

## 6 Combinatorics: the Art of Counting

### 6.1 Introduction

*Combinatorics* is the branch of Mathematics in which methods to solve counting problems are studied. Here is a list of questions that are considered combinatorial problems. As we will see, some of these problems in this list may *look* different but actually happen to be (instances of) the *same* problem.

1. What is the number of sequences of length  $n$  and constructed from the symbols 0 and 1 only, for any given natural  $n$ ?
2. What is the number of sequences of length  $n$  and constructed from the symbols 0 and 1 only, but containing no two 0's in succession, for any given natural  $n$ ?
3. What is the number of subsets of a given finite set?
4. What is the number of elements of the Cartesian product of two given finite sets?
5. What is the number of possible Hungarian license plates for cars? (A Hungarian license plate contains three letters followed by three digits.)
6. What is the number of relations on a given finite set?
7. What is the number of sequences of length  $n$  and containing each of the numbers  $0, 1, \dots, n-1$  (exactly) once, for given  $n: 1 \leq n$ ?
8. What is the number of sequences of length  $n$  and containing different objects chosen from a given collection of  $m$  different objects, for given  $n, m: 1 \leq n \leq m$ ?
9. What is the number of ways to select  $n$  objects from a given collection of  $m$  objects, for given  $n, m: 1 \leq n \leq m$ ?
10. What is the number of numbers, in the decimal system, the digits of which are all different?
11. What is the number of "words" consisting of 5 letters?
12. What is the number of ways in which 5 pairs can be formed from a group of 10 persons?
13. What is the number of  $n \times n$  matrices, all elements of which are 0 or 1? How many of these matrices have an odd determinant?
14. What is the number of steps needed to sort a given finite sequence?
15. What is the minimal number of "yes/no"-questions needed to determine which one out of a given (finite) set of possibilities is actually the case?

16. In a digital signaling system 8 wires are used, each of which may or may not carry a voltage. For the transmission of a, so-called, “code word” via these wires exactly 4 wires carry a voltage (and the other 4 wires carry no voltage). How many code words are thus possible?

To provide some flavor of the theory needed to answer these questions in a systematic way, we discuss some of these questions here, not necessarily to solve them already but, at least, to shed some light on them.

**question 1:**

What is the number of sequences of length  $n$  and constructed from the symbols 0 and 1 only, for any given natural  $n$ ? Well, a sequence of  $n$  symbols has  $n$  positions, each of which may contain a 0 or a 1. So, each position admits 2 possibilities, and the choice at each position is *independent* from the choice at every other position. For  $n=2$ , for instance, we have a total of  $2*2$  choices, whereas for  $n=3$  we have a total of  $2*2*2$  possibilities. Generally, for arbitrary  $n$  the number of possibilities is  $2^n$ .

This is about the simplest possible counting problem. A systematic way to solve it, which is also applicable to more difficult problems, is by means of induction on  $n$ . The only sequence of length 0 is the “empty” sequence, of which there is only one. So, for  $n=0$  the number of sequences equals 1. (And, if one wishes to avoid the notion of the empty sequence, one starts with  $n=1$  and finds the answer 2, because there are only 2 sequences of length 1, namely consisting of a single symbol 0 or 1.) Next, a sequence of length  $n+1$  can be viewed as the *extension* of a sequence of length  $n$  with an additional symbol, being 0 or 1. So, such an extension is possible in two ways, each yielding a new sequence. Hence, the number of sequences of length  $n+1$  is twice the number of sequences of length  $n$ . Thus, we also arrive at the conclusion that the number of sequences of length  $n$  equals  $2^n$ : for  $n=0$ , the number is  $2^0$ , and each extension doubles the answer, with  $2^n * 2 = 2^{n+1}$ .

**question 2:**

What is the number of sequences of length  $n$  and constructed from the symbols 0 and 1 only, but containing no two 0’s in succession, for any given natural  $n$ ? This question is harder than the previous one, because of the restriction that the sequences contain no two 0’s in succession: now different positions in the sequence are not independent anymore, and the question is not as easily answered as the previous one. As an abbreviation, we call the sequences constructed from the symbols 0 and 1 only, but containing no two 0’s in succession, “admissible”.

To answer this question it pays to introduce a *variable* – a name – for the answer. Because the answer depends on variable  $n$ , the parameter of the problem, we let this variable depend on  $n$ ; that is, we make it a function on  $\mathbb{N}$ , as  $n \in \mathbb{N}$ . So, we say: let  $a_i$  be the number of admissible sequences of length  $i$ , for all  $i \in \mathbb{N}$ . Then  $a$  is the function, with type  $\mathbb{N} \rightarrow \mathbb{N}$ . This enables us to try to formulate an equation for  $a$  which, hopefully, we can solve.

For our current problem we reason as follows. The one and only sequence of length 0 is the empty sequence, and it is admissible: as it contains no 0's at all, it certainly contains no two 0's in succession. Thus, we decide that  $a_0 = 1$ . Next, for  $i \in \mathbb{N}$ , a sequence of length  $i+1$  can now be obtained in two different ways: either by extending an admissible sequence of length  $i$  with a symbol 1, always resulting in an admissible sequence, or by extending an admissible sequence of length  $i$  with a symbol 0; this, however, yields an admissible sequence *only if* the sequence thus extended does not end with a 0 itself: if it did we would obtain two 0's in succession!

Hence, a new kind of sequences enter the game, namely "admissible sequences not ending with a symbol 0". Therefore, we introduce yet another name  $b$ , say: let  $b_i$  be the number of admissible sequences not ending with a symbol 0, for all  $i \in \mathbb{N}$ . Using  $b$ , we can now complete our equation for  $a$ ; we obtain:  $a_{i+1} = a_i + b_i$ . Notice that, in this formula,  $a_i$  is the number of admissible sequences of length  $i+1$  obtained by extending an admissible sequence with a symbol 1, and that  $b_i$  is the number of admissible sequences of length  $i+1$  obtained by extending an admissible sequence, not ending in a 0, with a symbol 0; their sum, then, is the total number of admissible sequences of length  $i+1$ .

Thus we obtain the following equation for our function  $a$ :

$$\begin{aligned} a_0 &= 1 \\ a_{i+1} &= a_i + b_i, \text{ for all } i \in \mathbb{N} \end{aligned}$$

This equation contains our new variable  $b$  and to be able to solve the equation we also need a similar equation for  $b$ . By means of precisely the same kind of reasoning we decide that the only admissible sequence, not ending in a 0, of length 0 is the empty sequence; hence,  $b_0 = 1$ . And, the only way to obtain an admissible sequence, not ending in a 0, of length  $i+1$  is by extending an admissible sequence of length  $i$  with a symbol 1; so,  $b_{i+1} = a_i$ . Thus we obtain the following equation for our function  $b$ :

$$\begin{aligned} b_0 &= 1 \\ b_{i+1} &= a_i, \text{ for all } i \in \mathbb{N} \end{aligned}$$

These two equations can now be combined into one set of equations for  $a$  and  $b$  together. Actually, they constitute a *recursive definition* for  $a$  and  $b$ , because, for any given  $n \in \mathbb{N}$ , they can be used as rewrite rules to calculate the values of  $a_n$  and  $b_n$  in a finite number of steps. Nevertheless, we also call this an "equation" because it does not give *explicit* expressions for  $a$  and  $b$ .

$$\begin{aligned} a_0 &= 1 \\ a_{i+1} &= a_i + b_i, \text{ for all } i \in \mathbb{N} \\ b_0 &= 1 \\ b_{i+1} &= a_i, \text{ for all } i \in \mathbb{N} \end{aligned}$$

Because these equations pertain to functions on  $\mathbb{N}$ , and because of their recursive nature, they are also called *recurrence relations*. Recurrence relations of this and similar forms can be solved in a systematic way. This is the subject of a separate section.

