Chapter 7

Sequences

Reason's last step is the recognition that there are an infinite number of things which are beyond it. Pascal

We write a sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ as $\{a_n\}$ and our interest is normally whether the sequence tends to a limit A written

- $a_n \to A$ as $n \to \infty$.
- or $\lim_{n\to\infty} a_n = A$

However there are many interesting sequences where limits are not the main interest. For example the Fibonacci sequence. In Fibonacci's Liber Abaci (1202) poses the following problem

> How Many Pairs of Rabbits Are Created by One Pair in One Year:
> A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also.

The resulting sequence is

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

and each term is the sum of the previous two terms. An interesting aside is that the nth Fibonacci number $F(n)$ can we written as

$$F(n) = [\phi^n - (1-\phi)^n] / \sqrt{5} \text{ where } \phi = (1+\sqrt{5})/2 \simeq 1.618\ldots$$

which is a surprise since $F(n)$ is an integer and the formula contains $\sqrt{5}$. For lots more on sequences see

http://www.research.att.com/ njas/sequences/

7.0.1 Limits of sequences

We turn our attention to the behaviour of sequences such as $\{a_n\}$ as n becomes very large.

1. A sequence may approach a finite value A. We say that it tends to a limit, so for example we write

$$1, \left(\frac{1}{2}\right), \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3 \cdots \left(\frac{1}{2}\right)^n, \ldots$$

   or

$$1.0000 \quad 0.5000 \quad 0.2500 \quad 0.1250 \quad 0.0625 \quad 0.0312 \quad 0.0156 \quad 0.0078 \quad 0.0039 \quad 0.0020 \ldots$$

   as

$$\left\{\left(\frac{1}{2}\right)^n\right\}$$

   and we shall see that

$$\left\{\left(\frac{1}{2}\right)^n\right\} \to 0 \text{ as } n \to \infty$$

2. If a sequence does not converge it may go to $\pm\infty$, that is keep increasing or decreasing.

$$1 \quad 2 \quad 4 \quad 8 \quad 16 \quad 32 \quad 64 \quad 128 \quad 256 \quad 512 \quad 1024 \ldots$$

   Informally $\{2^n\} \to \infty$ as $\to \infty$.

3. A sequence may just oscillate

$$1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1$$

Limit

We need a definition of a limit and after 2000 years of trying we use :
$\{a_n\} \to A$ as $\to \infty$ if and only if, given any number ϵ there is an N such that for $n \geq N \quad |a_n - A| < \epsilon$.

In essence I give you a guarantee that I can get as close as you wish to a limit (if it exists) for all members of the sequence with sufficiently large N, that is after N all the values of the sequence satisfy $| a_n - A | < \epsilon$. The idea is that if there is a limit then if you give me some tolerance, here ϵ, I can guarantee that for some point in the sequence all the terms beyond that *all* lie within ϵ of the limit.

Examples

- $\{\frac{1}{n}\} \to 0$.
- $\{x^n\} \to 0$ for $|x| < 1$.

- We argue as follows:
  Suppose you give me a (small) value for ϵ. I can then choose a value N where $N > 1/\epsilon$. We can do this as, for $\mid x \mid < 1$

$$\ldots \mid x \mid^4 < \mid x \mid^3 < \mid x \mid^2 < \mid x \mid$$

  It then follows that as $N > 1/\epsilon$ then $\epsilon > 1/N$. But if $n > N$ then $1/n < 1/N$ so we can say:
  if we choose $N > 1/\epsilon$ the when $N > n$ $\quad \mid 1/n - 0 \mid < \epsilon$ and so $1/n \to 0$

- We argue as follows:
  Suppose you give me a (small) value for ϵ. I can then choose a value N where $\mid x \mid^N < \epsilon$. Or $N \log \mid x \mid < \log \epsilon$. Rearranging

$$N > \frac{\log \epsilon}{\log \mid x \mid} \text{ beware the signs!}$$

  But if $\log \mid x \mid < 1$ then

$$\log \mid x \mid^2 < \log \mid x \mid, \log \mid x \mid^3 < \log \mid x \mid^2, \ldots \log \mid x \mid^n < \log \mid x \mid^{n+1}$$

  So we choose $N > \log \epsilon / \log \mid x \mid$ then when $N > n$ $\quad \mid x^n \mid = \mid x^n - 0 \mid < \epsilon$ and so $\mid x^n \mid \to 0$

Rules

Manipulating expressions like $| a_n - a |$ can be tricky so it is easier to develop some rules. Using these is very much easier as we shall see.

