



# Optimal liquidity management and bail-out policy in the banking industry

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## Abstract

We characterize the profit-maximizing reserves of a commercial bank, and the generated probability of a liquidity crisis, as a function of the penalty imposed by the Central Bank, the probability of depositors' liquidity needs, and the return on outside investment opportunities. We demonstrate that banks do not fully internalize the social cost associated with the bail-out policy if the liquidity needs of individuals are correlated, and that competitive inter-bank markets will induce banks to raise their reserves under reasonable conditions. The marginal benefits from an interbank market decrease as the correlation between the liquidity shocks of banks increases.

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## 1. Introduction

The literature on demand deposits has focused on liquidity crises generated by expectation-driven panics, but it has not offered any general method for calculating the probability of bank runs generated by a realization of liquidity needs by a large, but finite, number of depositors. Therefore, in this article we propose a method for calculating the probability of realizing a liquidity crisis and we characterize banks' optimal reserve ratio assuming that depositors face *real* liquidity needs as opposed to

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rumors or panics concerning a liquidity crisis. We determine the optimal response of a commercial bank to the interest rate (penalty rate) at which the Central Bank offers liquidity. We further delineate the socially optimal bail-out policy and show that the commercial bank does not fully internalize the social costs of a liquidity crisis. In particular, we establish that the socially optimal penalty rate is increasing in the correlation of the liquidity shocks facing depositors. Finally, we explore the implications of interbank markets. We prove that access to an interbank market will typically induce competing banks to raise their reserve holdings, as the interbank market offers an opportunity to benefit from potential liquidity needs of competing banks in a situation where the bank has excess reserves. However, the marginal benefits to banks from an interbank market are shown to decrease in response to an increase in the correlation between the liquidity shocks of banks.

The existing banking literature views the depository institutions as “pools of liquidity” providing consumers with insurance against idiosyncratic liquidity shocks. In the influential model by Diamond and Dybvig (1983), banks provide liquidity to depositors who are, *ex ante*, uncertain about their intertemporal preferences with respect to consumption sequences. They demonstrate how deposit contracts offer insurance to consumers and how such contracts can support a Pareto efficient allocation of risk. However, as they show, there exists a second, inefficient Nash equilibrium where the interaction between pessimistic depositor expectations generates a liquidity crisis. Such liquidity crises confronting individual banks may trigger socially costly bank panics. Against this background, most countries apply explicit or implicit deposit insurance policies as a mechanism for the elimination of inefficient Nash equilibria driven by pessimistic expectations. Despite the indisputable insurance benefits, empirical observations as well as theoretical research convincingly demonstrate how federal deposit insurance will encourage banks to engage in excessive risk taking and to keep lower levels of liquid reserves than what would be socially optimal (cf. Cooper and Ross, 1998). Consequently, researchers have systematically investigated mechanisms other than deposit insurance as instruments for reducing the instability of the banking system. Bhattacharya et al. (1998) categorize those regulatory measures.<sup>1</sup> In addition, all policy commitments relative to distressed financial institutions face a severe time-consistency problem as governments and Central Banks seem to have an incentive of bailing out distressed financial institutions with the intention of eliminating potential contagion problems (e.g. Chen, 1999). Freixas (1999) investigates such bail-out policies.

A meaningful evaluation of all policy measures directed towards the banking industry rely on the knowledge of how *ex-ante* uncertain liquidity needs translate into probabilities of realizing liquidity crises and of how the characteristics of this transmission mechanism interacts with banks’ optimal allocation of their portfolios between liquid low-yield assets and illiquid high-yield investments. In this paper we

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<sup>1</sup> Cordella and Yeyati (1999) offer a particularly rich study of bail-out policies which are designed to be contingent on the realization of well-specified states of nature. They show that such bailout policies might induce a risk-reducing value effect which can be so strong so as to outweigh the well-known moral hazard effect.

delineate the bank's optimal liquidity management and characterize how the profit-maximizing reserves adjust to the interest rate applied by the Central Bank for liquidity provision to a bank facing a liquidity crisis. In particular, the optimal reserves are found to be an increasing function of the correlation between the liquidity shocks facing individual depositors. As Holmström and Tirole (1998) show, the private sector cannot satisfy its own liquidity needs when aggregate uncertainty dominates the liquidity shocks. In our study we characterize the socially optimal interest rate (denoted penalty rate) to charge from banks facing a liquidity crisis. We find the socially optimal penalty rate to be an increasing function of the correlation between the liquidity shocks facing depositors. Thus, the private banking industry fails to fully internalize the full social costs of public liquidity provision.

Our study is organized as follows. Section 2 presents our model and characterizes the bank's optimal liquidity management. In Section 3 we carry out a welfare analysis and delineate the socially optimal bail-out policy. In Section 4 we explore the effects of introducing an interbank market. Section 5 concludes.

## 2. The model

Consider a three-period economy with one representative commercial bank and  $n$  depositors with known distributions of liquidity needs. Each depositor has  $d_i$  to deposit. Therefore, the total amount of money deposited in the bank is  $D = d_1 + \dots + d_n$ .

### 2.1. Timing

The economy operates in periods,  $t = 0, 1, 2$ . In period 0 consumer  $i$ ,  $i = 1, \dots, n$ , makes a deposit  $d_i$ , followed by the bank's decision of which reserve ratio to maintain and thereby which proportion of the accumulated deposits to allocate to an illiquid investment project. In period 1 depositors face uncertain liquidity needs, which may generate a liquidity crisis. In period 2 the bank collects the return on the outside investment project and pays a penalty if it borrows from the Central Bank in the event of a liquidity crisis in period 1.

