them being spot assets. These claims are in zero net supply and their real prices follow

$$ds(t, Y(t)) = I_s(t, Y(t))\mu_s(t, Y(t))dt + I_s(t, Y(t))\Sigma_s(t, Y(t))dZ(t), \quad (3)$$

where $I_s(\cdot)$ is a diagonal matrix the i th diagonal term of which is s_i , and $\mu(t, Y(t))$ and $\Sigma_s(t, Y(t))$ are general processes to be found *endogenously*.

In addition to equities and contingent claims, individuals have access to two money market accounts, both of which are in zero net supply.³ The real (index) savings account is denominated in the consumption/investment good and has an instantaneous return that is riskless in real terms and equal to the real interest rate $r(t)$. The nominal savings account is denominated in dollars and is riskless in nominal terms, its instantaneous yield being the nominal interest rate $R(t)$. Depending on the magnitude of H , the number of non-redundant contingent claims, the financial market is complete or incomplete, at will. We stress that *none* of our results depends on whether the financial market is complete or not.⁴ Note that if $H = (N + K - 2)$ then investors are able to trade $(H + 2) = (N + K)$ non-redundant assets with $(N + K)$ sources of uncertainty driving the economy. However, even in this case, the market still is incomplete since investment in the technology is subject to a short sale constraint.

Lastly, the Central Bank issues money and arbitrarily sets its nominal rate of return to zero. The reason why money is not strictly dominated by the nominally riskless money market account yielding $R(t)$ is that it helps reducing (implicit) transaction costs. It is therefore desired for the liquidity services it provides. The money supply process is exogenous to the model. The outstanding nominal quantity of money is not a state variable itself, but may be influenced by the state variables to which the Central Bank presumably reacts. To allow for this possible dependence, its dynamics is expressed as

$$dM(t) = M(t)\mu_M(t, Y(t))dt + M(t)\sigma_M(t, Y(t))'dZ(t), \quad (4)$$

³ Real (index) long-term bonds are not explicitly introduced into the analysis, as they are special cases of financial claims. Nor are nominal bonds, since they are claims to $1/P(t)$, rather than one, units of the consumption good, where $P(t)$ is the general price level at the bond maturity date t .

⁴ There is a minor difference in the assumption regarding individuals between the two cases. If the market is complete, we can assume that there is a representative individual. If not, the existence of a representative individual is more problematic and in general we have to resolve to the stronger assumption that individuals are identical. From now on, we will use the phrase “representative individual”, even though the market is incomplete.

where $\mu_M(t, Y(t))$ is a bounded function of t and Y and $\sigma_M(t, Y(t))$ is a bounded $(N + K) \times 1$ vector valued function of t and Y . These two functions are policy parameters and thus are *exogenous* to the model. The economic quantities to be derived endogenously will be, in particular, functions of $\mu_M(\cdot)$ and $\sigma_M(\cdot)$. Note however that the parameters of the production process, $\mu_q(t, Y(t))$ and $\sigma_q(t, Y(t))$, which depend on the state variables, do not depend directly on $\mu_M(t, Y(t))$ and $\sigma_M(t, Y(t))$. The initial stock of money $M(0)$ is compatible with the initial price level $P(0)$ but otherwise arbitrary.

The general price level, i.e. the money price of one unit of the consumption/investment good, $P(t)$, will be shown to follow

$$dP(t) = P(t)\mu_P(t, Y(t))dt + P(t)\sigma_P(t, Y(t))'dZ, \tag{5}$$

where $\mu_P(t, Y(t))$ and $\sigma_P(t, Y(t))$, the latter an $(N + K) \times 1$ vector valued function, are to be found *endogenously* as part of the solution to the equilibrium problem.

Finally, trading in all financial and monetary assets and in the technology takes place continuously in frictionless and arbitrage-free markets and at equilibrium prices only.

2.2. Preferences and the budget constraint

As stated in the introduction, we adopt the money-in-the-utility-function model originally developed by Sidrauski (1967) and frequently used ever since, e.g. by Poncet (1983) and Bakshi and Chen (1996). The infinitely-lived representative investor in this economy maximizes the expected utility of her intertemporal consumption $c(t)$ and real money balances holdings $m(t)$ under her budget constraint. Therefore, her consumption and portfolio decisions maximize

$$E \left[\int_t^\infty U(s, c(s), m(s)) ds \right], \tag{6}$$

where U is assumed to be a twice continuously differentiable, increasing and strictly concave utility function,⁵ E is the expectation operator conditional on current endowments and the state of the economy. Real money balances $m(t)$ are equal to $M(t)/P(t)$. When maximizing (6), the representative investor is assumed to limit her attention to admissible controls only.

From now on, we delete the explicit dependence of the variables on time and the state variables, unless stated otherwise. The budget constraint then reads

$$dw = w\alpha \frac{dq}{q} + w\theta' I_s^{-1} ds + w\delta \left(R dt + \frac{dP^{-1}}{P^{-1}} \right) + w\phi r dt - c dt + m \frac{dP^{-1}}{P^{-1}}, \tag{7}$$

where θ is the vector of proportions of real wealth invested in the contingent claims, δ (respectively, ϕ) is the proportion of real wealth invested in the nominal money

⁵ This implies the following classic conditions, where subscripts on U denote partial derivatives: $U_c > 0$, $U_m > 0$, $U_{cc} < 0$, $U_{mm} < 0$, $U_{cm} < 0$ and $U_{cc}U_{mm} - (U_{cm})^2 > 0$.

market account (respectively, the real money market account) and c is the consumption rate. The last term is the opportunity cost of holding money, given by the change in the real price of one unit of currency.

The wealth dynamics (7) deserves the two following comments. First, the ex post real return on the nominal money market account is equal to Rdt , the nominal interest rate, plus the realized rate of depreciation of the purchasing power of money from t to $t + dt$, (dP^{-1}/P^{-1}) . Second, the direct ex post cost of holding real balances m between date t and date $t + dt$ is, similarly, proportional to the decrease in the purchasing power of money. Thus, the exact ex post opportunity cost of holding one unit of real balances is *not* the nominal interest rate Rdt , but the sum $[r dt - dP^{-1}/P^{-1}]$. In a world of certainty, the latter sum would be equal to Rdt . However, under uncertain inflation, these two quantities differ. In particular Rdt is deterministic while $[r dt - dP^{-1}/P^{-1}]$ is stochastic. This difference will prove crucial for the non-neutrality of money.

Using $\varphi = 1 - \delta - \alpha - \theta' \mathbf{1}_H - m/w$, the wealth dynamics (7) can be rewritten as

$$\begin{aligned} dw = w\alpha \left[\frac{dq}{q} - r dt \right] + w\theta' [I_s^{-1} ds - \mathbf{1}_H r dt] + w\delta \left[R dt + \frac{dP^{-1}}{P^{-1}} - r dt \right] \\ + wr dt - c dt - m \left[r dt - \frac{dP^{-1}}{P^{-1}} \right]. \end{aligned} \quad (8)$$

Eq. (8) clearly demonstrates the need for a correct specification of the ex post opportunity cost of real money holdings. If Rdt were (wrongly) assimilated to $[r dt - dP^{-1}/P^{-1}]$, the third term in brackets on the RHS of the equation would vanish and, more importantly, the last term m would be multiplied by Rdt , a deterministic term. Most results would be then altered significantly.