**question 3:**

What is the number of subsets of a given finite set? Well, every element may or may not be an element of a subset: for every element of the given set we have a choice out of 2 possibilities, independently of (the choices for) the other elements. Therefore, if the given set has  $n$  elements, we have  $n$  independent choices out of 2 possibilities; hence, the number of subsets of a set with  $n$  elements equals  $2^n$ .

Actually, this is the very same problem as the one in question 1. The elements of any finite set can be ordered into a finite sequence, and now every subset of it can be represented as a sequence of length  $n$  consisting of symbols 0 and 1: a 1 in a given position encodes that the corresponding – according to the order chosen – element of the given set is an element of the subset, and a 0 in a given position encodes that the corresponding element of the given set is not an element of the subset. Thus, there is a one-to-one correspondence – that is: a bijection – between the set of all subsets of a given set of size  $n$  and the set of all sequences of length  $n$  consisting of symbols 0 and 1. Thus, question 3 and question 1 are essentially the same, and so are their answers.

**question 4:**

What is the number of elements of the Cartesian product of two given finite sets? Let  $V$  and  $W$  be finite sets with  $m$  and  $n$  elements, respectively. The Cartesian product  $V \times W$  is the set of all pairs  $(v, w)$  with  $v \in V$  and  $w \in W$ . Because  $v$  can be chosen out of  $m$  elements, and because, independently,  $w$  can be chosen out of  $n$  elements, the number of possible pairs equals  $m \times n$ . So, we have:  $\#(V \times W) = \#V * \#W$ .

Similarly, the number of elements of the Cartesian product of *three* finite sets is the product of the *three* sizes of these sets:  $\#(U \times V \times W) = \#U * \#V * \#W$ , and so on. The following two questions contain applications of this.

**question 5:**

What is the number of possible Hungarian license plates for cars? (A Hungarian license plate contains three letters followed by three digits.) Each letter is chosen from the alphabet of 26 letters, and each digit is one out of 10 decimal digits. Actually, with  $A$  for the alphabet and with  $D$  for the set of decimal digits, a Hungarian license plate number is an element of the Cartesian product  $A \times A \times A \times D \times D \times D$ ; hence, the number of possible combinations equals  $\#A * \#A * \#A * \#D * \#D * \#D$ , that is:  $26^3 * 10^3$ .

**question 6:**

What is the number of relations on a given finite set? A relation on a set  $V$  is a subset of the Cartesian product  $V \times V$ , so the number of relations on  $V$  equals the number of subsets of  $V \times V$ . If  $V$  has  $n$  elements then  $V \times V$  has  $n^2$  elements, so the number of relations equals  $2^{n^2}$ .

**question 7:**

What is the number of sequences of length  $n$  and containing each of the numbers  $0, 1, \dots, n-1$  (exactly) once? Notice that the requirement “containing each of the numbers  $0, 1, \dots, n-1$  (exactly) once” is rather *over specific*: that it is about the numbers  $0, 1, \dots, n-1$  is not very relevant, all that matters is that the sequence contains  $n$  given *different* objects. The element in the first position can be chosen out of  $n$  possible objects, and this leaves only  $n-1$  objects for the remainder of the sequence. Hence, the second element can be chosen out of these  $n-1$  objects, and this, in turn, leaves  $n-2$  objects for the (next) remainder of the sequence; and so on, until we are left with exactly one object to choose from for the last element of the sequence. Hence, the number of possible such sequences equals  $n * (n-1) * (n-2) * \dots * 2 * 1$ ; this quantity is usually denoted as  $n!$  – “ $n$  factorial” –.

Here, too, a recursive characterization is possible. The number of sequences of length 1, and containing one given object exactly once, equals 1. And, for any  $n \in \mathbb{N}$ , a sequence of length  $n+1$  containing each of  $n+1$  different objects exactly once can be constructed, firstly, by choosing one of these objects, which can be done in  $n+1$  ways, and, secondly, by constructing a sequence of length  $n$  containing each of the remaining  $n$  objects exactly once; finally, the sequence thus obtained is extended with the single object chosen in the first step. We conclude that the number of sequences of length  $n+1$  equals  $n+1$  times the number of sequences of length  $n$ . Thus we arrive at this, well-known, recursive definition of  $n!$ :

$$\begin{aligned} 1! &= 1 \\ (n+1)! &= (n+1) * n! \text{ , for all } n \in \mathbb{N}^+ \end{aligned}$$

**question 8:**

What is the number of sequences of length  $n$  and containing different objects chosen from a given collection of  $m$  different objects, for given  $n, m: 1 \leq n \leq m$ ? One way to approach this problem is to observe that the  $m$  given objects can be arranged into a sequence in  $m!$  ways, as we have seen in the previous question. The first  $n$  objects of such a sequence then constitute a sequence of length  $n$  and containing different objects chosen from the given collection of  $m$  objects. The order of the remaining  $m-n$  objects (in the sequence of length  $m$ ), however, is completely irrelevant: these remaining objects can be ordered in  $(m-n)!$  ways, so there are actually  $(m-n)!$  sequences of length  $n$ , that all begin with the same  $n$  objects in the same order, and that only differ in the order of their last  $m-n$  elements. So, to count the number of sequences of length  $n$  we have to *divide* the number of sequences of length  $m$  by  $(m-n)!$ . Thus we obtain as answer to our question:  $m! / (m-n)!$ .

**question 9:**

What is the number of ways to select  $m$  objects from a given collection of  $n$  objects, for given  $m, n: 1 \leq m \leq n$ ? One may well wonder if, and if so how, this question differs from the previous one. Well, the previous question was about *sequences*, whereas this

question is about *sets*. The question can be rephrased as: What is the number of subsets of size  $n$  of a given set of size  $m$ , for given  $n, m: 1 \leq n \leq m$ ? Well, every sequence of length  $n$  of the kind in the previous question represents such a subset, albeit that the order in which the objects occur in the sequence now is irrelevant. That is, every two sequences, of length  $n$ , containing the *same* objects in possibly *different* orders now represent the *same* subset. Actually, for every selection of  $n$  objects there are  $n!$  sequences containing these objects and all representing the same subset. Hence, the number of ways to form a subset of size  $m$  from a given set of size  $n$  equals the answer to the previous question divided by  $n!$ . Thus we obtain:  $m! / ((m-n)! * n!)$ . This quantity is called a *binomial coefficient* and it is usually denoted as  $\binom{m}{n}$ , which is defined by:

$$\binom{m}{n} = \frac{m!}{(m-n)! * n!} .$$

Binomial coefficients happen to have interesting properties which we will study later.

**question 10:**

What is the number of numbers, in the decimal system, the digits of which are all different? Obviously, such a number consists of *at most* 10 digits. It is questionable whether we allow such a number to start with the digit 0: on the one hand, “0123” does represent a number in the decimal system, on the other hand, we usually do not write down such leading zeroes because they are meaningless: “0123” represents the same number as “123”. So, let us *decide* to exclude meaningless leading zeroes. Then, the only number starting with the digit 0 is “0”.

In this case, the problem is most easily solved by distinguishing the possible numbers according to their length. We have already observed that the length of our numbers is at most 10. For numbers consisting of a single digit we have 10 possibilities, as each of the 10 decimal digits is permitted. For numbers of length  $n+1$ ,  $1 \leq n \leq 9$ , we can choose the first digit in 9 ways, 0 having been excluded as the first digit. Independently of how this first digit has been chosen we still have 9 digits left for the remaining  $n$  digits of the number, because now digit 0 is included again. So, for these remaining  $n$  digits the question becomes: what is the number of sequences of  $n$  digits, chosen from a set of 9 different digits? This question, however, is an instance of problem 8, with  $m := 9$ ; hence, the answer to this question is  $9! / (9-n)!$ . Thus, we conclude that the number of numbers, in the decimal system, the digits of which are all different, is 10 single-digit numbers, and  $9 * 9! / (9-n)!$  numbers consisting of  $n+1$  digits, for  $1 \leq n \leq 9$ .