### 2.2. The commercial bank

Let  $r$ ,  $0 \leq r \leq 1$ , be the reserve ratio which is set by the commercial bank. The bank keeps  $rD$  as reserves. The remaining amount,  $(1 - r)D$ , is invested into an outside investment project which bears a safe net return (gain) of  $g > 0$ . This investment project cannot be liquidated until period 2. Thus,  $g$  is the net gain on a two-period investment project. The depositors are paid a safe net return  $g^d$  ( $0 < g^d < g/2$ ) per period. We assume that the commercial bank considers itself unable to strategically affect  $g^d$ .

We consider a system in which the banks' liquidity management is centralized in the sense that the Central Bank acts as a counterpart and guarantees the finality of the banks' obligations (see Rochet and Tirole, 1996). More precisely, we assume that

the Central Bank maintains a deposit policy where it is committed to bail-out the bank in case of a liquidity crisis. The Central Bank imposes a penalty of  $\gamma$  for each unit of money it lends to the bank so as to make this survive the liquidity crisis. The bank has to compensate the Central Bank for this loan after it collects the return from the illiquid high-return investment project. This penalty serves as a general policy instrument with the practical interpretation as the interest rate the commercial bank has to pay in order to tap emergency resources from the Central Bank's discount window.<sup>2</sup>

Let  $X$  ( $0 \leq X \leq 1$ ) be the random withdrawal rate of the bank's deposits, with an associated strictly increasing and absolutely continuous distribution function  $F$ . The distribution function  $F$  is known by the bank and the withdrawal rate is further characterized in Section 2.3. In the presence of a bail-out policy the expected profit of the bank, is given by

$$E[\Pi] = (1-r)gD - \gamma DE[X-r]^+ - g^d DE[X] - ((1+g^d)^2 - 1)(1-E[X])D, \quad (1)$$

where

$$\gamma DE[X-r]^+ = \gamma D \int_r^1 (y-r) dF(y). \quad (2)$$

The first term in (1) measures the bank's profit (net return) generated by investing  $(1-r)D$  in the two-period illiquid investment project. The second term,  $\gamma DE(X-r)^+$ , which is spelled out in (2), measures the expected penalty imposed on the bank. This term is the product of the penalty rate,  $\gamma$ , and the expected amount withdrawn in  $t=1$  beyond the reserves held by the bank. The second line in (1) expresses the total interest paid to depositors by the commercial bank, in periods  $t=1$  and  $t=2$  respectively. The interest payments are realized at the end of period 2. Obviously, the penalty will depend on the size of reserves. We assume that the Central Bank will bail-out a bank facing a liquidity crisis at a penalty rate  $\gamma$ . Section 3 provides an analysis for determining the socially optimal penalty rate.

The bank chooses a reserve ratio,  $r$ , to maximize its profit given in (1). By applying Leibnitz' rule, the necessary and sufficient conditions for the profit-maximizing reserve ratio are implicitly given by<sup>3</sup>

$$\frac{1}{D} \cdot \frac{\partial E[\Pi]}{\partial r} = -g + \gamma(1 - F(r)) = 0, \quad (3)$$

meaning that the probability of a liquidity crisis is given by

<sup>2</sup> Of course, this is a very crude instrument to support a bail-out policy. Policy instruments like (partial) deposit insurance systems or costly bank charters would qualitatively serve the same purpose. However, the details associated with an analysis focusing specifically on these instruments would make the model much more complicated. Here we have designed the model in the simplest possible way, paying no attention to contractual fine-tuning.

<sup>3</sup> Sufficiency follows from the fact that  $F$  is increasing in  $r$ .

$$1 - F(r^*) = \frac{g}{\gamma}. \quad (4)$$

Reformulating (4) we find the profit-maximizing reserve ratio to satisfy

$$r^* = F^{-1}\left(\frac{\gamma - g}{\gamma}\right), \quad \text{for } \gamma > g, \quad (5)$$

where  $F^{-1}$  is the inverse of the distribution function. Eq. (4) implies that the commercial bank sets its reserve ratio so that the probability of a liquidity crisis,  $1 - F(r^*)$ , equals  $g/\gamma$ , which is the ratio between the return on the external illiquid investment project and the penalty rate. In order to induce the commercial bank to hold any reserves at all, the penalty rate  $\gamma$  must exceed the rate of return on the bank's investment project,  $g$ . We summarize the results in the following proposition.

**Proposition 1.** *The bank adjusts its profit-maximizing reserve ratio so that the probability of a liquidity crises is invariant to the liquidity need by a representative depositor. In addition, the bank's optimal reserve ratio is a decreasing function of the bank's two-period net investment return  $g$  and an increasing function of the penalty rate  $\gamma$ .*

The commercial bank adjusts its optimal reserve ratio so as to keep the probability of a liquidity crisis at the level  $g/\gamma$ . In fact, Clouse and Dow (2002) offer evidence (based on a day to day horizon) that banks adjust to increased volatility by increasing their average reserves so as to keep the probability of facing a liquidity shortage constant.

### 2.3. The aggregate distribution of withdrawals

In this section we analyze how the liquidity needs of a large number of representative depositors translates into an aggregate withdrawal rate. Assume that depositor  $i$  faces an uncertain liquidity need,  $d_i X_i$ , in period 1, where,  $X_i \in [0, 1]$  is the fraction of the deposit  $d_i$  that depositor  $i$  would like to withdraw in period 1. In the general case the distribution  $F$  of the random aggregate withdrawal rate facing the bank,  $X$ , is a convolution which has to be identified by the bank. It can be seen that

$$DE[X] = \sum_{i=1}^n d_i E[X_i] \quad \text{and} \quad D^2\sigma_X^2 = \sum_{i,j=1}^n \rho_{ij} d_i d_j \sigma_{X_i} \sigma_{X_j}, \quad (6)$$

where  $\sigma_X^2$  denotes the variance of  $X$  and  $\rho_{ij}$  denotes the correlation between the liquidity needs of depositors  $i$  and  $j$ . Under fairly mild conditions the distribution of the aggregate withdrawal rate  $X$  can be approximated by a normally distributed random variable <sup>4</sup>