Now, using (1), (3) and (5) yields the representative agent's wealth dynamics:

$$\begin{aligned} dw = \left[w\alpha(\mu_q - r) + w\theta'(\mu_s - \mathbf{1}_H r) + w\delta(R - r - \mu_p + \sigma'_p \sigma_p) + wr - c \right. \\ \left. - m(r + \mu_p - \sigma'_p \sigma_p) \right] dt + [w\alpha\sigma'_q + w\theta'\Sigma_s - (\delta w + m)\sigma'_p] dZ. \end{aligned} \quad (9)$$

We can now derive the general equilibrium in our monetary economy.

3. The general equilibrium

At this level of generality, completely closed-form solutions for the economic general equilibrium cannot be obtained, but major insights as to the transmission of monetary impulses to real and nominal variables and as to the behavior of interest rates can nevertheless be gained. In the remainder of this section, we drop the explicit dependence of the variables on time and the state variables to ease the exposition. We start by showing explicitly how the transmission mechanism of monetary impulses works, then derive the real and nominal rates of interest and the inflation risk premium.

3.1. The transmission mechanism

For the economy to be in equilibrium, the following market clearing conditions must be satisfied:

- (i) total wealth, taken without loss of generality to be that of the representative individual, must be equal to the total amount invested in the technology plus the real value of money balances held, i.e. $w\alpha + m = w$,
- (ii) net holdings in each of the two money market accounts and in each of the various contingent claims must be equal to zero, i.e. $\delta = 0$, $\varphi = 0$ and $\theta = \mathbf{0}_H$, and
- (iii) money supply must equal money demand, i.e. $M/P = m = w(1 - \alpha)$.

Under these conditions, we can derive the following proposition.

Proposition 1 (Real wealth). *In equilibrium, the aggregate real wealth evolves over time as*

$$\frac{dw}{w} = \mu_w dt + \sigma'_w dZ, \quad (10)$$

where its expected instantaneous growth rate μ_w is equal to

$$\begin{aligned} \mu_w = & -\frac{1-\alpha}{\alpha}\mu_M + \frac{1-\alpha}{\alpha^2}\sigma'_M\sigma_M + \mu_q - \frac{c}{\alpha w} + (-\mu_x + \sigma'_x\sigma_x) - \sigma'_x\sigma_q \\ & + \frac{2-\alpha}{\alpha}\sigma'_x\sigma_M - \frac{1-\alpha}{\alpha}\sigma'_q\sigma_M + \frac{1-\alpha}{\alpha^2}\sigma'_M\sigma_M \end{aligned} \quad (11)$$

and its instantaneous volatility σ_w is equal to

$$\sigma_w = \sigma_q - \frac{1-\alpha}{\alpha}\sigma_M - \sigma_x, \quad (12)$$

where μ_x and σ_x are the drift and the volatility, respectively, of the relative changes in the proportion of wealth devoted to real investment.

Results (11) and (12) are derived from the market clearing conditions and the representative agent's budget constraint only. First-order conditions for an optimum are not used at all at this point. It follows directly that these results hold regardless of the shape of the representative agent's utility function. In particular, they do not depend on consumption and money being separable in the investor's utility function. The fact that monetary parameters matter in (11) and (12) thus is inherent in a true monetary economy. Money cannot be (neutral or) superneutral in general and both the *systematic* (μ_M) and *unexpected* (σ_M) components of the money supply affect the wealth dynamics.

According to Eq. (11), an increase in the average growth rate of the money supply induces a decrease in the expected growth rate of real wealth, since the larger are money balances held at equilibrium, the higher is their opportunity cost. Inflation indeed is a tax on real money holdings. Moreover, real wealth average growth

increases with the variance of the money supply process $\sigma'_M \sigma_M$, because, as returns on real money holdings become more volatile, the real demand for money declines and real investment rises on average. It also increases, as anticipated, with the expected return on real investment μ_y and decreases with the consumption to investment ratio ($c/\alpha w$).

As to the behavior of the general equilibrium inflation rate, it is given in the following proposition.

Proposition 2 (Inflation). *In equilibrium, the price level follows*

$$\frac{dP(t)}{P(t)} = \mu_P(t, Y(t)) dt + \sigma_P(t, Y(t))' dZ, \quad (13)$$

where the expected instantaneous rate of inflation is equal to

$$\begin{aligned} \mu_P = & \frac{1}{\alpha} \mu_M + (-\mu_q + \sigma'_q \sigma_q) + \frac{c}{\alpha w} - \frac{1}{1-\alpha} (-\mu_x + \sigma'_x \sigma_x) + \frac{1}{1-\alpha} \sigma'_x \sigma_M \\ & + \frac{1}{(1-\alpha)^2} \sigma'_x \sigma_x - \frac{1}{\alpha} \sigma'_q \sigma_M - \frac{1}{1-\alpha} \sigma'_x \sigma_q \end{aligned} \quad (14)$$

and its instantaneous volatility is given by

$$\sigma_P = \frac{1}{\alpha} \sigma_M - \sigma_q + \frac{1}{1-\alpha} \sigma_x. \quad (15)$$

The expected rate of inflation given by Eq. (14) depends in a complex way on real and monetary factors. In addition to being positively related to the average growth rate of money μ_M , expected inflation is a decreasing function of the average productivity of real investment. It is a positive function of the consumption/investment ratio ($c/\alpha w$) since investment and expected output decrease when consumption increases. It is also decreasing in the expected change in $1/\alpha$, namely $(-\mu_x + \sigma'_x \sigma_x)$. Indeed, if the proportion of wealth devoted to real investment is expected to increase, $1/\alpha$ is expected to decrease, and this will cause an increase in expected inflation since the relative demand for money is expected to decrease. The additional terms in Eq. (14) reflect the second-order impact of the relevant covariances between technological returns, money growth and capital investment.⁶

Since real balances are part of real wealth, the optimal allocation of wealth between financial assets will differ from that obtained in a money-less economy. The

⁶ Note that, in their simpler (endowment) economy, Bakshi and Chen (1996) need to solve a (complex) valuation equation to derive the price level. It involves the (infinite sequence of) ratios of the marginal utilities of money and consumption and cannot be solved in closed form in general. In spite of the apparent difference, our Eq. (13) – written as a differential equation – is in fact similar to their Eq. (16) – written as an integral equation – in that it is not solved in closed form either and uses also (albeit implicitly) the ratio of marginal utilities through the (time varying) ratio α . Moreover, we can easily show that it can be expressed in exactly the same way as their Eq. (16). Closed form solutions for P (and α) can be obtained in special cases only, such as the one discussed in Section 4 below.

influence of money on the representative investor’s optimal allocation is given in the following proposition.