**question 11:**

What is the number of “words” consisting of 5 letters? This depends on what one considers a “word”. If we are asking for the number of 5-letter words in a specific language the question is a linguistic question, not a mathematical one. So, let us

decide that a “word” is just an arbitrary sequence consisting of letters from the 26-letter alphabet. Then, for every position in the sequence we have an independent choice out of 26 possibilities, so the number of 5-letter words equals  $26^5$ .

Notice that this question is very similar to questions 1 and 4.

**question 12:**

What is the number of ways in which 5 pairs can be formed from a group of 10 persons? This question does not admit a simple, straightforward answer. Therefore, as in the discussion of question 2, we introduce a variable to represent it: let  $a_i$  be the number of ways in which  $i$  pairs can be formed from a group of  $2*i$  persons, for all  $i \in \mathbb{N}^+$ . The original question, then, amounts to asking for the value of  $a_5$ .

For a group of 2 persons, the answer is quite simple: only one pair can be formed, so  $a_1 = 1$ . For a group of  $2*(i+1)$  persons, which is equal to  $2*i+2$ , one person can be matched to  $2*i+1$  other persons, and the remaining  $2*i$  persons can then be arranged into pairs in  $a_i$  ways. Thus we obtain:  $a_{i+1} = (2*i+1) * a_i$ . Combining these results we obtain the following recurrence relations for  $a$ :

$$\begin{aligned} a_0 &= 1 \\ a_{i+1} &= (2*i+1) * a_i, \text{ for all } i \in \mathbb{N}^+ \end{aligned}$$

So, informally, we have:  $a_i = (2*i-1) * (2*i-3) * \dots * 3 * 1$ , and the answer for a group of 10 persons is  $9 * 7 * 5 * 3 * 1$ . In some textbooks, the expression for  $a_i$  is denoted as  $(2*i-1)!!$ .

**question 13:**

What is the number of  $n \times n$  matrices, all elements of which are 0 or 1? How many of these have an odd determinant? The first question has no relation to linear algebra. An  $n \times n$  matrix has  $n^2$  elements, each of which may be 0 or 1. Hence, the number of such matrices equals  $2^{n^2}$ . This answer is the same as the answer to question 6, and this is no coincidence. Why?

The problem posed in the second question does belong to linear algebra. The answer, which we shall not explain here, is that the number of  $n \times n$  0/1-matrices with an odd determinant is given by this nice formula:

$$\left( \prod_{i: 0 \leq i < n} (2^n - 2^i) \right).$$

For  $n=3$ , for example, we thus find 168 matrices with an odd determinant, and 344, namely:  $2^{3^2} - 168$ , with an even determinant.

**question 16:**

In a digital signaling system 8 wires are used, each of which may or may not carry a voltage. For the transmission of a, so-called, “code word” via these wires exactly 4 wires carry a voltage (and the other 4 wires carry no voltage). How many code words are thus possible? Every code words corresponds to a particular selection of 4 wires

from the 8 available wires, so the number of possible code words equals the number of ways to select 4 objects out of a collection of 8. This is exactly question 9, with  $n := 4$  and  $m := 8$ . Hence, the answer is  $8!/(4!*4!)$ , which equals 70.

## 6.2 Recurrence Relations

### 6.2.1 An example

In the introduction we have discussed this question: What is the number of sequences of length  $n$  and constructed from the symbols 0 and 1 only, but containing no two 0's in succession, for any given natural  $n$ ? For the sake of brevity we have called such sequences “admissible”.

To solve this problem we have introduced two functions,  $a$  and  $b$ , on  $\mathbb{N}$ , with the following interpretation, for all  $i \in \mathbb{N}$ :

$$\begin{aligned} a_i &= \text{“the number of admissible sequences of length } i \text{ , and:} \\ b_i &= \text{“the number of admissible sequences of length } i, \\ &\quad \text{not ending with a 0” .} \end{aligned}$$

The answer to the above question then is  $a_n$ , and we already have formulated the following set of equations for  $a$  and  $b$ , also called *recurrence relations*:

$$\begin{aligned} (0) \quad a_0 &= 1 \\ (1) \quad a_{i+1} &= a_i + b_i \text{ , for all } i \in \mathbb{N} \\ (2) \quad b_0 &= 1 \\ (3) \quad b_{i+1} &= a_i \text{ , for all } i \in \mathbb{N} \end{aligned}$$

These recurrence relations form equations for *two* unknowns, namely  $a$  and  $b$ . We can, however, use (2) and (3) to *eliminate*  $b$  from equation (1): after all, we can view (2) and (3) as a definition for  $b$  in terms of  $a$ . Because of the case distinction between  $b_0$  and  $b_{i+1}$  we must apply a similar case distinction to (1); that is, we must split (1) into separate equations for  $a_1$  and  $a_{i+2}$ . In the equation for  $a_1$  we can now substitute 1 for  $b_0$  (and 1 for  $a_0$ ), and in the equation for  $a_{i+2}$  we can now substitute  $a_i$  for  $b_{i+1}$ . Thus we obtain a new set of equations for  $a$  in which  $b$  does not occur anymore:

$$\begin{aligned} (4) \quad a_0 &= 1 \\ (5) \quad a_1 &= 2 \\ (6) \quad a_{i+2} &= a_{i+1} + a_i \text{ , for all } i \in \mathbb{N} \end{aligned}$$

Thus, we have obtained a recurrence relation for  $a$  in isolation – that is, without  $b$ –; the “old” equations (2) and (3) can now be used to define  $b$  in terms of  $a$ , if so desired. That is to say, if we are able to derive an explicit – that is, non-recursive – definition for  $a$  then (2) and (3) provide an equally explicit definition for  $b$  in terms of  $a$ . Notice, however, the proviso “if so desired”: we may very well be interested in  $b$  too, but our original problem was about  $a$ , and we have only introduced  $b$  as an additional, auxiliary variable to be able to formulate equation (1).



\* \* \*

Equations (4) through (6) can be used to calculate as many values  $a_i$  as we like, preferably in the order of increasing  $i$ . For example, the first 7 values are:

$i$	$a_i$
0	1
1	2
2	3
3	5
4	8
5	13
6	21

We see that the values  $a_i$ , as a function of  $i$ , increase rather quickly. This is not so strange: equation (6) is very similar to the following, only slightly different, equation.

$$a_{i+2} = a_{i+1} + a_{i+1} \text{ , for all } i \in \mathbb{N} \text{ ,}$$

which is equivalent to  $a_{i+2} = 2 * a_{i+1}$ . Now for this equation, together with (4) and (5), it is quite easy to *guess* that the solution might be  $a_i = 2^i$ , for all  $i \in \mathbb{N}$ , and, once we have guessed this, it requires only ordinary Mathematical Induction to prove that our guess is correct.

In our case, however, we have to deal with equation (6). Because of the smaller index,  $i$  instead of  $i+1$ , in one of the terms of the right-hand side, we suspect that the solution does not increase as quickly as  $2^i$ , but, perhaps, it still increases exponentially? This idea deserves further investigation. Notice that we have tacitly decided to confine our attention to equation (6), and to ignore, at least for the time being, equations (4) and (5).

### 6.2.2 The characteristic equation

The last considerations give rise to the idea to investigate solutions of the form  $\gamma * \alpha^i$  (for  $a_i$ ), for some, non-zero, constants  $\gamma$  and  $\alpha$  yet to be determined. To investigate this we substitute this expression for  $a_i$  in equation (6), such that we can try to calculate solutions for  $\gamma$  and  $\alpha$ :

$$\gamma * \alpha^{i+2} = \gamma * \alpha^{i+1} + \gamma * \alpha^i \text{ , for all } i \in \mathbb{N} \text{ .}$$

Firstly, we observe that, unless  $\gamma = 0$ , which would make the equation useless, this equation does not really depend on  $\gamma$ ; for every  $\gamma \neq 0$ , this equation is equivalent to the following one:

$$\alpha^{i+2} = \alpha^{i+1} + \alpha^i \text{ , for all } i \in \mathbb{N} \text{ .}$$

Although this equation has to be met, for one-and-the-same  $\alpha$ , for *all* natural  $i$ , this is not as bad as it may seem; for  $\alpha \neq 0$ , this equation, in turn, is equivalent to the following one, in which  $i$  does not occur anymore:

$$\alpha^2 = \alpha^1 + \alpha^0 .$$

Using that  $\alpha^0 = 1$  and bringing all terms to one side of the equality we rewrite the equation into this form:

$$(7) \quad \alpha^2 - \alpha^1 - 1 = 0 .$$

This is called the *characteristic equation* of recurrence relation (6). What we have obtained now is the knowledge that if the solution to (6) is to be of the shape  $\gamma * \alpha^i$  then  $\alpha$  must satisfy (7).