If $\{a_n\}$ and $\{b_n\}$ are two sequences and $\{a_n\} \to A$ while $\{b_n\} \to B$ then

- $\{a_n \pm b_n\} \to A \pm B$
- $\{a_n b_n\} \to AB$
- $\{a_n/b_n\} \to A/B$ provided B is nonzero as are the $\{b_n\}$.
- For a constant c we have $\{ca_n\} \to cA$

also

- If $\{a_n\} \to \pm\infty$ then $\{1/a_n\} \to 0$
- If $\{a_n\} \to \pm\infty$ while $\{b_n\} \to B$ (finite B) then$\{a_n + b_n\} \to \pm\infty$
- If $\{a_n\} \to \infty$ while $\{b_n\} \to B$ (finite B) then$\{a_n b_n\} \to \pm\infty$ depending on the sign of B.

We can look at rational functions as follows

1. $$\left\{\frac{n+1}{n+13}\right\} = \left\{\frac{1+1/n}{1+13/n}\right\} \to \left\{\frac{1+0}{1+0}\right\} \to 1$$

2. $$\left\{\frac{n^2-3n+11}{n^4+13n^2-n+43}\right\} = \left\{\frac{1/n^2-3/n^3+11/n^4}{1+13/n^2-1/n^3+43/n^4}\right\} \to \left\{\frac{0-0+0}{1+0-0+0}\right\} \to 0/1 \to 0$$

3. $$\left\{\frac{n+1}{n+13x^n}\right\} = \left\{\frac{1+1/n}{1+13x^n/n}\right\} \to 1/1 \to 1 \qquad | x |< 1$$

4. $$\left\{\frac{3^n+1}{4^n+13}\right\} = \left\{\frac{(3/4)^n+1/4^n}{1+13(1/4)^n}\right\} \to 0/1 = 0 \to 1$$

Subsequence

A subsequence of a sequence $\{a_n\}$ is an infinite succession of its terms picked out in any way. Note that if the original series converges to A so does any subsequence. If $a_{n+1} \geq a_n$ we say the subsequence is increasing while if $a_{n+1} \leq a_n$ we say the subsequence is decreasing. Increasing or decreasing sequences are sometimes called monatonic.

Bounded

If an increasing sequence is bounded above then it must converge to a limit. Similarly If an decreasing sequence is bounded below then it must converge to a limit.

7.1 Series

A series is the sum of terms of a sequence written

$$u_1 + u_2 + u_3 + \cdots + u_N = \sum_{i=1}^{N} u_i$$

We use capital sigma (Σ) for sums and by

$$\sum_{i=a}^{b} u_i$$

we mean the sum of terms like u_i for i taking the values a to b. Of course there are many series we sum, for example we have met the Binomial series and we have the following useful results.

- $1 + 2 + 3 + 4 + \cdots + N = \sum_{i=1}^{N} = N(N+1)/2$
- $1^2 + 2^2 + 3^2 + 4^2 + \cdots + N^2 = \sum_{i=1}^{N} i^2 = N(2N+1)(N+1)/6$
- $1^3 + 2^3 + 3^3 + 4^3 + \cdots + N^3 = \sum_{i=1}^{N} i^3 = [N(N+1)/2]^2$
- $1\frac{1}{2} + \frac{1}{2}\frac{1}{3} + \cdots + \frac{1}{N}\frac{1}{N+1} = \sum_{i=1}^{N} \left(\frac{1}{i(i+1)}\right) = 1 - \frac{1}{N+1}$
- $1 + x + x^2 + x^3 + \cdots + x^N = \sum_{i=0}^{N} x^i = \left(1 - x^{N+1}\right)/(1-x)$

7.1.1 Infinite series

If we want the sum of the infinite series $\sum_{i=1}^{\infty} u_i$ -if such a thing exists - we need to be clear we mean. *Assume that all the terms in the series are non-negative, that is* $0 \leq u_i$. Consider the partial sums

$$S_1 = u_1$$

$$S_2 = u_1 + u_2$$
$$S_3 = u_1 + u_2 + u_3$$
$$S_4 = u_1 + u_2 + u_3 + u_4$$
$$\cdots$$
$$S_N = u_1 + u_2 + u_3 + \cdots + u_N$$
$$\cdots$$

If the sequence $\{S_n\}$ converges to a limit S then we say that the series $\sum_{i=1}^{\infty} u_i$ is convergent and the sum is S. Otherwise we say the series diverges or is divergent.

Examples

- $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.
  We can argue: Let

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \frac{1}{2} > 2$$

  and

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 3/2$$

$$S_{16} = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)$$

$$> 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} > 6/2$$

In general we can (with care show)

$$S_{2^k} > \frac{k}{2}+1$$

So we can make the partial sums of 2^k terms as large as we like and they are increasing and unbounded. Thus the series must be divergent.

This has an important consequence if $u_n \to 0$ **it does not mean that the sum is convergent**. It may be but it may not be!

- $\sum_{n=0}^{\infty} x^n$ is convergent for $|x| < 1$ and the sum is $1/(1-x)$. When $|x| > 1$ the series is divergent.