Proposition 3 (Demand for risky assets). *The equilibrium demand for risky assets is equal to*

$$\begin{pmatrix} \alpha \\ \theta \\ \delta \end{pmatrix} = (\Sigma \Sigma')^{-1} \begin{pmatrix} \mu_q - r \\ \mu_s - \mathbf{1}_H r \\ R - r - \mu_p + \sigma'_p \sigma_p \end{pmatrix} \frac{-J_w}{w J_{ww}} + (\Sigma \Sigma')^{-1} \Sigma \Sigma' \frac{-J_{wY}}{w J_{ww}} + \frac{m}{w} (\Sigma \Sigma')^{-1} \Sigma \sigma_p, \tag{16}$$

where

$$\Sigma' = (\sigma_q \quad \Sigma'_s \quad -\sigma_p). \tag{17}$$

The optimal dynamic trading strategy has three components. The first one is the usual mean–variance (myopic) speculative component while the second one is the traditional vector of Merton–Breedon hedges against the fluctuations of each and every state variable. The last component is novel and due to the presence of money. It is a minimum variance hedge ratio whose purpose is to hedge real balances against inflation risk. Individuals are bound to hold real balances for the services it provides, which entails in essence a constrained position that must be hedged.

Now, Merton–Breedon hedging components against the K state variables are known to lead to asset returns that contain K risk premiums at equilibrium in addition to the premium on the market portfolio. Unlike these components, the last term on the RHS of (16) does not depend on the utility parameter $(-J_{wY}/J_{ww})$, although it is not strictly speaking preference free since optimal money holdings (m) and wealth (w) depend on the investor’s utility. Nevertheless, this term obviously does not cancel out in equilibrium and thus commands an adjustment to the risk premium for each and every asset.

3.2. Real and nominal short-term rates

Since money is not neutral, the parameters of the money supply process affect the equilibrium real short-term rate, as shown in the following proposition.

Proposition 4 (Real interest rate). *In equilibrium, the instantaneous real rate of interest is equal to*

$$r = \mu_q - \left(-\frac{cU_{cc}}{U_c} \right) \sigma'_q \sigma_c - \left(-\frac{mU_{cm}}{U_c} \right) \sigma'_q \sigma_m \tag{18}$$

or, equivalently, to

$$\begin{aligned}
 r = & \mu_q - \left(\left(-\frac{cU_{cc}}{U_c} \right) \frac{w}{c} c_w + \left(-\frac{mU_{cm}}{U_c} \right) \right) \sigma'_q \sigma_q \\
 & + \left(\left(-\frac{cU_{cc}}{U_c} \right) \frac{w}{c} c_w + \left(-\frac{mU_{cm}}{U_c} \right) \right) \frac{1-\alpha}{\alpha} \sigma'_q \sigma_M \\
 & + \left(\left(-\frac{cU_{cc}}{U_c} \right) \frac{w}{c} c_w + \left(-\frac{mU_{cm}}{U_c} \right) \frac{1}{1-\alpha} \right) \sigma'_q \sigma_x \\
 & - \left(-\frac{cU_{cc}}{U_c} \right) \frac{1}{c} \sigma'_q \Sigma'_Y c_Y.
 \end{aligned}
 \tag{19}$$

Money non-neutrality is apparent in Eq. (18). To the best of our knowledge, this is the first time this expression is derived for the short-term real rate within a monetary economy in which money is not neutral. It is worth noting that non-neutrality is not solely related to the second risk premium on the RHS of the equation associated with real balances. This is because aggregate consumption, which appears in the first risk premium, also is affected by monetary parameters. Thus, the non-neutrality result remains valid if the representative investor’s utility function is separable in its two arguments, as in the particular case of log separable utility. This is because, as claimed above, non-neutrality does not depend on the shape of the utility function. The separability assumption ($U_{cm} = 0$) allows for an explicit solution to the general equilibrium, and the real short rate becomes

$$r = \mu_q - \left(-\frac{cU_{cc}}{U_c} \right) \sigma'_q \sigma_c.
 \tag{20}$$

Interestingly, r has the same structure as the real rate present in non-monetary economies such as CIR’s, with the crucial difference that (endogenous) consumption here is affected by monetary shocks. An important consequence of (20) is that, in partial equilibrium monetary models with separable utility function *and* exogenous consumption and output, the real rate is affected by real shocks only. This is for instance the case in Bakshi and Chen (1996). More generally, these endowment economy models cannot yield money non-neutrality endogenously. This is in sharp contrast with our more realistic production economy model.

As to the nominal interest rate and the inflation risk premium, they are the subjects of the following two propositions.

Proposition 5 (Nominal interest rate). *In equilibrium, the nominal short-term rate of interest is equal to*

$$R = \frac{U_m}{U_c}.
 \tag{21}$$

This result is standard in continuous time economies.⁷ The nominal rate is equal to the marginal rate of substitution between real money balances and consumption. The cost of holding one additional unit of money is the opportunity cost R (dollars),

⁷ See for instance Poncet (1983) and Bakshi and Chen (1996).

while the opportunity cost of consuming (destroying) one unit of consumption/investment good is one (dollar). The cost ratio, R , must at the optimum be equal to the ratio of marginal utilities.⁸ It is in general affected by both monetary and real shocks.

Proposition 6 (Inflation risk premium). *The inflation risk premium, defined as $\varepsilon \equiv R + E_t[dP^{-1}/P^{-1}] - r$, is equal to*

$$\varepsilon = -\frac{cU_{cc}}{U_c}(-\sigma_p)' \sigma_c - \frac{mU_{cm}}{U_c}(-\sigma_p)' \sigma_m. \quad (22)$$

The inflation risk premium is the reward to be granted to an investor, in addition to the real rate and to the expected rate of loss of the purchasing power of money, to induce her to invest into a nominal, rather than a real, money market account. The nominal money market account is a risky asset in real terms and thus commands such a premium at equilibrium. Two risk premiums, one related to consumption risk and the other to real balances risk, compensate the investor for the risk borne on the real return on the nominal money market account ($-\sigma_p$). Substituting for the volatility of the inflation rate given by Eq. (15), Eq. (22) clearly shows that, even when consumption and money are separable in the utility function ($U_{cm} = 0$), monetary policy, and in particular monetary uncertainty, still plays a major role in fixing the equilibrium value of the inflation risk premium. We will give explicit expressions of this variable in Section 4 to which we now turn.

4. A specialized economy

To derive explicit solutions, in particular for the dynamics of real and nominal interest rates, and provide some additional insights as to the scope of our results, we specialize the general economy presented in Sections 2 and 3. The simplifying assumptions are discussed first, and then the equilibrium is derived along with the values of real wealth, investment and consumption, the price level dynamics, the real and nominal interest rates and the inflation risk premium. The final subsection is devoted to the thorough analysis of the real and nominal interest rate dynamics in various situations.

4.1. The economy

We assume there are but two state variables ($K = 2$), a technological one that affects, as in CIR, the production process only (called hereafter the “real” state variable), and a monetary one that influences the dynamics of the money supply exclusively (dubbed hereafter the “nominal” state variable). Assuming more than one state variable for each process would contribute nothing to the economic

⁸ In discrete time, the cost ratio is equal to $R/(1 + R)$. The denominator simplifies to one in continuous time.

intuition underlying the results. Assuming that the technology is affected by monetary factors as well as real ones would not seriously impair the model's tractability, but would blur the message regarding money non-neutrality. In fact, by assuming that the technology is not affected by money, we make the presence of monetary factors in the dynamics of real variables truly endogenous. Nevertheless, the two state variables may be correlated.