Equation (7) is a quadratic one with two solutions  $\alpha_0$  and  $\alpha_1$ , say, given by:

$$(8) \quad \alpha_0 = (1 + \sqrt{5})/2 \text{ and: } \alpha_1 = (1 - \sqrt{5})/2 .$$

Apparently, we have two possibilities for  $\alpha$  here, so  $a_i = \alpha_0^i$  and  $a_i = \alpha_1^i$  are both solutions to our original equation (6), but there is more. We recall equation (6):

$$(6) \quad a_{i+2} = a_{i+1} + a_i , \text{ for all } i \in \mathbb{N} .$$

As we will discuss more extensively later, this is a so-called *linear* and *homogeneous* recurrence relation, which has the property that any *linear combination* of solutions is a solution as well. In our case, this means that for all possible constants  $\gamma_0$  and  $\gamma_1$ , the definition:

$$(9) \quad a_i = \gamma_0 * \alpha_0^i + \gamma_1 * \alpha_1^i , \text{ for all } i \in \mathbb{N} ,$$

provides a solution to equation (6). In fact, it can be proved that *all* solutions have this shape, so now we have obtained *all* solutions to equation (6).

Recurrence relations (4) through (6), however, which also constitute a recursive definition for  $a$ , suggest that the solution should be *unique*. After all, we are able to construct a table containing the values  $a_i$ , for increasing  $i$ , as we did in the previous section. So, which one out of the infinitely many solutions of the shape given by (9) is the one we are looking for? This means: what should be the values of constants  $\gamma_0$  and  $\gamma_1$  such that we obtain the correct solution? To answer this question we have to consider equations (4) and (5) again, the ones we have temporarily ignored:

$$(4) \quad a_0 = 1$$

$$(5) \quad a_1 = 2$$

If we now instantiate (9), with  $i := 0$  and  $i := 1$ , and using that  $\alpha^0 = 1$  and  $\alpha^1 = \alpha$ , for any  $\alpha$ , we obtain these two special cases:

$$a_0 = \gamma_0 + \gamma_1 \text{ and: } a_1 = \gamma_0 * \alpha_0 + \gamma_1 * \alpha_1 .$$

By substituting these values for  $a_0$  and  $a_1$  into equations (4) and (5) we obtain the following two new equations, in which  $\gamma_0$  and  $\gamma_1$  are the unknowns now:

$$\begin{aligned} \gamma_0 + \gamma_1 &= 1 \\ \gamma_0 * \alpha_0 + \gamma_1 * \alpha_1 &= 2 \end{aligned}$$

With  $\gamma_0$  and  $\gamma_1$  as the unknowns, these are just two linear equations that can be solved by standard algebraic means. In this case we obtain:

$$\gamma_0 = (2 - \alpha_1) / (\alpha_0 - \alpha_1) \quad \text{and:} \quad \gamma_1 = (\alpha_0 - 2) / (\alpha_0 - \alpha_1) \quad ,$$

where  $\alpha_0$  and  $\alpha_1$  are given, above, by definition (8). Thus we obtain, for all  $i \in \mathbb{N}$ :

$$(10) \quad a_i = ((3 + \sqrt{5}) / 2\sqrt{5}) * ((1 + \sqrt{5}) / 2)^i + ((\sqrt{5} - 3) / 2\sqrt{5}) * ((1 - \sqrt{5}) / 2)^i .$$

### 6.2.3 Linear recurrence relations

The recurrence relation studied in the previous section belongs to a class of recurrence relations known as *linear recurrence relations with constant coefficients*. They are called linear because function values, like  $a_{i+2}$ , are defined as linear combinations of other function values, like  $a_{i+1}$  and  $a_i$  in (6). Notice that equation (6) can be written, slightly more explicitly, as:

$$a_{i+2} = 1 * a_{i+1} + 1 * a_i \quad , \text{ for all } i \in \mathbb{N} \quad .$$

Any formula of the shape  $c_0 * x_0 + c_1 * x_1 + c_2 * x_2 + \dots$  is called a linear combination of the  $x$ s, with the  $c$ s being the coefficients. In the case of recurrence relations, we speak of constant coefficients if they *do not depend* on  $i$ . For example, in our example the coefficients, of  $a_{i+1}$  and  $a_i$ , are 1 and 1, respectively, which, indeed, do not depend on  $i$ .

The general shape of a linear recurrence relation with constant coefficients is the following one, in which the  $c_j$  are the coefficients, for all  $j: 0 \leq j < k$ , and in which  $k$  is a constant called the *order* of the recurrence relation:

$$(11) \quad a_{i+k} = c_{k-1} * a_{i+k-1} + \dots + c_1 * a_{i+1} + c_0 * a_i \quad , \text{ for all } i \in \mathbb{N} \quad .$$

So, in a  $k$ -th order recurrence relation value  $a_{i+k}$  is defined recursively in terms of its  $k$  direct predecessors, which are  $a_{i+k-1}, \dots, a_{i+1}, a_i$ , only.

Relation (11) defines  $a_{i+k}$  recursively in terms of  $a_{i+k-1}, \dots, a_{i+1}, a_i$ , as a linear combination, for all  $i \in \mathbb{N}$ , but it does not define the first  $k$  elements of function  $a$ ; that is, relation (11) gives no information on the values  $a_{k-1}, \dots, a_1, a_0$ . These values, therefore, may be, and must be, defined separately. So, a complete  $k$ -th order recurrence relation consists of relation (11) together with  $k$  separate definitions for the values  $a_i$ , for all  $i: 0 \leq i < k$ . These definitions also are known as the *initial conditions* of the recurrence relation.

\*                    \*                    \*

Relation (11) is *homogeneous*, which means that if  $\alpha_i$  is a solution for  $a_i$ , then so is  $\gamma * \alpha_i$ , for any constant  $\gamma$ . Relation (11) is *linear*, which means that if  $\alpha_i$  and  $\beta_i$  both are solutions for  $a_i$ , then so is  $\alpha_i + \beta_i$ . Combining these two observations we conclude that any linear combination of solutions to (11) is a solution as well. Note that this conclusion *only* pertains to equation (11) in isolation, so without regard for

the initial conditions. If the initial conditions are taken into account the recurrence relation has only one, unique, solution.

As in the example before, we now investigate to what extent equation (11) admits solutions of the shape  $\alpha^i$  for  $a_i$ . Notice that, because of the homogeneity, we do not need to incorporate a constant coefficient: as was the case with the solution to the example –Section 6.2.2–, this coefficient will drop out of the equation anyhow. Substitution of  $\alpha^i$  for  $a_i$  in (11) transforms it into the following equation for  $\alpha$ :

$$\alpha^{i+k} = c_{k-1} * \alpha^{i+k-1} + \dots + c_1 * \alpha^{i+1} + c_0 * \alpha^i, \text{ for all } i \in \mathbb{N} .$$

As we are looking for solutions with  $\alpha \neq 0$ , this equation can be further simplified into this, equivalent, form:

$$(12) \quad \alpha^k - c_{k-1} * \alpha^{k-1} \dots - c_1 * \alpha - c_0 * 1 = 0 .$$

Now  $\alpha^i$  is a solution for  $a_i$  in equation (11) if and only if  $\alpha$  is a solution to equation (12), which, as before, is called the *characteristic equation* of the recurrence relation. Notice that this is an algebraic equation of the same order as the order of the recurrence relation.