  We can argue that

$$\sum_{n=0}^{N} x^n = \frac{1-x^{N-1}}{1-x} \to 1/(1-x)$$

  and since we have an explicit form for the sum the result follows.

- $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and the sum is 1 since

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} - \sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{(n+1)}\right) = 1 - \frac{1}{N+1}$$

- $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ is divergent for $\alpha \geq 1$ and convergent otherwise.

Some Rules for series of positive terms

- If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are both convergent with sums S and T then $\sum_{n=1}^{\infty}(u_n \pm v_n)$ converges to $S \pm T$.
- If $\sum_{n=1}^{\infty} u_n$ converges then adding or subtracting a finite number of terms does not affect convergence, it will however affect the sum.

- If u_n does not converge to zero then $\sum_{n=1}^{\infty} u_n$ does not converge.
- The comparison test: If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are two series of positive terms and if $\{u_n/v_n\}$ tends to a non zero finite limit R then the series either both converge or both diverge.
- The Ratio test: If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms and suppose $\{u_{n+1}/u_n\} \to L$ then
  - If $L < 1$ the series converges.
  - If $L > 1$ the series diverges.
  - If $L = 1$ the question is unresolved.
- The integral test: Suppose we have $\sum_{n=1}^{\infty} u_n$ and $f(n) = u_n$ for some function f which satisfies
  1. $f(x)$ is decreasing as x increases.
  2. $f(x) > 0$ for $x \geq 1$

  Then
  1. $0 < \sum_{n=1}^{N} u_n - \int_1^{N+1} f(x)dx < f(1)$
  2. The sum converges if the integral $\int_1^{\infty} f(x)dx$ is finite and diverges if $\int_1^{\infty} f(x)dx$ is infinite.

Absolute Convergence

We say that $\sum_{n=1}^{\infty} u_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} \mid u_n \mid$ converges. If $\sum_{n=1}^{\infty} |u_n|$ does not converge but $\sum_{n=1}^{\infty} u_n$ does then we say the series is *conditionally convergent*. The nice thing about absolutely convergent series is we can rearrange the terms without affecting the convergence or the sum.

Alternating sign test

On simple test for non conditionally convergent series is the alternating sign test. Suppose we have a decreasing sequence of positive terms $\{u_n\}$ and let

$$S = u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^n u_n \ldots$$

Then S converges. For example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \ldots$$

Power series

A series of the form

$$S = a_0 + a_1 x + a_1 x + a_2 x^2 + a_3 x^3 \ldots = \sum_{n=0}^{\infty} a_n x^n$$

is called a *power series* Many power series only converge for values of x which satisfy $| x |< R$ for some value R. This value is called the *radius of convergence.* We can usually rind R using the ratio test, for example

$$S = 1 + \left(\frac{x}{5}\right) + \left(\frac{x}{5}\right)^2 + \left(\frac{x}{5}\right)^3 + \left(\frac{x}{5}\right)^4 \ldots$$

Then

$$| u_{n+1}/u_n |= \left|\left(\frac{x}{5}\right)^{n+1} / \left(\frac{x}{5}\right)^n\right| = \left|\left(\frac{x}{5}\right)\right|$$

for this to be less than 1 we need $| x |< 5$ You can then check $x \pm 5$ separately.

Exercises

1. Write down the first five terms of each of the sequences defined below
   (a) $a_n = 1 - (0.2)^n$
   (b) $a_n = 1 - (-0.2)^n$
   (c) $a_n = (n^2 + 1)/(n + 1)$.
   (d) $a_n = 3/a_{n-1} \qquad a_1 = -1$

2. Graph the sequences in question 1.

3. Decide which of the following sequences converges and find the limit if it exists.
   (a) $2 - (0.2)^n$
   (b) $2 - (-0.2)^n$
   (c) $(n + 1)/(n^2 + 1)$
   (d) $(4 + n)/(3n - 2)$
   (e) $(4 + n)$
   (f) $(n^2 - n + 2)/(5n^2 + 4n + 1)$
   (g) $2^n - \left(-\frac{1}{2}\right)$

4. How large must n be for $(1/3)^n$ to be less that

   (a) 0.01

   (b) 10^{-6}

5. Find a number N such that $n^2/2^n \leq 0.001$ if $n > N$.

6. Suppose $a_n = x^{1/n} \qquad x > 1$

   (a) Show that the sequence is decreasing.

   (b) Show that the sequence is bounded below.

   (c) Is the sequence convergent?

7. Show that

$$1 + 3 + 5 + \cdots + (2N - 1) = N^2$$

8. Find

$$\sum_{n=1}^{N} \frac{1}{(n+1)(n+2)}$$

9. Decide which of the following sums are convergent.

   (a) $\sum_{n=1}^{\infty} 1/(2n-1)$

   (b) $\sum_{n=1}^{\infty} 2/(n^2+3)$

   (c) $\sum_{n=1}^{\infty} 1/\sqrt{2n-1}$