<sup>4</sup> A normal approximation is justified if, for example,  $n$  is large and if the  $d_i X_i$ 's are identically distributed and  $0 < \rho_{ij} < 1$ , when  $i \neq j$ , or if  $X_i \sim_{\text{approx}} \mathbf{N}(E[X_i] \text{Var}[X_i])$ . This approximation also applies to an aggregation over a heterogeneous pool of depositors as long as there are sufficiently many depositors of each type.

$$X \sim_{\text{approx}} \mathbf{N}(E[X], \sigma_X^2). \tag{7}$$

With this approximation, the probability of a liquidity crisis (4) and the profit maximizing reserve ratio (5) simplify to

$$1 - \Phi\left(\frac{r^* - E[X]}{\sigma_X}\right) = \frac{g}{\gamma}, \tag{8}$$

and

$$r^* = E[X] + \sigma_X \cdot \Phi^{-1}\left(\frac{\gamma - g}{\gamma}\right), \tag{9}$$

where  $\Phi$  is the standard normal distribution function. The implications of Eq. (9) are summarized in the following proposition.<sup>5</sup>

**Proposition 2.** *For a normally distributed  $X$ , the profit-maximizing reserve ratio  $r^*$  is*

- (a) *linearly increasing in the depositors' expected liquidity need  $E[X]$ ,*
- (b) *linearly increasing in the standard deviation of the depositors' aggregated liquidity need for a sufficiently large penalty rate ( $\gamma \geq 2g$ ), and decreasing for lower penalty rates ( $\gamma < 2g$ ).*

We can intuitively explain Proposition 2 as follows. The illiquid investment can be regarded as a call option for the commercial bank and the reserves can be regarded as a put option. The value  $\gamma = 2g$ , which is the median of  $X$ , represents a threshold such that these option values are invariant to an increase in the variance  $\sigma_X^2$ . If the penalty rate  $\gamma$  is large enough ( $\gamma > 2g$ ), the option value associated with the illiquid investment is lower than the option value associated with holding reserves. For that reason an increase in the underlying variance  $\sigma_X^2$ , will increase the bank's optimal reserve ratio. Conversely, if the penalty rate is low enough ( $\gamma < 2g$ ), an increased variance will lower the commercial bank's incentives to hold reserves, promoting allocations towards the illiquid high-return investment.

Under the assumption of mutually identical, possibly correlated liquidity needs, it can be seen that the variance of total withdrawals can be written as

$$D^2 \sigma_X^2 = n^2 d^2 \sigma_X^2 = d^2 \sum_{i,j=1}^n \rho_{i,j} \sigma_{X_i} \sigma_{X_j} = nd^2(1 + (n - 1)\rho)\sigma_{X_i}^2, \tag{10}$$

where  $\rho = \rho_{i,j}$ ,  $0 < \rho < 1$  when  $i \neq j$ . Therefore, it holds that

$$\sigma_X^2 = (\rho + (1 - \rho)/n)\sigma_{X_i}^2. \tag{11}$$

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<sup>5</sup> A generalization to non-normal distributions is possible, but is not presented here for the purpose of keeping this paper as short as possible.

This variance is decreasing in the number of customers and it approaches

$$\lim_{n \rightarrow \infty} \sigma_X^2 = \rho \sigma_{X_i}^2. \quad (12)$$

Substitution of (11) into (9) yields the following finding.

**Proposition 3.** *When the depositors face mutually identical and correlated liquidity shocks, the profit-maximizing reserve ratio  $r^*$  is*

- (a) *increasing (decreasing) in the correlation of liquidity shocks,  $\rho$ , if  $\gamma \geq 2g$  ( $\gamma \leq 2g$ );*
- (b) *asymptotically decreasing (increasing) in the number of depositors,  $n$ , if  $\gamma \geq 2g$  ( $\gamma \leq 2g$ ).*

It is seen from (12) that the systematic component  $\rho \sigma_{X_i}^2$  becomes more significant when there are more depositors. As Proposition 3(a) demonstrates, the bank adjusts to an increased correlation among depositors by allocating more funds to the reserves. However, as will be shown in Section 3, from a social welfare point of view, the bank has insufficient incentives to increase the reserves in response to an increase in the correlation among the depositors' liquidity shocks.

### 3. The socially optimal bail-out policy

The Central Bank is assumed to have commitment power relative to the banking industry. As a rational policymaker it anticipates the commercial bank's reserve response  $r^*$ . The Central Bank determines the optimal penalty rate  $\gamma$  on lending intended to bail out the banking industry during a liquidity crisis.

Assume that society (as represented by the Central Bank) has to pay the interest rate,  $\delta$ , for the liquidity channeled to maintain the banking system operative. This interest rate is an increasing and convex function of the expected amount of the liquidity needed to bail out the bank. Hence, the social cost of liquidity is given by a function  $\delta = \delta(DE[X - r^*]^+)$ , where  $\delta', \delta'' > 0$ .