\*            \*            \*

The simplest case arises when the  $k$ -th order characteristic equation has  $k$  *different* real roots,  $\alpha_j$ , say, for  $j: 0 \leq j < k$ . Then, any linear combination of the powers of these roots, that is, for all possible coefficients  $\gamma_j$ , with  $0 \leq j < k$ , function  $a$  defined by:

$$(13) \quad a_i = (\sum j: 0 \leq j < k: \gamma_j * \alpha_j^i) , \text{ for all } i \in \mathbb{N} ,$$

is a solution to (11). Moreover, not only does this yield just *a* solution to (11), it can even be proved that *all* solutions are thus characterized.

As stated before, when the  $k$  initial conditions, that is, the defining relations for  $a_i$ , for  $0 \leq i < k$ , are taken into account, the solution to the recurrence relation is unique. This means that if the values for  $a_i$ , for  $0 \leq i < k$ , have been given, and once the  $\alpha_j$ , for  $0 \leq j < k$  haven been solved from (11), definition (13), for all  $i: 0 \leq i < k$ , is not a definition anymore but a *restriction* on the possible values for  $\gamma_j$ . That is, the set of relations:

$$(\sum j: 0 \leq j < k: \gamma_j * \alpha_j^i) = a_i , \text{ for all } i: 0 \leq i < k ,$$

now constitutes a system of  $k$  linear equations with unknowns  $\gamma_j$ , with  $0 \leq j < k$ , from which these can be solved by means of standard linear-equation solving techniques.

\*            \*            \*

A somewhat more complicated situation arises if the characteristic equation has *multiple* roots. A simple example of this phenomenon is the equation:

$$\alpha^2 - 2.8 * \alpha + 1.96 = 0 ,$$

which can be rewritten as:

$$(\alpha - 1.4)^2 = 0 .$$

This means that both roots  $\alpha_0$  and  $\alpha_1$  are equal to 1.4.

In a general  $k$ -th order algebraic equation a root may have a, so-called, multiplicity upto  $k$ . It can now be proved that, if a certain root  $\alpha_p$ , say, has, multiplicity  $q$ , say, then  $\alpha_p^i, i * \alpha_p^i, i^2 * \alpha_p^i, \dots, i^{q-1} * \alpha_p^i$  are  $q$  independent solutions for  $a_i$  in equation (11). These  $q$  different solutions account for the multiplicity, also  $q$ , of root  $\alpha_p$ . Thus, in total we still obtain  $k$  different basis solutions from which linear combinations can be formed to obtain, again, *all* solutions of the recurrence relation.

As a simple example, the characteristic equation:

$$\alpha^2 - 2.8 * \alpha + 1.96 = 0 ,$$

has a single root, 1.4, with multiplicity 2. Hence, all solutions to the (homogeneous) recurrence relation of which this is the characteristic equation are of the form  $\gamma_0 * 1.4^i + \gamma_1 * i * 1.4^i$ .

\* \* \*

The situation becomes even more complicated if the characteristic equation has less than  $k$ , possibly multiple, roots in  $\mathbb{R}$ : then the equation still has  $k$  roots, but some (or all) of them are complex numbers. A very simple example is the equation:

$$\alpha^2 - \alpha + 1 = 0 .$$

As the determinant, namely  $1 - 4$ , of this equation is negative, this equation has no roots in  $\mathbb{R}$  at all, but it has two complex numbers as its roots:  $(1 + \sqrt{-3})/2$  and  $(1 - \sqrt{-3})/2$ , which are usually written as -with  $i = \sqrt{-1}$  -:  $(1 + i * \sqrt{3})/2$  and  $(1 - i * \sqrt{3})/2$ .

These complex roots can be used, in the same way as described earlier, to obtain all solutions to the recurrence relation. Although, definitely, there is quite some mathematical beauty in this, such solutions are not very useful from a practical point of view. In Subsection ?? we present a more computational way to cope with such situations.

#### 6.2.4 Summary

Summarizing, for solving a recurrence relation of the shape

$$a_0 = n_0, a_1 = n_1, a_{i+2} = c_1 a_{i+1} + c_0 a_i \text{ for all } i \in \mathbb{N}$$

for given numbers  $n_0, n_1, c_0, c_1$ , we have the following approach:

- First ignore the requirements  $a_0 = n_0, a_1 = n_1$  and try to find a solution of  $a_{i+2} = c_1 a_{i+1} + c_0 a_i$  of the shape  $a_i = \alpha^i$  for all  $i$ . This yields the characteristic equation  $\alpha^2 = c_1 \alpha + c_0$ .
- Find the two solutions  $\alpha_0$  and  $\alpha_1$  of this equation, now for every two numbers  $A, B$ , the expression  $a_i = A\alpha_0^i + B\alpha_1^i$  is a solution of the equation. The numbers  $\alpha_0$  and  $\alpha_1$  may be real or complex numbers. In case  $\alpha_0 = \alpha_1$ , then the expression  $a_i = A\alpha_0^i + Bi\alpha_0^i$  is a solution of the equation.
- Finally, find values for  $A, B$  such that  $a_0 = n_0, a_1 = n_1$  hold.

We observe that if  $n_0, n_1, c_0, c_1 \in \mathbb{N}$  then we will obtain  $a_i \in \mathbb{N}$  for all  $i$ . In particular, this holds for counting problems as in our introduction, which are only about natural numbers. For integer  $n_0, n_1, c_0, c_1$  the result will always be integer. In the mean time we have obtained a solution of the form (10), in which both  $\alpha_0, \alpha_1$ , and the coefficients  $A, B$  are defined in terms of true –non-integer, even irrational– real numbers like  $\sqrt{5}$ , or even complex numbers. Apparently, however complicated the resulting formula is, its value is a natural number nevertheless. Isn't that strange?

We conclude that even for problems that are purely about natural or integer numbers, it turns out to be fruitful to make an excursion into  $\mathbb{R}$  or  $\mathbb{C}$  to be able to find a closed expression for the solution of the problem.

## 6.3 Binomial Coefficients

### 6.3.1 Factorials

We already have seen that the product  $n * (n-1) * (n-2) * \dots * 2 * 1$  is denoted as  $n!$  –“ $n$  factorial”–, for any  $n \in \mathbb{N}^+$ . Even for  $n=0$  this product is meaningful: then it is the *empty* product, consisting of 0 factors, for which the best possible definition is the identity element of multiplication, that is: 1. A recursive definition for  $n!$  is:

$$\begin{aligned} 0! &= 1 \\ (n+1)! &= (n+1) * n! \quad , \text{ for all } n \in \mathbb{N} \end{aligned}$$

Factorials occur in solutions to many counting problems. In Section 6.1 we already have argued that the number of ways to arrange  $n$  given, different objects into a sequence of length  $n$  equals  $n!$ . Here we repeat the argument, in a slightly more precise way. We do this recursively; for this purpose, we introduce a function  $a$  on  $\mathbb{N}$  with the idea that:

$$a_i = \text{“the number of sequences of length } i, \text{ containing } i \text{ different objects”} \quad , \\ \text{for all } i \in \mathbb{N} \quad .$$

The number of ways to arrange 0 different objects into a sequence of length 0 equals 1, because the only sequence of length 0 is the empty sequence, and it contains 0 objects. So, we conclude:  $a_0 = 1$ . Next, to arrange  $i+1$  different objects into a sequence of length  $i+1$  we can, firstly, select one object that will be the first element

of the sequence, and, secondly, arrange the remaining  $i$  objects into a sequence of length  $i$ . The first object can be selected in  $i+1$  different ways and, independently, the remaining  $i$  objects can be arranged in  $a_i$  ways into a sequence of length  $i$ . So, we obtain:  $a_{i+1} = (i+1) * a_i$ . Thus, we obtain as recurrence relation for  $a$ :

$$\begin{aligned} a_0 &= 1 \\ a_{i+1} &= (i+1) * a_i, \text{ for all } i \in \mathbb{N} \end{aligned}$$

The solution to this recurrence is  $a_i = i!$ , for all  $i \in \mathbb{N}$ .