The exogenous technology and money supply processes obey respectively

$$\frac{dq(t)}{q(t)} = \mu_q Y_q(t)^\lambda dt + Y_q(t)^\eta \sigma'_q dZ(t) \quad (23)$$

and

$$\frac{dM(t)}{M(t)} = \mu_M Y_M(t)^\beta dt + Y_M(t)^\nu \sigma'_M dZ(t), \quad (24)$$

where μ_q and μ_M are positive constants, σ_q and σ_M are $(N+2) \times 1$ vectors of positive constants, $Z(t)$ is an $(N+2) \times 1$ dimensional Wiener process in R^{N+2} , the constants λ , η , β and ν are positive, and Y_q and Y_M , respectively, are the real and the nominal state variables assumed to follow

$$dY_q(t) = \mu_{Y_q} Y_q(t) dt + \sqrt{Y_q(t)} \sigma'_{Y_q} dZ(t) \quad (25)$$

and

$$dY_M(t) = \mu_{Y_M} Y_M(t) dt + \sqrt{Y_M(t)} \sigma'_{Y_M} dZ(t), \quad (26)$$

where μ_{Y_q} and μ_{Y_M} are positive constants, σ_{Y_q} and σ_{Y_M} are $(N+2) \times 1$ vectors of positive constants. We normalize $q(0) = 1$ and assume that $M(0)$ is compatible with $P(0)$.

The Greek parameters λ , η , β and ν present in Eqs. (23) and (24) determine whether the exogenous processes are linear or non-linear functions of the underlying state variables. They play a crucial role as to the dynamics of interest rates. In particular, the nominal rate dynamics we obtain will be shown to be general enough to encompass most of the dynamics present in the literature, from Vasicek and CIR to recent quadratic and, more generally, non-linear interest rate models. By assigning different values to these parameters, we provide a direct link between the linearity or non-linearity in interest rates to the behavior of the fundamental economic processes (technology and money).

We further assume that the representative investor has a log separable utility function, since it is a benchmark in finance theory:⁹

$$U(t, c(t), m(t)) = e^{-\rho t} [\phi \ln c(t) + (1 - \phi) \ln m(t)]. \quad (27)$$

We can now derive the equilibrium values of all the relevant nominal and real variables.

⁹ See for instance Arrow (1964), Samuelson (1969), Hakansson (1970), Merton (1971), Kraus and Litzenberger (1975), Rubinstein (1976), Fischer (1979), CIR (1985a) and Bakshi and Chen (1996).

4.2. The equilibrium

Solving for the model under the previous assumptions yields the following set of propositions.

Proposition 7 (Real investment). *In the special economy, the proportion of wealth invested in the technology is given by*

$$\alpha = f(Y_M), \tag{28}$$

where f solves

$$\begin{aligned} f + \left(\frac{Y_M^v \sqrt{Y_M} \sigma'_{Y_M} \sigma_M - \mu_{Y_M} Y_M}{\rho + \mu_M Y_M^\beta - Y_M^{2v} \sigma'_M \sigma_M} \right) f' - \frac{1}{2} \left(\frac{Y_M \sigma'_{Y_M} \sigma_{Y_M}}{\rho + \mu_M Y_M^\beta - Y_M^{2v} \sigma'_M \sigma_M} \right) f'' \\ = \frac{\phi \rho + \mu_M Y_M^\beta - Y_M^{2v} \sigma'_M \sigma_M}{\rho + \mu_M Y_M^\beta - Y_M^{2v} \sigma'_M \sigma_M} \end{aligned} \tag{29}$$

and where f' and f'' denote the first and second derivatives of f , respectively.

The ordinary differential equations (28) and (29) is not solvable in general because its coefficients depend themselves on the state variable Y_M . However, a striking result is that the proportion of real wealth invested in equities depends solely on monetary factors, not on technological ones. This follows from assuming a log separable utility function.

A worth stressing, particular, solution to (29) occurs when the only state variable is the “real” one ($K = 1$). The monetary process (24) then reduces to a mere geometric Brownian motion and the investment to wealth ratio α simplifies to

$$\alpha = \frac{\phi \rho + \mu_M - \sigma'_M \sigma_M}{\rho + \mu_M - \sigma'_M \sigma_M}, \tag{30}$$

α then is a constant for given monetary parameters, but nonetheless remains affected by both the systematic and the surprise elements of monetary policy. In this subsection, we will refer to this important situation as “the special case”.

Eq. (28) implies that a necessary condition for the investment to wealth ratio α to be stochastic is that the parameters of the money supply process vary over time through their dependence upon a state variable. Casual observation of the dynamics of the money supply and of the investment to wealth ratio in major economies shows that this is the rule rather than the exception. Therefore allowing for time variation in the drift and volatility of the money supply process, and thus in the (endogenous) dynamics of α , is a desirable property. This impinges on the money neutrality issue. When the parameters of the money supply process and those of the change rate process for the ratio α are stochastic, monetary factors affect both the level and the change rate of real investment. When the parameters of the money supply process and the ratio α are constant, monetary policy still influences the optimal allocation of wealth between real money holdings and investment, as shown by Eq. (30). However, only the *level* of investment is affected, not its *expected rate of growth*. In any

case, money can never be neutral vis-à-vis investment. Similar conclusions as to consumption and real wealth can be derived from the following proposition.

Proposition 8 (Real wealth and consumption). *In the specialized economy, the representative agent’s real wealth is given by*

$$w(t) = w(0) e^{-\rho t} q(t) \xi_M(t)^{-1} \xi_x(t)^{-1}, \tag{31}$$

where the processes ξ_M and ξ_x are defined by ¹⁰

$$\frac{d\xi_M}{\xi_M} = \left(\frac{1 - f(Y_M)}{f(Y_M)} \right) \sigma'_M dZ \tag{32}$$

and

$$\frac{d\xi_x}{\xi_x} = - \frac{1 - f(Y_M)}{f(Y_M)} \frac{f'(Y_M)}{f(Y_M)} Y_M^{\nu+0.5} \sigma'_{Y_M} \sigma_M dt + \frac{f'(Y_M)}{f(Y_M)} \sqrt{Y_M} \sigma'_{Y_M} dZ. \tag{33}$$

Furthermore, optimal consumption is strictly proportional to real wealth:

$$c = \phi \rho w. \tag{34}$$

All real variables (wealth, consumption and investment) are affected by monetary factors, since money non-neutrality is preserved in this specialized setting. Perhaps more surprisingly, this influence remains quite complex. This is mainly due to the way money affects the proportion of real wealth invested in equity.

Equilibrium real wealth is affected by monetary policy through the processes $\xi_M(t)$ and $\xi_x(t)$ which are both affected by monetary factors only. $\xi_M(t)$ is a martingale related to money supply uncertainty while $\xi_x(t)$ is related to the uncertainty regarding the investment to wealth ratio due to the nominal state variable affecting the parameters of the money supply process. Thus, monetary uncertainty influences equilibrium in the real sector through two distinct channels. Note however that only one of them is sufficient to insure that money is not superneutral vis-à-vis wealth and consumption. Indeed, if $\sigma_{Y_M} = 0$, the process $\xi_x(t)$ reduces to the constant one but the process $\xi_M(t)$ still is a martingale, and, if $\sigma_M = 0$, the process $\xi_M(t)$ reduces to the constant one, but $\xi_x(t)$ is a martingale. Only if the money growth rate and the nominal state variable are both deterministic ($\sigma_M = \sigma_{Y_M} = 0$) is money superneutral vis-à-vis real wealth and, because of Eq. (34), consumption (but not investment).

