Chapter 2

The statement calculus and logic

"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic.
Lewis Carroll

You will have encountered several languages - your native language or the one in which we are currently communicating(English) and other natural languages such as Spanish, German etc. You may also have encountered programming languages like Python or C. You have certainly met some mathematics if you have got this far.

A language in which we describe another language is called a *metalanguage*. For almost all of mathematics, the metalanguage is English with some extra notation.

In computing we need to define, and use, languages and formal notation so it is essential that we have a clear and precise metalanguage. We begin by looking at some English expressions which we could use in computing. Most sentences in English can be thought of as a series of statements combined using connectives such as "and", "or", "if . . . then . . ."

For example the sentence "if it is raining and I go outside then I get wet" is constructed from the three simple statements:

1. "It is raining."
2. "I go outside."
3. "I get wet."

Whether the original sentence is true or not depends upon the truth or not of these three simple statements.

If a statement is true we shall say that its logical value is true, and if it is false, its logical value is false. As a shorthand we shall use the letter T for true and F for false.

We will build compound statements from simple statements like "it is raining", "it is sunny" by connecting them with *and* and *or* In order to make things shorter and we hope more readable, we introduce symbolic notation.

1. Negation will be denoted by $\neg$.
2. "and" by $\wedge$.
3. "or" by $\vee$.

We now look at these connectives in a little more detail.

Negation $\neg$

The negation of a statement is false when the statement is true and is true if the statement is false. So a statement and its negation always have different truth values. For example "It is hot" and "It is not hot." In logic you need to be quite clear about meanings so the negation of,

"All computer scientists are men"

is

"Some computer scientists are men"

NOT

"No computer scientists are men."

The first and third statement are both false!

In symbolic terms if p is a statement, say " it is raining" , then $\neg p$ is its negation. That is $\neg p$ is the statement "it is not raining". We summarize the truth or otherwise of the statements in a *truth table*, see table 2.1.

p	$\neg p$
T	F
F	T

Table 2.1: Truth table for negation ($\neg$)

In the truth table 2.1 the first row reads in plain English - "If p is true then $\neg p$ is false" and row two "If p is false then $\neg p$ is true'.

Conjunction $\wedge$

Similarly, if p and q are statements, then $p \wedge q$ is read as "p and q" . This (confusingly) is called the *conjunction* of p and q.

So if p is the statement " it is green" while q is the statement " it is an apple" then

$$p \wedge q \text{ is the statement "It is green and it is an apple "}$$

We often write this in the shorter form:
If p=" it is green" and q = " it is an apple" then $p \wedge q$ = "It is green and it is an apple "

Clearly this statement is true only when both p and q are true. If either of them is false then the compound statement is false. It will be helpful if we have a precise definition of $\wedge$ and we can get one using a truth table.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2.2: The truth table for $\wedge$

From table 2 we see that if p and q are both true then $p \wedge q$ is also true. If p is true and q is false then $p \wedge q$ is false.

Disjunction $\vee$

Suppose we now look at "or". In logic we use $p \vee q$ as a symbolic way of writing p or q. The truth table in this case is given in table 2.3 This version of "or" , which

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 2.3: The truth table for $\vee$

is the common one used in logic is sometimes known as the "inclusive or" because we can have $p \vee q$ true if either one of p and q is true or if *both* are true.

You could of course define the exclusive or , say $\not\equiv$ as having the truth table in 2.4

p	q	$p \not\equiv q$
T	T	F
T	F	T
F	T	T
F	F	F

Table 2.4: The truth table for $\not\equiv$

The Conditional $\Rightarrow$

A rather more interesting connective is "implies" as in p "implies" q. This can be written many ways, for example

- p implies q
- If p then q
- q if p
- p is a sufficient condition for q

I am sure you can think of other variants. We shall use the symbolic form $p \Rightarrow q$ and the truth table for our definition is given in table 2.5.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 2.5: The truth table for $\Rightarrow$

We sometimes call p the *hypothesis* and q the *consequence* or conclusion. Many people find it confusing when they read that " p only if q" is the same as "If p then q". Notice that " p only if q" says that p cannot be true when q is not true, in other words the statement is false if p is true but q is false. When p is false q may be true or false.