Notice that, while we are speaking here of “different objects”, their actual nature is irrelevant: all that matters is that they are different. Actually, we even have used this tacitly in the above argument: after we have selected one object from a collection of  $i+1$  different objects, the remaining collection, after removal of the object selected, is a collection of  $i$  different objects, independently of which object was selected to be removed.

\* \* \*

A slightly more complicated problem we also have discussed in Section 6.1 was: what is the number of ways to select and arrange  $n$  different objects from a given collection of  $m$  different objects into a sequence of length  $n$ , for given  $n, m: 0 \leq n \leq m$ ?

One way to approach this problem is to observe that, as we now know, the  $m$  given objects can be arranged into a sequence in  $m!$  ways. The first  $n$  objects of such a sequence then constitute a sequence of length  $n$  that contains  $n$  different objects chosen from the given collection of  $m$  objects. The order of the last  $m-n$  objects (in the sequence of length  $m$ ), however, is completely irrelevant: these remaining objects can be ordered in  $(m-n)!$  ways without affecting the first  $n$  objects in the sequence, so there are actually  $(m-n)!$  sequences of length  $n$ , that all begin with the same  $n$  objects in the same order, and that only differ in the order of their last  $m-n$  elements. So, to count the number of sequences of length  $n$  we have to *divide* the number of sequences of length  $m$  by  $(m-n)!$ . Thus we obtain as solution:  $m! / (m-n)!$ .

### 6.3.2 Binomial coefficients

The number of ways to select an (unordered) *collection* – instead of an (ordered) *sequence* – of  $n$  different objects from a given collection of  $m$  different objects, for  $n, m: 0 \leq n \leq m$ , can be determined by the following argument. The number of ways to arrange  $n$  different objects into a sequence is, as we know,  $n!$ . If we are only interested in a collection of such objects, their order is irrelevant, and this means that all  $n!$  such arrangements are equivalent. So, to obtain the number of possible collections we must, again, divide the answer to the previous problem by  $n!$ . Thus, we obtain as solution:

$$\frac{m!}{n! * (m-n)!} .$$

It so happens that this is an important quantity that has many interesting properties. These quantities are called *binomial coefficients*, and we introduce a notation for it, which is pronounced as “ $m$  over  $n$ ”:

$$\binom{m}{n} = \frac{m!}{n! * (m-n)!} \quad , \text{ for all } n, m : 0 \leq n \leq m \quad .$$

\*                    \*                    \*

For example, we consider all sequences of length  $m$  consisting of (binary) digits – “bits”, for short – 0 or 1. As every bit in such a sequence is either 0 or 1 –so, one out of two possibilities–, independently of the other bits in the sequence, the total number of bit sequences of length  $m$  equals  $2^m$ . If we number the positions in such a sequence from and including 0 and upto and excluding  $m$ , a one-to-one correspondence –that is, a bijection– exists between the collection of all subsets of the interval  $[0..m)$  and the collection of all bit sequences of length  $m$ : number  $i$  is in a given subset if and only if the corresponding sequence contains a bit 1 at position  $i$ , for all  $i \in [0..m)$ .

The collection of bit sequences of length  $m$  can be *partitioned* into  $m+1$  disjoint classes, according to the number of bits 1 in the sequence. That is, we consider the bit sequences, of length  $m$ , containing exactly  $n$  bits 1, for  $n, m : 0 \leq n \leq m$ . In the subset  $\leftrightarrow$  bit-sequence correspondence every bit sequence containing exactly  $n$  bits 1 now corresponds to a subset with exactly  $n$  elements.

We already know that the number of subsets containing exactly  $n$  elements chosen from a given collection of  $m$  elements equals  $\binom{m}{n}$ . Hence, because of the one-to-one correspondence, the number of bit sequences of length  $m$  and containing exactly  $n$  bits 1 also equals  $\binom{m}{n}$ .

Because the bit sequences of length  $m$  and containing exactly  $n$  bits 1, for all  $n$  with  $0 \leq n \leq m$ , together are all bit sequences of length  $m$ , we obtain the following, quite interesting result:

$$\left( \sum_{n: 0 \leq n \leq m} \binom{m}{n} \right) = 2^m \quad , \text{ for all } m \in \mathbb{N} \quad .$$

\*                    \*                    \*

Binomial coefficients satisfy an interesting recurrence relation, which is also known as *Pascal's triangle*. As a basis for the recurrence we have, for all  $m \in \mathbb{N}$ :

$$\binom{m}{0} = 1 \quad \text{and:} \quad \binom{m}{m} = 1 \quad ,$$

and for all  $n, m$  with  $1 \leq n \leq m$  we have:

$$\binom{m+1}{n} = \binom{m}{n} + \binom{m}{n-1} \quad .$$



By means of these relations the binomial coefficients can be arranged in an (infinite) table of triangular shape –hence the name “Pascal’s triangle”–, such that, for any  $m \in \mathbb{N}$ , row  $m$  in the table contains the  $m+1$  binomial coefficients  $\binom{m}{n}$  in the order of increasing  $n$ . For instance, here are the first 7 rows of this triangular table:

			1						
			1	1					
			1	2	1				
			1	3	3	1			
			1	4	6	4	1		
			1	5	10	10	5	1	
			1	6	15	20	15	6	1

### 6.3.3 The Shepherd’s Principle

Sometimes it is difficult to count the things we wish to count directly, and it may be easier to count the elements of a larger set that is so tightly related to our set of interest that we can obtain the count we want by means of a straightforward correction.

This is known as the *Shepherd’s Principle*, because of the following metaphor. To count the number of sheep in a flock of sheep may be difficult: when the sheep stick closely together one observes a single, cluttered mass of wool in which no individual sheep can be distinguished easily. But, knowing –and assuming!– that every sheep has 4 legs, we can count the legs in the flock and conclude that the number of sheep equals this number of legs divided by 4.

As a matter of fact, we have already applied this technique, in Subsection 6.3.1: to count the number of sequences of length  $n$  and containing  $n$  different objects from a given collection of  $m$  different objects –the sheep–, we have actually counted the number of sequences of length  $m$  –the legs– and divided this by the number of ways the irrelevant  $m-n$  remaining objects can be ordered –the number of legs per sheep–.

\*            \*            \*

To demonstrate the Shepherd’s Principle we discuss a few simple examples. Firstly, the number of anagrams of the word “FLIPJE” just equals the numbers of ways in which the 6 different letters can be arranged into a sequence of length 6. This is simple because the 6 letters are different, and the answer just is  $6!$ .

Secondly, what is the number of anagrams of the word “SHEPHERD”? As this word contains 8 letters there would be, if all letters would be different,  $8!$  ways to arrange them into an anagram. But the letters are not different: the word contains 2 letters “H” and 2 letters “E”; here we assume, of course, that the 2 letters “H” are

indistinguishable and also that the 2 letters “E” are indistinguishable. Well, we make them different *temporarily*, by *tagging* them, for example, by means of subscripts: “SH<sub>0</sub>E<sub>0</sub>PH<sub>1</sub>E<sub>1</sub>RD”. Now all letters are different, and the number of anagrams of this word – the legs – equals 8!. The number of legs per sheep now is the number of ways in which the tagged letters can be arranged: E<sub>0</sub> and E<sub>1</sub> can be arranged in 2! ways and, similarly, H<sub>0</sub> and H<sub>1</sub> can be arranged in 2! ways. So, the number of legs per sheep equals 2! \* 2!, and, hence, the number of anagrams of “SHEPHERD” – the number of sheep – equals 8! / (2! \* 2!).

Thirdly, what is the number of anagrams of the word “HAHAHAH”? Well, this 7-letter word contains 4 letters “H” and 3 letters “A”; so, on account of the Shepherd’s Principle, the answer is 7! / (4! \* 3!), which equals  $\binom{7}{4}$ . Is this a surprise? No, because the word is formed from only 2 two different letters, “H” and “A”; so, the number of anagrams of “HAHAHAH” is equal to the number of 7-letter words, formed from letters “H” and “A” only and containing exactly 4 letters “H”.