The behavior of the price level is now described in the following proposition.

Proposition 9 (Price level). *In the specialized economy, the equilibrium price level is given by*

$$P(t) = e^{\rho t} \frac{M(t)}{(1 - f(Y_M))w(0)q(t)} \xi_M(t) \xi_x(t). \tag{35}$$

¹⁰ $\xi_M(0)$ is equal to one.

The price level is by definition the ratio of nominal money stock $M(t)$ over real money balances $[e^{-\rho t}(1 - f(Y_M))w(0)P(t)\xi_M(t)^{-1}\xi_z(t)^{-1}]$, the complement to wealth of real investment. Since monetary policy affects both the latter and $M(t)$, the relationship between the price level dynamics and the money supply process is highly non-linear.

The one-to-one relationship generally assumed between the inflation rate and the expected money growth rate thus does not hold in a general equilibrium model, even in this specialized version. To illustrate, in the special case where Eq. (30) holds, the expectation of the inflation rate and its volatility read, respectively:

$$\mu_P = \rho - (-\mu_M + \sigma'_M \sigma_M) + (-\mu_q + \sigma'_q \sigma_q) + \frac{\rho + \mu_M - \sigma'_M \sigma_M}{\phi\rho + \mu_M - \sigma'_M \sigma_M} (\sigma'_M \sigma_M - \sigma'_q \sigma_q), \tag{36}$$

$$\sigma_P = \frac{\rho + \mu_M - \sigma'_M \sigma_M}{\phi\rho + \mu_M - \sigma'_M \sigma_M} \sigma_M - \sigma_q. \tag{37}$$

Both parameters thus depend on the two components of monetary policy. These findings, whose in-depth analysis is beyond the scope of this paper, have obvious implications for the design and conduct of monetary policy.

Proposition 10 (Nominal interest rate). *In the specialized economy, the equilibrium nominal short-term rate is equal to*

$$R = \frac{(1 - \phi)\rho}{1 - f(Y_M)}. \tag{38}$$

In the special case of a constant alpha (for a given set of monetary parameters), its expression simplifies further to

$$R = \rho + \mu_M - \sigma'_M \sigma_M. \tag{39}$$

The nominal rate thus is independent of technological parameters and is affected by monetary factors only (in addition to the investor's preference parameters). In the special case of a constant α , R is equal to the representative individual's impatience rate minus the expected relative change in $1/M$ (equal to $-\mu_M + \sigma'_M \sigma_M$). As expected, the nominal rate increases with an expansionary monetary policy, i.e. a decrease in the anticipated growth rate of $1/M$. This is consistent with the empirical evidence recently documented by Evans and Marshall (2001) according to which most of the long-term variability in nominal interest rates of all maturities is due to macroeconomic impulses. More specifically, monetary shocks impact the slope of the term structure consistently and significantly, while fiscal impulses do not seem to have any significant influence.

Proposition 11 (Real interest rate). *In the specialized economy, the equilibrium real short-term rate is equal to*

$$r = \mu_q Y_q^\lambda - Y_q^{2\eta} \sigma'_q \sigma_q + \frac{1 - f(Y_M)}{f(Y_M)} Y_M(t)^\nu Y_q^\eta \sigma'_q \sigma_M + \frac{f'(Y_M)}{f(Y_M)} \sqrt{Y_M} Y_q^\eta \sigma'_q \sigma_{Y_M}. \quad (40)$$

In the special case of a constant alpha, its expression simplifies to

$$r = \mu_q - \sigma'_q \sigma_q + \frac{(1 - \phi)\rho}{\phi\rho + \mu_M - \sigma'_M \sigma_M} \sigma'_q \sigma_M. \quad (41)$$

The real rate depends, as expected, on two “real” factors, namely the representative investor’s preference parameters, ρ and ϕ , and the characteristics of the technological returns, μ_q and σ_q . It also depends on monetary and “nominal” factors, namely the parameters of the money growth rate and those of the nominal state variable. Money non-neutrality is brought about by two sources of risk, the covariance between technological returns and the money growth rate on the one hand, and the covariance between technological returns and changes in the nominal state variable (which also influence the monetary policy parameters) on the other. Money affects the real interest rate through the interaction between monetary (or nominal) and real uncertainties. The intuition behind this result is as follows. The real rate is equal to the expected return from the technology minus the required risk premium, the latter depending on the covariance between consumption and equity returns. Since consumption is affected by monetary factors, so is the real rate.

The sign and magnitude of the impact of monetary impulses on the real short rate level are directly related to the sign and magnitude of the covariances above. For instance, in the special case of Eq. (41), the change in r due to an increase in the average monetary growth rate μ_M has a sign opposite to the sign of the covariance $\sigma'_q \sigma_M$ for the following reason. Assuming for example that the return on real investment is positively correlated with the money growth rate, an increase in the latter will make the equilibrium rate r decrease since the demand for real savings will increase due to a positive wealth effect. This feature is vindicated by the result that, *ceteris paribus*, the real rate covaries negatively with the expected inflation rate. This can be seen by eliminating the term $(\mu_q - \sigma'_q \sigma_q)$ from Eq. (41) using Eq. (36), which yields the term $-\mu_p$ on the RHS of Eq. (41). This finding is consistent with an empirical study by Pennacchi (1991) who showed that the real interest rate and the inflation rate follow joint diffusion processes and that the instantaneous real rate and the instantaneous rate of expected inflation are significantly negatively correlated.

In the special case where the money supply is deterministic, Eq. (41) simplifies to $r = \mu_q - \sigma'_q \sigma_q$. We recover money superneutrality with respect to r . Since real wealth and consumption are independent of the money supply, the risk premium associated with the covariance between consumption and real investment returns does not depend on the money growth rate. Another important implication of (41) is that a sufficient condition for the real rate not to be affected by monetary policy is that real and monetary shocks are uncorrelated. Recall that money affects the real rate through the risk premium required on the technology returns. When equity returns and money growth are not correlated, money leaves this risk premium, hence the real interest rate, unaffected.