You need to be aware that " q only if p" is *NOT* a way of expressing " $p \Rightarrow q$. We see this by checking the truth values. The truth value in line 3 of table 2.5 is the critical difference.

You might like to check that " $\neg p \vee q$ is equivalent to $p \Rightarrow q$, see the table below

p	$\neg p$	q	$\neg p \vee q$
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T

Table 2.6: The truth table for $\Rightarrow$

Notice that our definition of implication is rather broader than the usual usage.

Typically you might say

"if the sun shines today we will have a barbecue" .

The hypothesis and the conclusion are linked in some sensible way and the statement is true unless it is sunny and we do not have a barbecue. By contrast the statement

"If the sun shines today 19 is prime"

is true from the definition of an implication because the conclusion is always true no matter if it is sunny or not. If we consider

"if the sun shines today 8 is prime"

The statement is obviously false if today is sunny because 8 is never prime. However the whole statement is true when the sun does not shine today even though 8 is never prime. Of course we are unlikely to make statements like these in real life.

The Biconditional $\iff$

Suppose p and q are two statements. Then the statement "p if and only if q" is called the *biconditional* and denoted by $p \iff q$ or iff. Yes there are two f's! It is true only when p and q have the same logical values, i.e., when either both are true or both are false.

You may also meet the equivalent

- p iff q
- p is necessary and sufficient for q

The truth table is shown in figure 2.7. For example we might say

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 2.7: The truth table for $\iff$

You can go to the match if and only if you buy a ticket.

This sort of construction is not very common in ordinary language and it is often hard to decide whether a biconditional is implied in ordinary speech. In mathematics or computing you need to be clear if you are dealing with implication $p \Rightarrow q$ or the biconditional $p \iff q$

Converse, contrapositive and inverse

Propositional logic has lots of terminology. So If $p \Rightarrow q$ then

- $q \Rightarrow p$ is the converse.
- $\neg q \Rightarrow \neg p$ is the contrapositive.
- $\neg p \Rightarrow \neg q$ is the inverse.

Truth tables

It is probably obvious that we aim to use logic to help us in checking arguments. We hope to be able to translate from English to symbols. Thus if p is "John learns to cook" and q is " John will find a job" then $p \Rightarrow q$ represents . "If John learns to cook" and then John will find a job" In problems like these the truth table, while cumbersome can be very helpful in giving a mechanical means of checking the truth values of arguments.

To construct tables for compound statements such as $p \vee \neg q \Rightarrow (p \wedge q)$ we need to think about the order we work out the truth values of symbols. The table 2.8 gives the order of precedence.

Precedence	1(Highest)	2	3	4	5(Lowest)
Operator	$\neg$	$\wedge$	$\vee$	$\Rightarrow$	$\iff$

Table 2.8: Operator precedence

So we negate first, then and etc. As in algebra we also use brackets to indicate that we evaluate the terms in brackets first. Thus for $(p \vee q) \wedge r$ we evaluate the term in brackets $(p \vee q)$ first. Thus

	p	q	$(p \vee q)$	$\neg p$	$(p \vee q) \vee \neg p$
	T	T	T	F	T
	T	F	T	F	T
	F	T	T	T	T
	F	F	F	T	T
precidence	-	-	1	2	3

The vital point about logical statements and about truth tables is :
Two symbolic statements are equivalent if they have the same truth table.
and two statements $p1$ and $p2$ are equivalent, we will write $p1 \iff p2$.