We define the economy's welfare function as the sum of the commercial bank's expected profit and the depositors' expected yield minus the cost for the Central Bank of providing the banking system with the required liquidity. Formally, the benevolent Central Bank determines the penalty rate in order to maximize expected social welfare, determined by

$$\begin{aligned} E[W(\gamma, \cdot)] &= (1 - r^*)gD - \gamma DE[X - r^*]^+ - \delta(DE[X - r^*]^+) + \gamma DE[X - r^*]^+ \\ &= (1 - r^*)gD - \delta \left( D \int_{r^*}^1 (y - r^*) dF(y) \right). \end{aligned} \quad (13)$$

The expected social welfare,  $EW$ , is the difference between the expected profit of the commercial bank (1) and the social cost of raising funds so as to support the bail-out policy. Eq. (13) captures the idea that the social cost of emergency funding from the Central Bank will in general depend on external parameters as well as

institutional distortions in the economy. As we show below, a higher penalty rate  $\gamma$  will increase the reserves and reduce the expected amount of liquidity needed to bail out the bank. The additional cost imposed on society for the bail-out policy is  $\delta(DE[X - r]^+) - D\gamma E[X - r]^+$ . The expected net interest paid to the depositors does not enter into the welfare function (13), since it represents only a transfer from the bank to consumers. In order to transform the social welfare function into a more tractable form we use the following lemma.<sup>6</sup>

**Lemma 4.** For  $X \sim N(E[X], \sigma_X^2)$  it holds that

$$E[X - r^*]^+ = \int_{r^*}^1 (y - r^*) dF(y) = \sigma_X(\phi(\underline{z}) - \underline{z}g/\gamma), \tag{14}$$

where  $\underline{z}(\gamma) = \Phi^{-1}\left(\frac{r-g}{\gamma}\right)$  and  $\phi$  is the standard normal density function. Furthermore,

$$\frac{\partial E[X - r^*]^+}{\partial \gamma} = \frac{\partial}{\partial \gamma}(\phi(\underline{z}) - \underline{z}g/\gamma) = -\underline{z}'g/\gamma < 0. \tag{15}$$

Relation (15) in Lemma 4 captures the economic intuition that a higher penalty rate induces the commercial bank to hold higher reserves. Maximizing (13) and taking (15) into account yields the following optimality condition:

$$\begin{aligned} \frac{\partial E[W]}{\partial \gamma} &= -D\underline{z}'g - \delta'(D\sigma_X(\phi(\underline{z}(\gamma)) - \underline{z}(\gamma)g/\gamma)) \cdot (-\underline{z}'g/\gamma) \cdot D\sigma_X \\ &= D\underline{z}'g(\sigma_X\delta' - 1) = 0. \end{aligned} \tag{16}$$

Sufficiency follows from the fact that  $\partial\delta'/\partial\gamma = \delta''(\cdot) \cdot (-\underline{z}'g/\gamma) \cdot D\sigma_X < 0$ . Now we are ready to state the central proposition of this section. Let  $\gamma^*$  be determined by the optimality condition (16).

**Proposition 5.** The socially optimal penalty rate,  $\gamma^*$ , imposed on the commercial bank is an increasing function of the correlation between the depositors' withdrawals. Formally,  $\partial\gamma^*/\partial\rho > 0$ .

**Proof.** As  $\sigma_X\delta'(D\phi(\underline{z}(\gamma)) - \underline{z}(\gamma)g/\gamma) = 1$  the implicit function theorem implies that

$$\delta'(\cdot) \frac{\partial\sigma_X}{\partial\rho} d\rho = \delta''(\cdot)\underline{z}'g\gamma^{*-1} D\sigma_X^2 d\gamma^*. \tag{17}$$

By (11) it holds that  $\sigma_X^2 = (\rho + (1 - \rho)/n)\sigma_{X_i}^2$ , which implies

$$\frac{\partial\sigma}{\partial\rho} = \frac{\partial\sigma_X^2}{2\sigma\partial\rho} = \frac{(n - 1)\sigma_{X_i}^2}{2n\sigma_X}. \tag{18}$$

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<sup>6</sup> The proof of the lemma is standard stochastic calculus, and available from the authors upon request.

Hence,

$$\frac{\partial \gamma^*}{\partial \rho} = \frac{\delta'}{\delta''} \cdot \frac{(n-1)\gamma^* \sigma_{X_i}^2}{2nD(\cdot) \underline{z}' g \sigma_X^3} > 0. \quad \square \quad (19)$$

As Proposition 5 shows, the socially optimal penalty rate is an increasing function of the correlation between the liquidity shocks facing the depositors. This is an intuitive result in light of the fact that an increased probability of a liquidity crisis makes it increasingly costly for the policymaker to raise the funds necessary to support the bail-out program required to avoid a liquidity crisis. Furthermore, Proposition 3(a) implies that a private banking industry adjusts the reserve ratio to an increase in the withdrawal correlation, but this adjustment is insufficient from the social point of view. In other words, the private banking industry does not fully internalize the costs of raising emergency liquidity needed to support the bail-out program.<sup>7</sup>

In qualitative terms Proposition 5 means that the correlation of depositor withdrawals should be an essential element in the design of the regulatory framework for the banking industry. In fact, Proposition 5 can be viewed as a formalization of the policy conclusion that an increased systemic risk, measured as an increased degree of correlation of liquidity shocks among depositors, calls for stronger intervention in the sense of higher penalty rates. This conclusion can be seen to complement the analysis in Rochet and Tirole (1996) and Freixas et al. (2000), which analytically explore how the design of interbank markets will impact on the vulnerability of the banking industry to systemic liquidity risks.

#### 4. Interbank markets

Our analysis so far has relied on the assumption of a single representative bank without access to any interbank market. In this section we relax this assumption and study the effect of introducing an interbank market. We focus on an interbank market opening up access to lending by a bank with excess reserves as a way to complement a bail-out policy as a source of emergency liquidity. For simplicity, and without loss of generality, we assume that there are two banks, A and B. In our implementation, we focus on bank A, and the competing bank B represents all other banks in the industry. Bank A has a market share  $\alpha$  and the competing bank B has a market share  $1 - \alpha$ . The distributions for the banks' aggregate withdrawal rates are denoted  $F_A$  and  $F_B$  respectively. Each depositor is assumed to maintain a single account in one of the banks only. Bank A has  $n_A = \alpha n$  customers, whereas its competitor B has  $n_B = (1 - \alpha)n$  customers. All depositors are identical, and the mutual withdrawal correlation with any other depositor is  $\rho$ .