As the exact nature of the objects is irrelevant – it really does not matter whether we use “H” and “A” or “1” and “0” –, this last question is the same as the question for the number of 7-bit sequences containing exactly 4 bits 1. Hence, the answers are the same as well.

### 6.3.4 Newton’s binomial formula

In subsection 6.3.2 we have derived the following relation for binomial coefficients:

$$(\sum_{n: 0 \leq n \leq m} \binom{m}{n}) = 2^m, \text{ for all } m \in \mathbb{N} .$$

Actually, this is an instance of the following, more general relation, for all – integer, rational, real, ... – numbers  $x, y$ :

$$(14) \quad (x+y)^m = (\sum_{n: 0 \leq n \leq m} \binom{m}{n} * x^n * y^{m-n}) , \text{ for all } m \in \mathbb{N} .$$

This is known as *Newton’s binomial formula*, although it appears to be much older than Newton. Newton, however, has generalized the formula by not restricting  $m$  to the naturals but by allowing it even to be a complex number.

We will not elaborate this here; instead we confine ourselves to the observation that there is a connection between the recurrence relation for binomial coefficients we have seen earlier:

$$\binom{m+1}{n} = \binom{m}{n} + \binom{m}{n-1} ,$$

and relation (14), by means of the recurrence relation for exponentiation when applied to  $(x+y)$ :

$$(x+y)^{m+1} = (x+y) * (x+y)^m .$$

## 6.4 A few examples

We conclude this chapter with a few more examples, some of which we have already solved, albeit in a slightly different form.

### example 1:

A vase contains  $m$  balls which have been numbered  $0, 1, \dots, m-1$ . From this vase we draw, at random,  $n$  balls, for some  $n: 0 \leq n \leq m$ . What is the number of possible results if the order in which the balls are drawn is relevant?

Notice that, although the balls themselves may not be distinguishable by their shapes, the fact that they have been numbered makes them distinguishable. So, what matters here is that we have a collection of  $m$  different objects. The problem at hand now is the same as a question we have already answered earlier: What is the number of sequences of length  $n$  and containing different objects chosen from a given collection of  $m$  different objects, for given  $n, m: 1 \leq n \leq m$ ? As we have seen, the answer to this question is:  $m! / (m-n)!$ .

### example 2:

A vase contains  $m$  balls which have been numbered  $0, 1, \dots, m-1$ . From this vase we draw, at random,  $n$  balls, for some  $n: 0 \leq n \leq m$ . What is the number of possible results if the order in which the balls are drawn is *not* relevant? We recall that this question is equivalent to the question: What is the number of subsets of size  $n$  of a given finite set of size  $m$ ? As we have seen, the answer is  $\binom{m}{n}$ .

We also have seen that an alternative way to answer this question is to apply the Shepherd's Principle: the  $n$  balls drawn from the vase can be ordered in  $n!$  ways; as this order is irrelevant we have to divide the answer to the previous question by this very  $n!$ ; as a result, we obtain the same answer, of course:  $\binom{m}{n}$ .

### example 3:

A vase contains  $m$  balls which have been numbered  $0, 1, \dots, m-1$ , for some positive natural  $m$ . From this vase we draw, at random,  $n$  balls, for some  $n \in \mathbb{N}$ , but now every ball drawn is placed back into the vase immediately, that is, before any next balls are drawn. What is the number of possible results if the order in which the balls are drawn is relevant? (Notice that in this example we do not need the restriction  $n \leq m$ .) The result of this game simply is a sequence, of length  $n$ , of numbers in the range  $[0..m)$ , and the numbers in this sequence are mutually independent. Hence, the answer is:  $m^n$ .

### example 4:

A vase contains  $m$  balls which have been numbered  $0, 1, \dots, m-1$ , for some positive natural  $m$ . From this vase we draw, at random,  $n$  balls, for some  $n \in \mathbb{N}$ , but now

every ball drawn is placed back into the vase immediately, that is, before any next balls are drawn. What is the number of possible results if the order in which the balls are drawn is *not* relevant? This example is new, and more difficult than the previous ones.

An effective technique to solve this and similar problems is to choose the “right” *representation* of the possible results of the experiment, namely in such a way that these results can be counted. In the current example, we might write down the numbers of the balls drawn, giving rise to sequences, of length  $n$ , and containing numbers in the range  $[0..m)$ . In this case, however, this representation is not *unique*, because the order of these numbers is irrelevant. For instance, the sequence “3011004” is a possible result, but so are the sequences “4011030” and “0001134”; because the order of the numbers in the sequences is irrelevant and because these three sequences contain the same numbers, albeit in different orders, these sequences actually represent only *one* result, which should be counted only once. So, we need a *unique* representation, because only then the number of possible results is equal to the number of possible representatives of these results.

We obtain a unique representation by writing down the numbers of the balls drawn in *ascending order*: then every result corresponds in a unique way to an ascending sequence, of length  $n$ , and containing numbers in the range  $[0..m)$ . For instance, from the three example sequences “3011004”, “4011030”, and “0001134”, the last one is ascending: it uniquely represents the, one-and-only, common result corresponding to these sequences.

So, the answer to our original question equals the number of *ascending sequences*, of length  $n$ , and containing numbers in the range  $[0..m)$ . To be able to count these smoothly, yet another representation happens to be convenient, based on the following observation. The sequence “0001134”, for instance, consists of 3 numbers 0, followed by 2 numbers 1, followed by 0 numbers 2, followed by 1 number 3, followed by 1 number 4, followed by 0 numbers 5, where we have assumed  $m=6$  and  $n=7$ . So, by counting, for every number in the range  $[0..m)$ , how often it occurs in the ascending sequence we can also represent the result of the experiment. For the sequence “0001134”, for instance, the corresponding sequence of counts is “320110”: it contains  $m$  numbers, namely one count for every number in the range  $[0..m)$ ; each count itself is a number in the range  $[0, m]$  and the sum of all counts equals  $n$ , which was the length of the ascending sequence.

The restriction that the sum of the counts equals  $n$  is a bit awkward but can be made more manageable by means of the following *coding trick*: we represent each count by a sequence of as many zeroes as the count – in the *unary* representation, so to speak – and we separate these sequences of zeroes by means of a digit 1. For instance, again with  $m=6$ , the counts of the original ascending sequence “0001134” are 3, 2, 0, 1, 1, and 0, which in the zeroes-encoding become “000”, “00”, “”, “0”, “0”, and “”. Concatenated and separated by digits 1 we obtain: “000100110101”. This is a bit sequence containing exactly  $n$  digits 0, because now the sum of the counts simply equals the total number of zeroes, and containing exactly  $m-1$  digits 1, because we have  $m$  counts separated by  $m-1$  digits 1. Consequently, the length of this bit sequence is  $m-1+n$ .

Generally, every ascending sequence of length  $n$  and containing numbers in the range  $[0..m)$  can be represented *uniquely* by a bit sequence of length  $m-1+n$  containing exactly  $n$  digits 0 and  $m-1$  digits 1. So, also the results of our balls drawing experiment are uniquely represented by these bit sequences. Therefore, the number of all possible results equals the number of these bit sequences, which is:

$$\binom{m-1+n}{n}.$$

#### 6.4.1 Summary

Summarizing, for counting the number of possibilities to draw  $k$  copies from a set of  $n$  objects, there are four possible results, depending on the context:

- If every copy is drawn from the full set, so the drawn copies are put back, and drawings in a different order are considered to be different, then the number of possibilities is

$$n^k.$$

- If every copy is drawn from the remaining set, and no copies are put back, and drawings in a different order are considered to be different, then the number of possibilities is

$$n(n-1)(n-2)\cdots(n-k+1) = n!/(n-k)! = k! \binom{n}{k}.$$

- If every copy is drawn from the remaining set, and no copies are put back, and drawings in a different order are considered to be the same, then the number of possibilities is

$$\binom{n}{k}.$$

- If every copy is drawn from the full set, so the drawn copies are put back, and drawings in a different order are considered to be the same, then the number of possibilities is

$$\binom{n-1+k}{k}.$$

### 6.5 Exercises

1. Let  $V$  be a finite set with  $n$  elements.
  - (a) What is the number of functions in  $V \rightarrow \{0, 1\}$ ?
  - (b) What is the number of functions in  $V \rightarrow \{0, 1, 2\}$ ?