Thus, for example, the statements $(p \vee q) \wedge \neg p$ and $\neg p \wedge q$ are equivalent. We can deduce this from the truth tables, see table 2.9

p	q	$p \vee q$	p	$(p \vee q) \wedge \neg p$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

p	$\neg p$	q	$\neg p \wedge q$
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F

Table 2.9: The truth tables for $(p \vee q) \wedge \neg p$ and$(\neg p \wedge q)$

The reader can use truth table to verify the following equivalences.

1. $\neg(p \vee q) \iff \neg p \wedge \neg q$
2. $\neg(p \wedge q) \iff \neg p \vee \neg q$

One can avoid writing truth tables in table 2.9 and verify the first equivalence as follows:
$p \vee q$ is false only when both p and q are false. Therefore $\neg(p \vee q)$ is true only when both p and q are false. Similarly, $\neg p \wedge \neg q$ is true only when both $\neg p$ and $\neg q$ are true, which is when p and q are false. This proves the equivalence.

Exercise

Construct truth tables for

1. $\neg(p \wedge q)$
2. $\neg(p \vee q) \wedge \neg(q \vee p)$
3. $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
4. $(p \vee q \Rightarrow r) \wedge (r \Rightarrow s)$
5. $(p \vee q \Rightarrow r) \wedge (r \Rightarrow s) \Rightarrow (p \Rightarrow r)$

Arguments

We now look briefly at logical arguments and begin with some definitions. Definition:

- A statement that is always true is called a *tautology.*
- A statement that is always false is called a *contradiction.*

So a statement is

1. A tautology if its truth table has no value F.
2. A contradiction if its truth table has no value T.

Notice you may find some writers who say that a formula (in the statement calculus we have just described) is *valid* rather than use the term tautology. The symbol $\models$ A is often used as a shorthand for "A is a tautology" or " A is valid".

Examples

1. The statement $\mathsf{p} \vee \neg \mathsf{p}$ is a tautology, while the statement $\mathsf{p} \wedge \neg \mathsf{p}$ is a contradiction.
2. The statement $((\mathsf{p} \vee \mathsf{q}) \wedge \mathsf{p}) \iff \mathsf{p}$ is a tautology.
3. Two statements $\mathsf{p1}$ and $\mathsf{p2}$ are *equivalent* when $\mathsf{p1} \iff \mathsf{p2}$ is a tautology, and so $\mathsf{p1} \equiv \mathsf{p2}$ when $\mathsf{p1} \iff \mathsf{p2}$ is a tautology.

Definition 1: Given two statements $\mathsf{p1}$ and $\mathsf{p2}$ we say that $\mathsf{p1}$ *implies* $\mathsf{p2}$ if $\mathsf{p1} \Rightarrow \mathsf{p2}$ is a tautology.

In everyday life we often encounter situations where we make conclusions based on evidence. In a courtroom the fate of the accused may depend the defence proving that the opposing side's arguments are not valid. A typical task in theoretical sciences is to logically come to conclusions given premises. That is to provide principles for reasoning.

A scientist might say

> "if all the premises are true then we have the following conclusion."

Thus they would assert that the conditional

> "if all the premises are true then we have the following conclusion"

is a tautology, or that the premises imply his/her conclusion. If his/her reasoning is correct we say that his argument is valid.

Definition 2: A conditional of the form

(a conjunction of statements) implies c

where c is a statement, is called an *argument*. Symbolically

$$p_1, p_2, \ldots, p_m \Rightarrow c$$

The statements in the conjunction on the left side of the conditional are called *premises*, while c is called the *conclusion*.

An argument is valid if it is a tautology, that is, if the premises imply the conclusion (every line of the truth table is T), otherwise it is invalid. So we might have a sequence of premises $p_1, p_2, p_3, \ldots, p_m$ for which c is a valid consequence, symbolically

$$p_1, p_2, p_3, \ldots, p_m \models c$$

You should note that

1. A conjunction of several statements is true only when all the statements are true.
2. A conditional is false only when the antecedent (the left hand side) is true and the consequent (the right hand side) is false.
3. Therefore, an argument is invalid only when there is a situation where all the premises are true, but the conclusion is false. If such a situation cannot occur, the argument is valid.