<sup>7</sup> This result is generally in line with Holmström and Tirole (1998). However, these authors study a completely different environment and they do not explicitly focus on the correlation between the depositors liquidity needs.

Obviously, the total variance of the total withdrawals in the entire banking system remains the same as before,  $D^2\sigma_X^2 = nd^2(1 + (n - 1)\rho)\sigma_{X_i}^2$  (see (10)). The correlation of the withdrawal rates between banks A and B is

$$\begin{aligned} \rho_{A,B} &= \frac{n(1 + (n - 1)\rho) - n_A(1 + (n_A - 1)\rho) - n_B(1 + (n_B - 1)\rho)}{2\sqrt{n_A(1 + (n_A - 1)\rho)n_B(1 + (n_B - 1)\rho)}} \\ &= \sqrt{\frac{n^2\alpha(1 - \alpha)\rho^2}{\alpha(1 - \alpha)n^2\rho^2 - (n - 1)\rho^2 + (n - 1)\rho + (1 - \rho)}}. \end{aligned} \tag{20}$$

From (20) it can be verified that the interbank correlation has the following properties:

$$\forall \rho > 0: \quad \lim_{n \rightarrow \infty} \rho_{A,B} = 1, \tag{21}$$

and as  $n^2\alpha(1 - \alpha) \geq n - 1$ ,

$$\forall n \in \{3, 4, \dots\} \quad \forall \rho \in (0, 1) : \quad \rho_{A,B} > \rho. \tag{22}$$

Hence, the interbank correlation is always higher than the correlations between the individual depositors, and when the number of depositors increases the correlation between the aggregate withdrawals of the banks becomes almost perfect.

If bank A cannot fulfil its liquidity needs in the interbank market, it has access to a bail-out policy. Its expected profit has a structure similar to (1) as captured by

$$\begin{aligned} E[\Pi_A] &= (1 - r_A)g\alpha D - \gamma\alpha DE[X_A - r_A]^+ \\ &\quad - g^d\alpha DE[X_A] - ((1 + g^d)^2 - 1)(1 - E[X_A])\alpha D, \end{aligned} \tag{23}$$

and its profit-maximizing reserve ratio (with the normal distribution) would be

$$r_A^* = E[X_A] + \sigma_A \cdot \Phi^{-1}\left(\frac{\gamma - g}{\gamma}\right) > r^*, \tag{24}$$

where  $\sigma_A^2 = (\rho + (1 - \rho)/n_A)\sigma_{X_i}^2$ . The inequality in (24) follows from the fact that bank A faces an aggregate shock with a higher standard deviation,  $\sigma_A > \sigma$ , than a single bank operating in the absence of competition. This effect captures the idea that a smaller bank is forced to survive with a lower degree of diversification. Therefore, a small bank needs relatively more reserves than a big bank.

When bank A has access to the interbank market it has the opportunity to lend (some share of) its slack reserves in period 1 to other banks through the interbank market at an interest rate,  $\theta$ . From the point of view of bank A, the volume of the interbank market is restricted by its supply,  $D\alpha(r_A - X_A)^+$ , as well as by the demand of liquidity by its competitor,  $D(1 - \alpha)(X_B - r_B)^+$ . The factor  $D \min(\alpha(r_A - X_A)^+, (1 - \alpha)(X_B - r_B)^+)$  represents the realized volume of transactions in the interbank market with bank A on the lending side and bank B on the borrowing side. In the following this volume is denoted  $D\varepsilon_A$ . Thus the expected volume of bank A's lending to the interbank market is  $DE[\varepsilon_A(r_A, r_B, \rho)]$ .  $\varepsilon_B$  is defined analogously; i.e.  $\varepsilon_B = \min(\alpha(X_A - r_A)^+, (1 - \alpha)(r_B - X_B)^+)$ . Overall, the expected volume of the interbank market is given by  $DE[\varepsilon_A + \varepsilon_B]$ . The interbank market is complemented by a

bail-out policy whereby the Central Bank provides liquidity at rate  $\gamma$  in case of insufficient supply of liquidity in the interbank market.<sup>8</sup> With access to the interbank market, the profit for bank A associated with lending in the interbank market is given by

$$E[\Pi_A] = (1 - r_A)g\alpha D + \theta DE[\mathcal{E}_A] + (\gamma - \theta)DE[\mathcal{E}_B] - \gamma D\alpha E[X_A - r_A]^+ - g^d \alpha DE[X_A] - ((1 + g^d)^2 - 1)(1 - E[X_A])\alpha D. \quad (25)$$

By comparing (25) with (1) we can identify a twofold effect for bank A from the presence of the interbank market. The term  $\theta DE[\mathcal{E}_A]$  captures bank A's expected revenues from lending its excess reserves at the interest rate  $\theta$ . In addition, bank A may gain from the interbank market as it may be able to extract reserves from this market at a lower interest rate than the penalty rate,  $\gamma$ , applied by the Central Bank. The term  $(\gamma - \theta)DE[\mathcal{E}_A]$  measures the expected benefit of access to reserves at the interbank rate  $\theta$  rather than at the penalty rate  $\gamma$  maintained by the Central Bank.

There are alternative mechanisms for how the interest rate could be determined in the interbank market. It must hold that  $\theta \in [0, g)$ , because if it were true that  $\theta \geq g$  the bank would have no incentive to finance illiquid projects. In addition, as observed in our discussion subsequent to (5) it must hold that  $g < \gamma$ ; hence,  $0 \leq \theta < g < \gamma$ . The interest rate  $\theta$  could be determined through Nash bargaining between banks with excess supply and excess demand of liquidity. Alternatively, the interest rate in the interbank market could be the result of a Central Bank operating with a target for this interest rate and prepared to intervene so as to support such a target.<sup>9</sup> In this context we do not pursue to explore the interesting and important details of the interest rate determination as our present purpose is to briefly outline the consequences for the reserve holdings of the presence of an interbank market.<sup>10</sup>

The most important characterizations of how Bank A's expected lending activities in the interbank market depend on the reserve holdings of both banks are summarized in the following lemma.