- (c) What is the number of functions in  $V \rightarrow W$ , with  $\#W = m$ ?
2. (a) What is the number of “words” consisting of 5 *different* letters?  
 (b) What is the number of *injective* functions in  $\{0, 1, 2, 3, 4\} \rightarrow \{a, b, c, \dots, z\}$ ?  
 (c) What is the number of *injective* functions in  $V \rightarrow W$ , with  $\#V = n$  and  $\#W = m$ ?
3. Let  $w_i$  be the number of sequences, of length  $i$ , consisting of letters from  $\{a, b, c\}$ , in which each two successive letters are different, for all  $i \in \mathbb{N}$ .  
 (a) Formulate a recurrence relation for  $w$ .  
 (b) Derive an explicit definition for  $w$ .
4. Let  $v_i$  be the number of sequences, of length  $i$ , consisting of letters from  $\{a, b, c\}$ , in which no two letters  $a$  occur in direct succession, for all  $i \in \mathbb{N}$ . It is given that  $v_{i+2} = 2 * v_{i+1} + 2 * v_i$ , for all  $i \in \mathbb{N}$ .  
 (a) Determine  $v_0$  and  $v_1$ .  
 (b) Values  $c_0, c_1, \alpha_0$ , and  $\alpha_1$  exist such that  $v_i = c_0 * \alpha_0^i + c_1 * \alpha_1^i$ , for all  $i \in \mathbb{N}$ . Determine  $c_0, c_1, \alpha_0$ , and  $\alpha_1$ .
5. Let  $a_i$  be the number of 0/1-sequences, of length  $i$ , not containing isolated zeroes, for all  $i \in \mathbb{N}$ . Such sequences are called “admissible”. For example, “1” and “0011” are admissible, but “0”, “0110”, and “0100” are not. The empty sequence – with length 0 – is considered admissible.  
 (a) Formulate a recurrence relation for  $a$ .  
 \* (b) Solve this recurrence relation.
6. (a) What is the number of 2-letter combinations in which the 2 letters occur in alphabetical order? For example: “kx” is such a combination, whereas “xk” is not.  
 (b) What is the number of 3-letter combinations in which the 3 letters occur in alphabetical order?  
 (c) What is the number of  $n$ -letter combinations in which the  $n$  letters occur in alphabetical order, for every natural  $n: n \leq 26$ ?
7. We consider (finite) sequences  $g$ , of length  $n$ , consisting of natural numbers, with the additional property that  $g_i \leq i$ , for all  $i: 0 \leq i < n$ . (Here  $g_i$  denotes the element at position  $i$  in the sequence, where positions are numbered from 0, as usual.)  
 (a) What is the number of this type of sequences, as a function of  $n \in \mathbb{N}^+$ ?  
 (b) Define a bijection between the set of these sequences and the set of all *permutations* of  $[0..n)$ .

8. This exercise is about strings of beads. The answers to the following questions depend on which strings of beads one considers “the same”: therefore, give this some thought to start with.
- What is the number of ways to thread  $n$  different beads onto a string?
  - We have  $2 * n$  beads in  $n$  different colors; per color we have exactly 2 beads, which are indistinguishable. What is the number of ways to thread these beads onto a string?
  - The same question, but now for  $3 * n$  beads, with 3 indistinguishable beads per color.
  - What is the number of ways to thread  $m$  red and  $n$  blue beads onto a string, if, again, beads of the same color are indistinguishable?
9. The sequence of, so-called, *Fibonacci numbers* is the function  $F$  defined recursively by:  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_{i+2} = F_{i+1} + F_i$ , for  $i \in \mathbb{N}$ . The sequence  $S$  of, so-called, *partial sums* of  $F$  is defined by:  $S_n = (\sum i: 0 \leq i < n: F_i)$ , for all  $n \in \mathbb{N}$ . Prove that  $S_n = F_{n+1} - 1$ , for all  $n \in \mathbb{N}$ .
10. What is the number of anagrams of the word “STRUCTURES”?
11. Prove that  $(\sum j: 0 \leq j < i: 2^j) = 2^i - 1$ , for all  $i \in \mathbb{N}$ .
12. A given function  $a$  on  $\mathbb{N}$  satisfies:  $a_0 = 0$  and  $a_{i+1} = a_i + i$ , for all  $i \in \mathbb{N}$ . Derive an explicit definition for  $a$ .
13. A given function  $a$  on  $\mathbb{N}$  satisfies:  $a_0 = 0$  and  $a_{i+1} = 2 * a_i + 2^i$ , for all  $i \in \mathbb{N}$ . Derive an explicit definition for  $a$ .
14. A given function  $a$  on  $\mathbb{N}$  satisfies:  $a_0 = 0$  and  $a_{i+1} = 2 * a_i + 2^{i+1}$ , for all  $i \in \mathbb{N}$ . Derive an explicit definition for  $a$ .
15. A given function  $a$  on  $\mathbb{N}^+$  satisfies:  $a_1 = 1$  and  $a_{i+1} = (1 + 1/i) * a_i$ , for all  $i \in \mathbb{N}^+$ . Derive an explicit definition for  $a$ .
16. At the beginning of some year John deposits 1000 Euro in a savings account, on which he receives 8% interest at the end of every year. At the beginning of each next year he withdraws 100 Euro. How many years can John maintain this behavior, that is, after how many years the balance of his account is insufficient to withdraw yet another 100 Euro?
17. Solve the following recurrence relations for a function  $a$ ; in each of the cases we have  $a_0 = 0$  and  $a_1 = 1$ , and for all  $i \in \mathbb{N}$ :
- $a_{i+2} = 6 * a_{i+1} - 8 * a_i$ ;
  - $a_{i+2} = 6 * a_{i+1} - 9 * a_i$ ;
  - $a_{i+2} = 6 * a_{i+1} - 10 * a_i$ .

18. Solve the recurrence relation  $a_{i+2} = 6 * a_{i+1} + 9 * a_i$ , for all  $i \in \mathbb{N}$ , with  $a_0 = 6$  and  $a_1 = 9$ .
19. The, so-called, Lucas sequence is the function  $L$  defined recursively by:  $L_0 = 2$ ,  $L_1 = 1$ , and  $L_{i+2} = L_{i+1} + L_i$ , for all  $i \in \mathbb{N}$ . Determine the first 8 elements of this sequence, and derive an explicit definition for  $L$ .
20. Let  $V$  be a finite set with  $m$  elements. Let  $n$  satisfy:  $0 \leq n \leq m$ . What is the number of functions in  $V \rightarrow \{0, 1\}$ , with the additional property that  $\#\{v \in V \mid f(v) = 1\} = n$ ?
21. A chess club with 20 members must elect a board consisting of a chairman, a secretary, and a treasurer.
  - (a) What is the number of ways to form a board (from members of the club) if the three positions must be occupied by different persons?
  - (b) What is the number of ways if it is permitted that a person occupies several positions?
  - (c) What is the number of ways if the rule is that every person occupies at most 2 positions?
22. What is the number of sequences, of length 27, consisting of exactly 9 symbols 0, 9 symbols 1, and 9 symbols 2?
23. In a two-dimensional plane, a robot walks from grid point  $(0, 0)$  to grid point  $(12, 5)$ , by successively taking a unit step, either in the positive  $x$ -direction or in the positive  $y$ -direction; it never takes a step backwards. What is the number of possible routes the robot can walk?
24. What is the number of 8-digit decimal numbers in which the digits occur in *descending* order only? For example, "77763331" is such a number, whereas "98764511" is not.