Proposition 12 (Inflation risk premium). *In the specialized economy, the equilibrium inflation risk premium is given by*

$$\begin{aligned} \varepsilon = & -\frac{2-f(Y_M)}{f(Y_M)} Y_q^\eta Y_M^v \sigma'_M \sigma_q + Y_q^{2\eta} \sigma'_q \sigma_q + \frac{1-f(Y_M)}{f(Y_M)^2} Y_M^{2v} \sigma'_M \sigma_M \\ & -\frac{2-f(Y_M)}{1-f(M)} Y_q^\eta \frac{f'(Y_M)}{f(Y_M)} \sqrt{Y_M} \sigma'_M \sigma_q + \frac{2}{f(Y_M)} Y_M^v \frac{f'Y(M)}{f(Y_M)} \sqrt{Y_M} \sigma'_M \sigma_M \\ & + \frac{1}{1-f(Y_M)} \left(\frac{f'(Y_M)}{f(Y_M)} \right)^2 Y_M \sigma'_M \sigma_{Y_M}. \end{aligned} \tag{42}$$

In the special case of a constant alpha, it reduces to

$$\begin{aligned} \varepsilon = & -\frac{(2-\phi)\rho + \mu_M - \sigma'_M \sigma_M}{\phi\rho + \mu_M - \sigma'_M \sigma_M} \sigma'_M \sigma_q + \sigma'_q \sigma_q \\ & + \frac{(1-\phi)\rho(\rho + \mu_M - \sigma'_M \sigma_M)}{(\phi\rho + \mu_M - \sigma'_M \sigma_M)^2} \sigma'_M \sigma_M. \end{aligned} \tag{43}$$

Money non-neutrality is again brought about by two sources of risk, the stochastic money growth rate on the one hand, and stochastic changes in the nominal state variable (that also influence the monetary policy parameters) on the other. However, while for money to affect the real interest rate there should be an interaction between monetary and real uncertainties (non-zero covariances), this is not necessary for the inflation risk premium. The variance of the money supply process and/or the variance of the state variable affect directly this premium.

The inflation risk premium depends on real factors through the variance of the technology productivity. Since an increase in the latter enlarges the variance of the inflation rate, ε is a positive function of $\sigma'_q \sigma_q$. Monetary policy affects the premium through three channels: the systematic part μ_M that bears on α , and the surprise parts σ_M and $\sigma'_q \sigma_M$. Although Eq. (43) is complex, the inflation risk premium is expected to be positive and is obviously so when the covariance between equity returns and money growth ($\sigma'_q \sigma_M$) is non-positive. Even if monetary policy were deterministic, the premium would still be positive under output uncertainty, because, as observed previously, the inflation rate would remain stochastic. In addition, were real returns not stochastic or not correlated with monetary growth, the inflation risk premium would still remain (positively) affected by the variance of the money growth rate, as intuition suggests.

4.3. Interest rate dynamics

We now turn to the in-depth analysis of interest rate dynamics. Here, we examine a case that is more general than that of the preceding subsection where α was a constant for a given set of monetary parameters. This case now allows for both the real and the nominal state variables to be present ($K = 2$) but is restricted to the situation where Eq. (29) can be meaningfully approximated by

$$\alpha(t) = f(Y_M(t)) \approx \frac{\phi\rho + \mu_M Y_M^\beta(t) - Y_M^{2v}(t)\sigma'_M\sigma_M}{\rho + \mu_M Y_M^\beta(t) - Y_M^{2v}(t)\sigma'_M\sigma_M}. \quad (44)$$

One way to assess that this approximation is reasonable is to check that, in absence of state variables, Eq. (44) does reduce to the exact solution (30).

Eq. (38) for the nominal rate then reads

$$R(t) = \rho + \mu_M Y_M^\beta(t) - Y_M^{2v}(t)\sigma'_M\sigma_M \quad (45)$$

which is a slight but important generalization of Eq. (39). Interestingly, the level of the nominal rate is independent of the parameters of the state variable processes, as is the level of the real rate in CIR. However, it does logically depend on the level of the nominal state variable.

The literature regarding the proper way to model the dynamics of nominal short-term rates is vast and expanding rapidly. This is no wonder as interest rates lie at the core of many economic and financial issues, the solution to which require a reliable model for the stochastic behavior of interest rates. However, as stated in the introduction, no consensus has been reached yet. In particular, the standard and popular affine term structure models such as Vasicek (1977), CIR (1985b) or Duffie and Kan (1996), which are still much in use among practitioners, have been shown to perform rather poorly by Chan et al. (1992), possibly because of the implicit restrictions they impose on the term structure volatility. The new strands of models developed by Ait-Sahalia (1996), Stanton (1997) and Boudoukh et al. (1999) among others that apply non-parametric estimation techniques to nominal short-term rates reveal strong non-linearities in the drift functions of those rates. The various quadratic term structure models recently offered by Ahn et al. (2002) and Leippold and Wu (2002) account for these non-linearities as well as non-linearities in the diffusion functions. At the empirical level, two approaches have been adopted: regime switching for the underlying state variable(s) or stochastic volatility. According to Ahn and Gao (1999) and Ahn et al. (2002), whose model encompasses older models by Longstaff (1989), Longstaff and Schwartz (1992) and Constantinides (1992), quadratic or more generally non-linear models seem to outperform affine models, although Chapman and Pearson (2000) tend to disagree and Dai and Singleton (2000, 2002) offer a more mitigated evidence.

We show below that our model is general enough to embed all the dynamics for the nominal short-term rate of interest proposed above. These dynamics depend in a crucial way on the values taken by the β and v parameters. Although the latter have no direct economic meaning, they reflect fundamental linearities or non-linearities between the state variables and the observable macroeconomic variables under scrutiny. We provide three types of models of various generality. We start from a model that remains in the spirit of CIR, and then generalize it to the case where the money process parameters are proportional to a function of the state variable rather than to the state variable itself. Finally, we relax CIR's assumption of a square root process for the nominal state variable.

4.3.1. CIR-style dynamics

Setting $\beta = 1$ and $v = 0.5$ in a CIR-like manner leads to the following dynamics for the nominal short-term rate.

Proposition 13 (Affine structure for the nominal rate). *Assuming $\beta = 1$ and $v = 0.5$, the nominal short-term rate is equal to*

$$R(t) = \rho + (\mu_M - \sigma'_M \sigma_M) Y_M(t) \tag{46}$$

and its dynamics obeys

$$dR(t) = \kappa_1(R(t) - \rho) dt + \sqrt{\kappa_2(R(t) - \rho)} dz(t), \tag{47}$$

where $z(t)$ is a standard one-dimensional Wiener process and

$$\begin{aligned} \kappa_1 &\equiv \mu_{Y_M}, \\ \kappa_2 &\equiv \sigma'_{Y_M} \sigma_{Y_M} (\mu_M - \sigma'_M \sigma_M). \end{aligned}$$

It follows that when the monetary supply process follows a dynamics such that both the drift and the variance are proportional to the nominal state variable, the nominal short rate essentially inherits the dynamics of the latter. The square root process (47) resembles that of CIR, but is slightly more general due to the presence in κ_2 of the parameters of the money supply process. This provides a long awaited theoretical justification to the dozens of papers adopting standard affine models that assume that the dynamics of the nominal short-term rate is given by CIR's dynamics, although CIR derived their results for the real rate. In our setting, the latter rate and the inflation risk premium can be explicitly computed, as shown in the following proposition.