**Lemma 6.** *Bank A's expected lending to the interbank market,  $E[\mathcal{E}_A(r_A, r_B, \rho)]$ , has the following properties (when  $\rho_{A,B} < 1$ ):*

- (a)  $\partial E[\mathcal{E}_A]/\partial r_A > 0$ ,
- (b)  $\partial E[\mathcal{E}_A]/\partial r_B < 0$ ,
- (c)  $\partial^2 E[\mathcal{E}_A]/(\partial r_A \partial r_B) < 0$ , and
- (d)  $E[\mathcal{E}_A]$  is monotonically decreasing in the correlation of the withdrawals between the banks  $\rho_{A,B}$  and is bounded from below by the following limits:

<sup>8</sup> One can make the interpretation that the interbank market operates under the restriction determined by the Central Bank's money market target rate. Here, the rate ceiling is  $\gamma$  and for simplicity we take the rate floor to be 0.

<sup>9</sup> These mechanisms were in fact suggested to us by an anonymous referee.

<sup>10</sup> Of course, a detailed study of the interest rate determination would be an important and challenging topic for further research within the context of our model.

- (i) If  $F_A(r_A) > F_B(r_B)$  then bank A is more frequently on the lending side in the interbank market than B, and

$$E[\Xi_A]/D \downarrow \int_{F_A^{-1}(F_B(r_B))}^{r_A} \min \{ \alpha(r_A - x)^+, (1 - \alpha)(y - r_B)^+ \} dF_A(x),$$

when  $\rho_{A,B} \uparrow 1$ , and

- (ii) if  $F_A(r_A) \leq F_B(r_B)$  then  $E[\Xi_A] \downarrow 0$  when  $\rho_{A,B} \uparrow 1$ .

**Proof.** In Appendix A.  $\square$

Lemma 6(a) captures the natural idea that bank A will be able to benefit more from the interbank market the higher is its reserves. Part (b) of the Lemma establishes that there is a conflict of interest in the interbank market between the two rival banks. Consequently, the expected benefit to bank A is decreasing as a function of the reserves held by its rival, because this means that bank A will face a reduced expected demand for liquidity. Lemma 6(c) exhibits that the reserve holdings of the two rival banks serve as strategic substitutes in the interbank market. This means that bank A has a strategic incentive to be aggressive in its reserve holdings, because that will induce the rival bank to reduce its reserves. This optimal response from the rival bank makes the interbank market more profitable for bank A as it will add to the rival’s demand for liquidity. By applying the same type of strategic arguments to the rival bank we can conclude that strategic opportunities for lending opportunities in the interbank market will induce the banks to increase their reserves. Access to interbank borrowing works in the opposite direction. The total effect on the optimal reserves is dependent on which of these effects dominates. Formally, it is dependent on the sign of the derivative  $\partial\{\theta DE[\Xi_A] + (\gamma - \theta)DE[\Xi_B]\}/\partial r_A$ . From Lemma 6(a) and (b) we can infer that

$$\frac{\partial^2}{\partial r_A \partial \theta} \{ \theta DE[\Xi_A] + (\gamma - \theta)DE[\Xi_B] \} > 0. \tag{26}$$

From (26) we can conclude that a higher interbank rate will increase the incentives for holding reserves. A higher interbank rate, increases the optimal reserve ratio. Deeper intuition for this is achieved by studying a duopoly market with symmetric banks ( $\alpha = 1/2$ ) in the following:

**Proposition 7.** Consider a symmetric equilibrium with  $r_A = r_B$  when the banks maintain equal market shares ( $\alpha = 1/2$ ). Then  $\theta > \gamma/2$  implies that the equilibrium reserve levels held by the banks are higher than those held without access to the interbank market.

Proposition 7 can be understood as follows. Lemma 6(a)–(c) in combination with  $\theta > \gamma/2$  and  $r_A = r_B$  imply that  $\partial\{\theta DE[\Xi_A] + (\gamma - \theta)DE[\Xi_B]\}/\partial r_A > 0$  and  $\partial^2\{\theta DE[\Xi_A] + (\gamma - \theta)DE[\Xi_B]\}/\partial r_A \partial r_B > 0$ . Therefore, both banks have an incentive to increase their reserves above the optimal level held without an interbank market.

In general, in a symmetric equilibrium it holds that  $\partial E[\Xi_A]/\partial r_A = \partial E[\Xi_B]/\partial r_A$  as we can see from (A.3) and (A.4). For that reason the sign of

$$\theta \frac{\partial E[\Xi_A]}{\partial r_A} = (\gamma - \theta) \frac{\partial E[\Xi_B]}{\partial r_A} \quad (27)$$

is determined by the sign of  $\theta - \gamma/2$ . Intuitively, this captures the idea that the revenue-enhancing effect of increased reserve holdings associated with lending in the interbank market exceeds the revenue-reducing effect of more extensive borrowing activities when the interest rate is sufficiently high, i.e.  $\theta > \gamma/2$ . For interest rates below  $\gamma/2$  this relationship is reversed.

In general, it appeals to intuition that the marginal expected benefit from an additional unit of reserves is decreasing as a function of the correlation of the liquidity shocks facing the two competing banks. This intuition is formally verified by Lemma 6(d). The expected benefit from the interbank market approaches zero as the correlation in the liquidity shocks facing the two banks approaches one, if the banks adjust their reserves symmetrically, in the sense that their (unconditional) probabilities of the liquidity needs are the same. With perfect correlation, bank A can benefit from the interbank market only if it has a lower probability of liquidity needs than the competitor B, i.e.  $F_A(r_A) > F_B(r_B)$ . This asymptotic result is equivalent to the qualitative conclusion that the expected benefit from an interbank market tends to vanish if the banks face liquidity risks which approach perfect correlation. This limitation of the benefits from the interbank market seems to complement the associated characterizations in Rochet and Tirole (1996) or Freixas et al. (2000).