Exercise s:

1. Is the following argument valid?

   All birds are mammals and the platypus is a bird. Therefore, the platypus is a mammal.

   Note the premises may be wrong but we are interested in the argument.

2. Sketch how you might show that the statements below below imply that "It rained". Beware this is a big truth table so you are probably best to ensure you understand the method.

   If it does not rain or if it is not foggy then the regatta will be held and the lifeboat demonstration will go on. If the regatta is held then the trophy will be awarded.

   and

the trophy was not awarded.

3. Show that the following argument is valid.

   Blodwin works hard. If Blodwin works hard then she is a dull girl. If Blodwin is a dull girl she will not get the job therefore Blodwin will not get the job.

So far we have used truth tables only to determine the validity of arguments that are given in symbolic form. However, we can do the same with other arguments by first rewriting them in symbolic form. This is illustrated in the following example.

> Either I shall go home or stay and have a drink. I shall not go home. Therefore I stay and have a drink.

Suppose p= I shall go home and q = I shall stay and have a drink. The argument is $\neg p \Rightarrow q$.

p	$\neg p$	q	$\neg p \Rightarrow q$
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F

Table 2.10: The truth table for $\Rightarrow$

From the truth table table 2.10 we have a F and so the argument is not valid is , we do not have a tautology. We summarize the process of determining the validity of arguments as follows.

2.0.4 Analyzing Arguments Using Truth Tables

- Step 1: Translate the premises and the conclusion into symbolic form.
- Step 2: Write the truth table for the premises and the conclusion.
- Step 3: Determine if there is a row in which all the premises are true and the conclusion is false. If yes, the argument is invalid, otherwise it is valid.

However truth table can become unwieldy if we have several premises. Consider the following

$$p, r, (p \wedge q) \rightarrow \neg r \vDash \neg q$$

Given we have p, q and r we need 8 rows (2^3) in our table 2.11 as we need all combinations of p, q and r. If we examine line 3 in table 2.11 we can see that when $p, r, (p \wedge q) \rightarrow \neg r$ are all true (we can ignore q) then the result $\neg q$ is true and we have a tautology.

p	q	r	$p \wedge q \Rightarrow \neg r$	$\neg q$	
T	T	T	F	F	
T	T	F	T	F	
T	F	T	T	T	←
T	F	F	T	T	
F	T	T	T	T	
F	T	F	T	F	
F	F	T	T	T	
F	F	F	T	T	

Table 2.11: Truth table with p, q and r

Now suppose we have p, q, r, s and t. Our table will have $2^5 = 32$ rows.
Take as an example :
If I go to my first class tomorrow , then I must get up early, and if I go to the dance tonight, I will stay up late. If I stay up late and get up early, then I will be forced to exist on only five hours sleep. I cannot exist on five hours of sleep. Therefore I must either miss my fist class tomorrow or not go to the dance.

- Let p be " I go to my first class tomorrow"
- Let q be " I must get up early"
- Let r be " I go to the dance "
- Let s be " I stay up late ".

- Let t be "I can exist on five hours sleep".

The premises are

$$(p \Rightarrow q) \wedge (r \Rightarrow t), s \wedge q \Rightarrow t, \neg t$$

and the conclusion is $\neg p \vee \neg r$. We will prove that $\neg p \vee \neg r$ is a valid consequence of the premises.

Of course we could write out a truth table, however we can try to be cunning.

1. Take the consequence $\neg p \vee \neg r$ and assume that it is FALSE.
2. Then both p and r must be TRUE.
3. The first premise $(p \Rightarrow q) \wedge (r \Rightarrow t)$ implies that q and t are true.
4. So t is true and the last premise is $\neg t$ is assumed TRUE so we have a contradiction.
5. Thus our premise is valid.