Proposition 14 (Real rate and inflation risk premium). *Assuming $\lambda = \beta = 1$ and $\eta = v = 0.5$, the real short-term rate is equal to*

$$r = (\mu_q - \sigma'_q \sigma_q) Y_q + \frac{(1 - \phi)\rho}{\phi\rho + (\mu_M - \sigma'_M \sigma_M) Y_M} (\sigma'_q \sigma_M + (\mu_M - \sigma'_M \sigma_M) \sigma'_q \sigma_{Y_M}) \sqrt{Y_M} \sqrt{Y_q} \tag{48}$$

and its dynamics obeys

$$dr = h(r, R) dt + \psi(r, R) d\bar{z}, \tag{49}$$

where \bar{z} is a standard one-dimensional Wiener process and

$$\begin{aligned} \psi(r, R) &= \left((\mu_q - \sigma'_q \sigma_q) \sqrt{a(r, R)} + \frac{1}{2} g(R) \right) \sigma'_{Y_q} \sigma_{Y_q} + (g')^2 a(r, R) \frac{R - \rho}{\mu_M - \sigma'_M \sigma_M} \sigma'_{Y_M} \sigma_{Y_M} \\ &+ 2 \left((\mu_q - \sigma'_q \sigma_q) \sqrt{a(r, R)} + \frac{1}{2} g(R) \right) g' \sqrt{a(r, R)} \sqrt{\frac{R - \rho}{\mu_M - \sigma'_M \sigma_M} \sigma'_{Y_q} \sigma_{Y_M}}, \end{aligned}$$

$$\begin{aligned}
 h(r, R) = & (\mu_q - \sigma'_q \sigma_q) \mu_{Y_q} a(r, R) + g' \sqrt{a(r, R)} \mu_{Y_M} \frac{R - \rho}{\mu_M - \sigma'_M \sigma_M} + \frac{1}{2} \sqrt{a(r, R)} g \mu_{Y_q} \\
 & + \frac{1}{2} g'' \sqrt{a(r, R)} \frac{R - \rho}{\mu_M - \sigma'_M \sigma_M} \sigma'_{Y_M} \sigma_{Y_M} - \frac{1}{8} g \left(\sqrt{a(r, R)} \right)^{-1} \sigma'_{Y_q} \sigma_{Y_q} \\
 & + \frac{1}{2} \sqrt{\frac{R - \rho}{\mu_M - \sigma'_M \sigma_M}} g' \mu_{Y_q} \sigma'_{Y_q} \sigma_{Y_M},
 \end{aligned}$$

$$g(R) = \frac{(1 - \phi)\rho}{R - (1 - \phi)\rho} (\sigma'_q \sigma_M + (\mu_M - \sigma'_M \sigma_M) \sigma'_q \sigma_{Y_M}) \sqrt{\frac{R - \rho}{\mu_M - \sigma'_M \sigma_M}},$$

and $a(r, R)$ is the positive root of

$$0 = -r + (\mu_q - \sigma'_q \sigma_q)x + g(R)\sqrt{x}.$$

Under the same set of assumptions, the inflation risk premium is equal to

$$\begin{aligned}
 \varepsilon = & -\frac{(1 - \phi)\rho + R}{(\phi - 1)\rho + R} \sqrt{a(r, R)} \sqrt{\frac{R - \rho}{\mu_M - \sigma'_M \sigma_M}} \sigma'_M \sigma_q + a(r, R) \sigma'_q \sigma_q \\
 & + \frac{(1 - \phi)\rho(R - \rho)}{((\phi - 1)\rho + R)^2} \sigma'_M \sigma_M + 2 \frac{(1 - \phi)\rho(R - \rho)}{((\phi - 1)\rho + R)^2} \sigma'_{Y_M} \sigma_M \\
 & - \frac{(\mu_M - \sigma'_M \sigma_M)((1 - \phi)\rho + R)}{R((\phi + 1)\rho + R)} \sqrt{a(r, R)} \sqrt{\frac{R - \rho}{\mu_M - \sigma'_M \sigma_M}} \sigma'_{Y_M} \sigma_q \\
 & + \frac{(1 - \phi)\rho(\mu_M - \sigma'_M \sigma_M)(R - \rho)}{R((\phi - 1)\rho + R)^2} \sigma'_{Y_M} \sigma_{Y_M}.
 \end{aligned} \tag{50}$$

Interestingly, the real rate process does not recover the linearity found for the nominal rate process. Hence, somewhat paradoxically, CIR’s square root process is more appropriate for modeling the nominal rate than for modeling the real one, in spite of their economy being purely real.

4.3.2. Generalized CIR dynamics

We now assume more generally that the drift and the variance of the money supply process are proportional to a function of the state variable rather than to the state variable itself. As will be shown, the interest rate dynamics changes noticeably. We focus here on the nominal rate, since it is the concern of empirical research and professional applications and the expressions for the real rate become rapidly too complex. We provide two different examples leading to a non-linear and a quadratic structure for the nominal rate, respectively.

Proposition 15 (Non-linear structure). *Assuming $\beta = 2\nu$, the nominal short-term rate becomes*

$$R(t) = \rho + (\mu_M - \sigma'_M \sigma_M) Y_M(t)^\beta \tag{51}$$

and its dynamics obeys

$$dR(t) = (\kappa_5 + \kappa_6 R(t) + \kappa_7 (R(t) - \rho)^\gamma) dt + \sqrt{\kappa_8 (R(t) - \rho)^{1+\gamma}} d\hat{z}(t), \tag{52}$$

where \hat{z} is a standard one-dimensional Wiener process and

$$\begin{aligned} \gamma &\equiv 1 - \frac{1}{\beta}, \quad \kappa_5 \equiv -\beta\mu_{Y_M}\rho, \quad \kappa_6 \equiv \beta\mu_{Y_M}, \\ \kappa_7 &\equiv \frac{1}{2}\beta(\beta - 1)\sigma'_{Y_M}\sigma_{Y_M}(\mu_M - \sigma'_M\sigma_M)^{1/\beta}, \quad \kappa_8 \equiv \beta^2\sigma'_{Y_M}\sigma_{Y_M}(\mu_M - \sigma'_M\sigma_M)^{1/\beta}. \end{aligned}$$

The dynamics of the nominal interest rate is far more complicated than the simple square root process. In particular, the drift and the volatility are highly non-linear functions of the instantaneous rate. Therefore, without assuming exogenous non-linearities in the spot rate or in the state variables, our model delivers non-linearities in the nominal rate dynamics in a simple way. An interesting result is that the drift and the volatility of the nominal rate process react differently to its current value. Several recent non-linear models such as Ahn and Gao’s (1999) thus are recovered.

Our model also embeds the important quadratic term structure models developed by Leippold and Wu (2002) and Ahn et al. (2002). The latter provided a theoretical foundation for such models based on equilibrium properties of a real economy. The following proposition shows that this is but a particular case of our monetary economy.