We summarize the expected benefits from an interbank market as characterized by properties expressed by Lemma 6(a)–(d) in the following proposition.

**Proposition 8.** *The marginal benefits from an interbank market decrease as the correlation between the liquidity shocks of banks increase. If  $F_A(r_A) \leq F_B(r_B)$  bank A's lending through the interbank market disappears as we approach the limit case of perfect correlation.*

From Propositions 7 and 8 we can draw the general conclusion that interbank markets serve as a mechanism to introduce competition with respect to liquidity provision between banks. This competition has positive welfare implications as it raises the reserve holdings of the banking industry as long as the interbank rate is not too small relative to the penalty rate. However, this effect diminishes when the correlation,  $\rho_{A,B}$ , increases. It is still left as an issue for future research to explore the relationship between the industry equilibrium of reserves generated by competition and the socially optimal reserve holding. In this respect the perspective recently offered by Gorton and Huang (2002) seems particularly interesting. Namely, within a slightly different framework Gorton and Huang (2002) argue that there is a welfare-enhancing role for public supply of liquidity because the government can issue government securities backed by (future) tax revenue.

## 5. Concluding comments

This paper analyzes a banking industry with liquidity risks caused by depositors facing uncertain liquidity needs. We develop a method for calculating the profit-maximizing amount of reserves of a representative bank, and characterize the associated probability of a liquidity crisis. We show that the only information needed to predict the probability of a liquidity crisis is the cost of maintaining reserves and the penalty rate charged to a bank facing a run.

Within the framework of our welfare analysis we delineated a characterization of the socially optimal penalty rate, which, by taking the optimal response of the banking industry into account, will determine the socially optimal fraction of reserves. Importantly, this socially optimal penalty rate was found to be an increasing function of the correlation between the liquidity shocks facing depositors. Indeed, as was established in our analysis, the private banking industry will have an incentive to adjust the reserves upwards when facing an increased correlation, but this incentive will for structural reasons be too weak from a social point of view. Namely, the banking industry does not fully internalize the increasing social costs associated with a need to raise additional liquidity in order to support a more extensive bail-out program. We further demonstrated that access to an interbank market will induce competing banks to raise their reserve holdings under reasonable conditions. However, the marginal benefits from an interbank market was shown to decrease as a function of the correlation between the liquidity shocks of banks.

The expected profit of the bank (1) could be formulated in alternative ways. An example of a plausible reformulation of (1) would be:

$$E[\Pi] = (1 - r)gD - \gamma D \int_{r/(1+g^d)}^1 (y(1 + g^d) - r) dF(y) - E[X]g^d D - ((1 + g^d)^2 - 1)(1 - E[X])D, \quad (28)$$

where the interest on the principal withdrawn in period 1 would be paid at the end of period one. In such an alternative setting the probability of a bank run would be unchanged, but the optimal reserves must be scaled up with the coefficient  $(1 + g^d)$ . However, the qualitative findings of our analysis are easily adapted to such an alteration.

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## Appendix A. Proof of inequalities in Lemma 6

Define the set where  $\Xi_A$  is supply restricted by  $\bar{S}$  and that where it is demand restricted by  $\bar{S}^c$  so that

$$\begin{aligned} \bar{S} &= \{\alpha(r_A - X_A)^+ \leq (1 - \alpha)(X_B - r_B)^+\} \quad \text{and} \\ \bar{S}^c &= \{\alpha(r_A - X_A)^+ > (1 - \alpha)(X_B - r_B)^+\}. \end{aligned} \tag{A.1}$$

Now, using indicator functions, bank A’s expected lending through the interbank market can be rewritten as

$$E[\mathcal{E}_A] = \int_0^{r_A} \int_{r_B}^1 \left\{ \alpha(r_A - x)^+ \mathbf{1}_{\bar{S}} + (1 - \alpha)(y - r_B)^+ \mathbf{1}_{\bar{S}^c} \right\} dF_B(y|x) dF_A(x). \tag{A.2}$$

Next, we apply Leibnitz’ rule to find the partial derivatives of (A.2) with respect to  $r_A$  and  $r_B$ :

$$\begin{aligned} \frac{1}{D} \cdot \frac{\partial E[\mathcal{E}_A]}{\partial r_A} &\stackrel{\text{a.s.}}{=} \int_0^{r_A} \int_{r_B}^1 \alpha \mathbf{1}_{\bar{S}} dF_B(y|x) dF_A(x) \\ &+ \int_{r_B}^1 (1 - \alpha)(y - r_B)^+ \mathbf{1}_{\bar{S}^c} dF_B(y|r_A) f_A(r_A) > 0 \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} \frac{1}{D} \cdot \frac{\partial E[\mathcal{E}_A]}{\partial r_B} &\stackrel{\text{a.s.}}{=} \int_0^{r_A} \int_{r_B}^1 -(1 - \alpha) \mathbf{1}_{\bar{S}^c} dF_B(y|x) dF_A(x) \\ &- \int_0^{r_A} \alpha(r_A - x)^+ \mathbf{1}_{\bar{S}} f_B(r_B|x) dF_A(x) < 0, \end{aligned} \tag{A.4}$$

which validates the claims in Lemma 6(a) and (b). Finally, Lemma 6(e)  $\partial^2 E[\mathcal{E}_A] / \partial r_A \partial r_B = \partial^2 E[\mathcal{E}_A] / \partial r_B \partial r_A < 0$  can be shown by observing that

$$\frac{1}{D} \cdot \frac{\partial E[\mathcal{E}_A]}{\partial r_B \partial r_A} \stackrel{\text{a.s.}}{=} - \int_{r_B}^1 (1 - \alpha) \mathbf{1}_{\bar{S}^c} dF_B(y|r_A) f_A(r_A) - \int_0^{r_A} \alpha \mathbf{1}_{\bar{S}} f_B(r_B|x) dF_A(x) < 0. \tag{A.5}$$

