# Preface

This book helps students to master the material of a standard US undergraduate first course in Linear Algebra.

The material is standard in that the subjects covered are Gaussian reduction, vector spaces, linear maps, determinants, and eigenvalues and eigenvectors. Another standard is book's audience: sophomores or juniors, usually with a background of at least one semester of calculus. The help that it gives to students comes from taking a developmental approach — this book's presentation emphasizes motivation and naturalness, using many examples as well as extensive and careful exercises.

The developmental approach is what most recommends this book so I will elaborate. Courses at the beginning of a mathematics program focus less on theory and more on calculating. Later courses ask for mathematical maturity: the ability to follow different types of arguments, a familiarity with the themes that underlie many mathematical investigations such as elementary set and function facts, and a capacity for some independent reading and thinking. Some programs have a separate course devoted to developing maturity but in any case a Linear Algebra course is an ideal spot to work on this transition to more rigor. It comes early in a program so that progress made here pays off later but also comes late enough so that the students in the class are serious about mathematics. The material is accessible, coherent, and elegant. There are a variety of argument styles, including proofs by contradiction and proofs by induction. And, examples are plentiful.

Helping readers start the transition to being serious students of mathematics requires taking the mathematics seriously so all of the results here are proved. On the other hand, we cannot assume that students have already arrived and so in contrast with more advanced texts this book is filled with examples, often quite detailed.

Some books that assume a not-yet sophisticated reader begin with extensive computations, including matrix multiplication and determinants. Then, when vector spaces and linear maps finally appear and definitions and proofs start, the abrupt change brings students to an abrupt stop. While this book begins with linear reduction, from the start we do more than compute. The first chapter includes proofs, such as that linear reduction gives a correct and complete solution set. With that as motivation the second chapter starts with real vector spaces. In the schedule below this happens at the beginning of the third week.

Another example of the emphasis here on motivation and naturalness is that the chapter on linear maps does not begin with the definition of homomorphism. Instead it begins with the definition of isomorphism, which is natural — students themselves observe that some spaces are "the same" as others. After that, the next section takes the reasonable step of isolating the operation-preservation idea to define homomorphism. This loses some mathematical slickness but it is a good trade because it gives to students a large gain in sensibility.

A student progresses most in mathematics while doing exercises. The problem sets start with simple checks and range up to reasonably involved proofs. Since instructors often assign about a dozen exercises I have aimed to typically put two dozen in each set, thereby giving a selection. There are even a few that are puzzles taken from various journals, competitions, or problems collections. These are marked with a '**?**' and as part of the fun I have retained the original wording as much as possible.

That is, as with the rest of the book, the exercises are aimed to both build an ability at, and help students experience the pleasure of, *doing* mathematics. Students should see how the ideas arise and should be able to picture themselves doing the same type of work.

**Additional topics.** Applications and computing are interesting and vital aspects of the subject. Consequently, each chapter closes with a selection of topics in those areas. These give a reader a taste of the subject, discuss how Linear Algebra comes in, point to some further reading, and give a few exercises. They are brief enough that an instructor can do one in a day's class or can assign them as projects for individuals or small groups. Whether they figure formally in a course or not, they help readers see for themselves that Linear Algebra is a tool that a professional must have.

**Availability.** This book is Free. In particular, instructors can run off copies for students and sell them at the bookstore. See this book's web page http://joshua.smcvt.edu/linearalgebra for the license details. That page also has the latest version, exercise answers, beamer slides, and LaTeX source.

**Acknowledgements.** A lesson of software development is that complex projects need a process for bug fixes. I welcome such reports from both instructors and students and I periodically issue revisions. My contact information is on the web page.

I thank Gabriel S Santiago for the cover colors. I am also grateful to Saint Michael's College for supporting this project over many years.

And, I thank my wife Lynne for her unflagging encouragement.

**If you are reading this on your own.** This book's emphasis on motivation and development, and its availability, make it widely used for self-study. If you are an independent student then good for you, I admire your industry. However, you may find some advice useful.

While an experienced instructor knows what subjects and pace suit their class, a suggested semester's timetable may help you estimate how much time sections typically take. This schedule was graciously shared by George Ashline.

| *week* | *Monday* | *Wednesday* | *Friday* |
|---|---|---|---|
| 1 | One.I.1 | One.I.1, 2 | One.I.2, 3 |
| 2 | One.I.3 | One.III.1 | One.III.2 |
| 3 | Two.I.1 | Two.I.1, 2 | Two.I.2 |
| 4 | Two.II.1 | Two.III.1 | Two.III.2 |
| 5 | Two.III.2 | Two.III.2, 3 | Two.III.3 |
| 6 | EXAM | Three.I.1 | Three.I.1 |
| 7 | Three.I.2 | Three.I.2 | Three.II.1 |
| 8 | Three.II.1 | Three.II.2 | Three.II.2 |
| 9 | Three.III.1 | Three.III.2 | Three.IV.1, 2 |
| 10 | Three.IV.2, 3 | Three.IV.4 | Three.V.1 |
| 11 | Three.V.1 | Three.V.2 | Four.I.1 |
| 12 | EXAM | Four.I.2 | Four.III.1 |
| 13 | Five.II.1 | –THANKSGIVING BREAK– | |
| 14 | Five.II.1, 2 | Five.II.2 | Five.II.3 |

This supposes that you already know Section One.II, the elements of vectors. Note that in the above course, in addition to the shown exams and to the final exam that is not shown, students must do take-home problem sets that include proofs. That is, the computations are important but so are the proofs.

In the table of contents I have marked subsections as optional if some instructors will pass over them in favor of spending more time elsewhere.

As enrichment, you might pick one or two topics that appeal to you from the end of each chapter or from the lab manual. You'll get more from these if you have access to software for calculations. I recommend *Sage*, freely available from http://sagemath.org.

My main advice is: do many exercises. I have marked a good sample with ✓'s in the margin. Do not simply read the answers — you must actually try the problems and quite possibly struggle with some of them. For all of the exercises, you must justify your answer either with a computation or with a proof. Be aware that few people can write correct proofs without training. Try to find a knowledgeable person to work with you on these.

Finally, a caution for all students, independent or not: I cannot overemphasize that the statement, "I understand the material but it is only that I have trouble with the problems" shows a misconception. Being able to do things with the ideas is their entire point. The quotes below express this sentiment admirably (I have taken the liberty of formatting them as poetry). They capture the essence of both the beauty and the power of mathematics and science in general, and of Linear Algebra in particular.

*I know of no better tactic*
*than the illustration of exciting principles*
*by well-chosen particulars.*
*–Stephen Jay Gould*

*If you really wish to learn*
*then you must mount the machine*
*and become acquainted with its tricks*
*by actual trial.*
*–Wilbur Wright*

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*Author's Note.* Inventing a good exercise, one that enlightens as well as tests, is a creative act and hard work. The inventor deserves recognition. But texts have traditionally not given attributions for questions. I have changed that here where I was sure of the source. I would be glad to hear from anyone who can help me to correctly attribute others of the questions.