Proposition 16 (Quadratic structure). *Assuming $\beta = v = 1$, the nominal short-term rate is equal to*

$$R(t) = \rho + \mu_M Y_M(t) - \sigma'_M \sigma_M Y_M(t)^2, \tag{53}$$

and, if the parameters of the money supply process are such that there is only one positive root to (53), its dynamics is such that

$$\begin{aligned} dR(t) &= \left(\kappa_9 + \kappa_{10} R(t) + \kappa_{11} \sqrt{\kappa_{12} + \kappa_{13} R(t)} \right) dt + \left| \kappa_{14} + 2R(t) + \kappa_{15} \sqrt{\kappa_{12} + \kappa_{13} R(t)} \right| \\ &\quad \times \left(\kappa_{16} \sqrt{\kappa_{12} + \kappa_{13} R(t)} - \kappa_{15} \right)^{-1/2} \kappa_{17} d\hat{z}(t), \end{aligned} \tag{54}$$

where \hat{z} is a standard one-dimensional Wiener process and

$$\begin{aligned} \kappa_9 &\equiv -2\rho\mu_{Y_M} - (\sigma'_M\sigma_M\sigma'_{Y_M}\sigma_{Y_M} + \mu_M\mu_{Y_M})\frac{\mu_M}{2\sigma'_M\sigma_M}, \\ \kappa_{10} &\equiv 2\mu_{Y_M}, \quad \kappa_{11} \equiv -(\sigma'_M\sigma_M\sigma'_{Y_M}\sigma_{Y_M} + \mu_M\mu_{Y_M})\frac{1}{2\sigma'_M\sigma_M}, \\ \kappa_{12} &\equiv \mu_M^2 + 4\sigma'_M\sigma_M\rho, \quad \kappa_{13} \equiv -4\sigma'_M\sigma_M, \quad \kappa_{14} \equiv -2\rho - \frac{\mu_M^2}{2\sigma'_M\sigma_M}, \\ \kappa_{15} &\equiv -\frac{\mu_M}{2\sigma'_M\sigma_M}, \quad \kappa_{16} \equiv \frac{1}{2\sigma'_M\sigma_M}, \quad \kappa_{17} \equiv \sqrt{\sigma'_{Y_M}\sigma_{Y_M}}. \end{aligned}$$

One advantage of the specification (54) is that it remains a one-factor model, although it exhibits a quadratic structure for the level of R .

4.3.3. Other dynamics

In fact, our framework is rich enough to generate for the interest rate dynamics many classes of processes. For instance, an alternative to the results above can be obtained by relaxing CIR's assumption of a square root process for the state variable. This assumption was required in CIR to ensure a positive interest rate: since, in equilibrium, their (real) interest rate is proportional to their (real) state variable, the latter must be positive almost surely. It turns out that other processes for the state variables not only lead to positive (nominal) interest rates but also to empirically interesting rate dynamics. For instance, let us assume that the monetary state variable follows

$$dY_M(t) = \mu_{Y_M} Y_M(t) dt + Y_M(t) \sigma'_{Y_M} dZ(t) \quad (55)$$

while the general dynamics for the technology return and the money supply growth rate are preserved. Then the level of the nominal rate is still given by Eq. (51) but its dynamics is quite different. We provide two specific examples. The first one is in the spirit of the lognormal model of Miltersen et al. (1997) where simple (not compounded) forward rates follow lognormal diffusions.

Proposition 17 (Lognormal rate). *Assuming $\beta = 2\nu$, the nominal rate is given by*

$$R(t) = \rho + (\mu_M - \sigma'_M \sigma_M) Y_M(t)^\beta \quad (56)$$

and its dynamics is such that

$$dR(t) = \kappa_{18}(R(t) - \rho) dt + \kappa_{19}(R(t) - \rho) d\hat{z}(t), \quad (57)$$

where \hat{z} is a standard one-dimensional Wiener process and

$$\kappa_{18} \equiv \beta \mu_{Y_M} + \frac{1}{2} \beta (\beta - 1) \sigma'_{Y_M} \sigma_{Y_M}, \quad \kappa_{19} \equiv \beta \sqrt{\sigma'_{Y_M} \sigma_{Y_M}}.$$

Consequently, the drift and the variance of the process are linear functions of the nominal rate level even though the rate itself is not a linear function of the state variable. This is not the case in Eq. (52), although Eqs. (51) and (56) are identical.

Proposition 18 (Linear-quadratic). *Assuming $\beta = \nu$, the nominal interest rate is given by*

$$R(t) = \rho + \mu_M Y_M(t)^\beta - \sigma'_M \sigma_M Y_M(t)^{2\beta} \quad (58)$$

and its dynamics is such that

$$dR(t) = (\kappa_{20} + \kappa_{21}v(t) + \kappa_{22}R(t)) dt + v(t) \sigma'_{Y_M} dZ(t), \quad (59)$$

where

$$v(t) = \beta\mu_M Y_M(t)^\beta - 2\beta\sigma'_M \sigma_M Y_m(t)^{2\beta}, \quad \kappa_{20} \equiv \frac{1}{2}\beta^2 \sigma'_{Y_M} \sigma_{Y_M} \rho,$$

$$\kappa_{21} \equiv \mu_{Y_M} + \frac{1}{2}(\beta - 1) + \frac{1}{2}\beta\sigma'_{Y_M} \sigma_{Y_M}, \quad \kappa_{22} \equiv -\frac{1}{2}\beta^2 \sigma'_{Y_M} \sigma_{Y_M}.$$

The dynamics (59) for the nominal rate is reminiscent of the result obtained by Longstaff and Schwartz (1992). The major difference is that their interest rate is the real one. Note that their model already embeds CIR's model in that the term structure is driven by two factors (the rate level and its volatility) instead of one. In Eq. (59), our two factors are directly linked to the sources of uncertainty, namely the two monetary policy parameters μ_M and σ_M affected by the nominal state variable Y_M . This is, again, consistent with Evans and Marshall's (2001) findings that monetary impulses have a sizeable and lasting influence on the nominal term structure.

5. Conclusion

We have derived the general equilibrium dynamics of the main real and nominal aggregate variables in a monetary economy affected by technological and monetary shocks. The level and dynamics of any real variable, in particular the short-term real rate of interest, is inherently driven by both monetary and real factors. Money non-neutrality thus is generic, as it does not stem from any friction such as price or wage stickiness, asymmetric information and restricted participation or from a particular utility function. Non-neutrality obtains because the ex ante cost of real money holdings is random due to inflation uncertainty. In a specialized version of this economy in which the state variables follow particular processes, and the representative investor has a log separable utility function, we have explicitly derived the level and dynamics of the short-term real and nominal interest rates. These two kinds of rates in fact behave in very different manners, as the inflation risk premium is diversely affected by real and nominal shocks. The processes obtained for the nominal interest rate encompass most of the dynamics offered in the literature, from the standard affine models to the recent quadratic and non-linear models, and lead to new, more general, nominal term structures.

The proposed setting is sufficiently flexible so that many interesting issues of interest for both academics and practitioners can be addressed. At the microeconomic level, these range from fixed income instrument pricing, option pricing and hedging, asset–liability management, value-at-risk assessment and other interest risk measurements to the valuation of floaters, interest rate and currency swaps, forwards and futures, and swaptions, to name a few. At the macroeconomic level, they range from the conduct of monetary policy and its impact on real income, investment, consumption and wealth and on the inflation, interest and exchange rates to the relationship between asset returns and inflation, the equity premium puzzle, the current hotly debated stock return predictability issue, and, last but not foremost, the term structure estimating issue.

This work could be extended in a number of ways. First, instead of adopting the money-in-the-utility approach, one could consider a cash-in-advance economy with a credit good in addition to the cash good. Second, an explicit reaction function on the part of the monetary authorities could be modeled rather than assuming an exogenous process for the money supply. This could allow for the comparative study of the influence of various policy rules on the aggregate variables deemed relevant. Third, a government sector with autonomous expenditures and nominal taxes could be introduced, for instance to assess the impact of fiscal policies on the real and nominal yield curves.

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