Next, we analyze the behavior of  $E[\mathcal{E}_A]$  with respect to the interbank correlation. We begin by solving the extremal when  $\rho \uparrow 1$ . Denote  $\bar{z}_A = (r_A - E[X_A]) / \sigma_A$  and  $\bar{z}_B = (r_B - E[X_B]) / \sigma_B$ . When the correlation increases towards one the probability mass of the conditional distribution  $F_B(y|x)$  of the competitor becomes concentrated to one point,

$$F_B(y|x) \rightarrow \mathbf{1}_{\{\bar{z}_B \geq \bar{z}_A\}} \quad \text{when } \rho_{A,B} \uparrow 1. \tag{A.6}$$

Therefore, there is a jump in  $F_B$  at the point  $\bar{z}_B = \bar{z}_A$ . Hence, when  $\rho_{A,B} = 1$ , there exists no lending opportunities for bank A if  $\bar{z}_B \geq \bar{z}_A$ . When  $\bar{z}_A > \bar{z}_B$  and  $\rho_{A,B} = 1$

$$\begin{aligned} \frac{E[\mathcal{E}_A]}{D} &= \int_0^{r_A} \int_{r_B}^1 \min \{ \alpha(r_A - x)^+, (1 - \alpha)(y - r_B)^+ \} dF_B(y) dF_A(x) \\ &= \int_{F_A^{-1}(F_B(r_B))}^{r_A} \min \{ \alpha(r_A - x)^+, (1 - \alpha)(y - r_B)^+ \} dF_A(x) \end{aligned} \tag{A.7}$$

$$\approx \int_{\bar{z}_B}^{\zeta} (1 - \alpha) \sigma_B (z - \bar{z}_B) d\Phi(z) + \int_{\zeta}^{\bar{z}_A} \alpha \sigma_A (\bar{z}_A - z) d\Phi(z), \tag{A.8}$$

where  $\zeta$  is the threshold associated with reserve holdings such that A's supply equals B's demand, i.e. the root to the Eq.  $\alpha\sigma_A(\bar{z}_A - \zeta) = (1 - \alpha)\sigma_B(\zeta - \bar{z}_B)$ . Trivially, if  $\bar{z}_A \leq \bar{z}_B$  then  $\Xi_A|_{\rho_{A,B}=1} = 0$ . Finally, we provide a graphical proof which also explains why the adjustment towards the lower limit ((A.8) or zero) is monotonically decreasing in  $\rho_{A,B}$ . Figs. 1 and 2, illustrate the random outcomes in the case of zero correlation and perfect correlation, respectively. In Fig. 1 the niveau curves for a bivariate normal density function are circular as the correlation is zero. The northwest rectangle in the graph expresses the outcomes where bank A lends to bank B. In the horizontal direction the rectangle is restricted by bank A's potential supply and in the vertical direction by bank B's potential demand. All realizations where supply exceeds demand are shifted to the right, and all realizations where demand

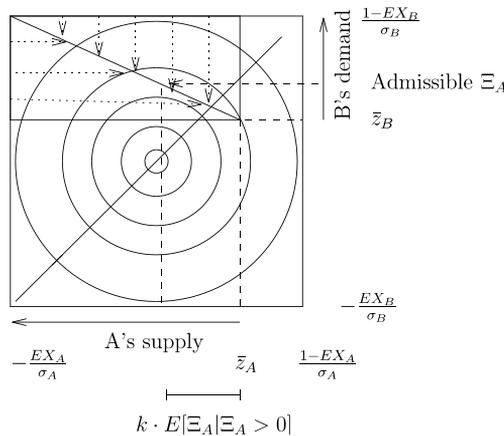


Fig. 1. Bank A's lending opportunities when  $\bar{z}_A > \bar{z} > 0$  and  $\rho_{A,B} = 0$ .

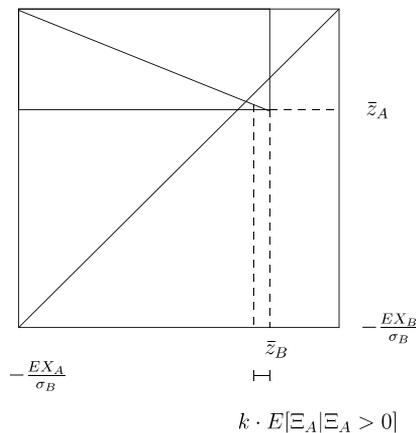


Fig. 2. Bank A's lending opportunities where  $\bar{z}_A > \bar{z}_B > 0$  and  $\rho_{A,B} = 1$ .

exceeds supply are shifted down towards the downward sloping diagonal line, which indicates the admissible  $\Xi_A$ . In the figures A's expected lending is represented by  $k \cdot E[\Xi_A | \Xi_A > 0]$ , where the scaling coefficient  $k$  is  $k = 1/(\sigma_A \alpha D)$ .

When the correlation is increased the bivariate density function is tilted towards the  $45^\circ$  upward sloping line. Therefore, the expected lending opportunities shrink when  $\rho_{A,B}$  grows. Fig. 2 demonstrates the lower limit when the interbank correlation is perfect. Here,  $\Xi_A$  is concentrated on the small segment of the downward sloping line, which lies southeast of the  $45^\circ$  upward sloping line. In this case  $E[\Xi_A | \Xi_A > 0]$  is minimal, because when the correlation is smaller than one, part of there probability is will be to northwest of the  $45^\circ$  upward sloping line in the graph. This would raise  $E[\Xi_A | \Xi_A > 0]$  above its minimum.

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