I think you might agree that this is a good deal shorter than using truth tables!.

Exercises

Show that

1. $\models (p \Rightarrow q) \Rightarrow ((q \Rightarrow r)) \Rightarrow (p \Rightarrow r))$
2. $\models p \Rightarrow (\neg q \Rightarrow \neg p) \Rightarrow q)$

We add some tables of tautologies which enable us to eliminate conditionals and biconditionals.

1. $\models p \Rightarrow q \iff \neg p \vee q$
2. $\models p \Rightarrow q \iff \neg(p \vee \neg q)$
3. $\models p \vee q \iff \neg p \to q$
4. $\models p \vee q \iff \neg(p \Rightarrow \neg q)$
5. $\models p \vee q \iff \neg p \to q$

6. $\models p \vee q \iff \neg p \to q$

7. $\models p \wedge q \iff \neg(p \Rightarrow \neg q)$

8. $\models p \wedge q \iff \neg(\neg p \vee \neg q)$

9. $\models (p \iff q) \iff (p \Rightarrow q) \wedge (q \Rightarrow p)$

Normal forms

A statement is in *disjunctive normal form* (DNF) if it is a disjunction i.e. a sequence of $\vee$'s consisting of one or more *disjuncts.* Each disjuncts is a conjunction, $\wedge$, of one or more literals (i.e., statement letters and negations of statement letters. For example

1. p

2. $(p \wedge q) \vee (p \wedge \neg r)$

3. $(p \wedge q \wedge \neg r) \vee (p \wedge \neg q)$

4. $p \vee (q \wedge r)$

However $\neg(p \vee q)$ is not a disjunctive normal form($\neg$ is the outermost operator) nor is $p \vee (q \wedge (r \vee s)$ as a $\vee$ is inside a $\wedge$. Converting a formula to DNF involves using logical equivalences, such as the double negative elimination, De Morgan's laws, and the distributive law. All logical formulas can be converted into disjunctive normal form but conversion to DNF can lead to an explosion in the size of of the expression.

A formula is in *conjunctive normal form* (CNF) if it is a conjunction of clauses, where a clause is a disjunction of literals. Essentially we have the same form as a DNF but we use $\wedge$ rather than $\vee$. As a normal form, it is useful (as is the DNF) in theorem proving.

We leave with some ideas which are both important and common in mathematics.

2.0.5 Contradiction and consistency

We say a contradiction is a formula that always takes the value F, for example $p \wedge \neg p$. Then a set of statements $p_1, p_2, \ldots, p_n$ is *inconsistent* if a contradiction can be drawn as a valid consequence of this set.

$p_1, p_2, \ldots, p_n \vDash q \wedge \neg q$ for some formula b if a contradiction can be derived as a valid consequence of $p_1, p_2, \ldots, p_n \vDash q$ and $\neg q$

Mathematics is full of proofs by contradiction or Reductio ad absurdum (Latin for "reduction to the absurd"). For example

There are infinitely many prime numbers.

Assume to the contrary that there are only finitely many prime numbers, and all of them are listed as follows: $n_1, n_2 \ldots, p_m$. Consider the number

$$q = n_1 \times n_2 \times \ldots \times p_m + 1$$

Then the number q is either prime or composite. If we divided any of the listed primes n_i into q, there would result a remainder of 1 for each $i = 1, 2, ..., m$ Thus, q cannot be composite. We conclude that q is a prime number, not among the primes listed above, contradicting our assumption that all primes are in the list $n_1, n_2 \ldots, n_m$. Thus there are and infinite number of primes.

there is no smallest rational number greater than 0

Remember that a ration can be written as the ratio of two integers p/q say.

Assume $n_0 = p/q$ is the smallest rational bigger that zero. Consider $n_0/2$. It is clear that $n_0/2 < n_0$ and n_0 is rational. Thus we have a contradiction and can assume that there is no smallest rational number greater than 0.