

# Chapter 1

## Sets and Notation

### 1.1 Defining sets

**Definition.** A *set* is an unordered collection of distinct objects. The objects in a set are called the *elements*, or *members*, of the set. A set is said to *contain* its elements.

A set can be defined by simply listing its members inside curly braces. For example, the set  $\{2, 4, 17, 23\}$  is the same as the set  $\{17, 4, 23, 2\}$ . To denote membership we use the  $\in$  symbol, as in  $4 \in \{2, 4, 17, 23\}$ . On the other hand, non-membership is denoted as in  $5 \notin \{2, 4, 17, 23\}$ .

If we want to specify a long sequence that follows a pattern, we can use the ellipsis notation, meaning “fill in, using the same pattern”. The ellipsis is often used after two or more members of the sequence, and before the last one, as follows:  $\{1, 2, \dots, n\}$ . The pattern denoted by the ellipsis should be apparent at first sight! For instance,  $\{1, \dots, n\}$  is generally regarded as underspecified (that is, too ambiguous). Of course, even  $\{1, 2, \dots, n\}$  is still ambiguous—did we mean all integers between 1 and  $n$ , all powers of two up to  $n$ , or perhaps the set  $\{1, 2, 25, n\}$ ?—but is generally sufficient, unless you really do mean all powers of two up to  $n$ , in which case  $\{2^0, 2^1, 2^2, \dots, 2^k\}$  for an appropriate  $k$  is a better choice. The ellipsis can also be used to define an infinite set, as in the following.

**Definition.** The set of *natural numbers* or *nonnegative integers*, denoted by  $\mathbb{N}$ , is defined as  $\{0, 1, 2, \dots\}$ .

To avoid ambiguities it is often useful to use the *set builder* notation, which lists on the right side of the colon the property that any set element, specified on the left side of the colon, has to satisfy. Let’s define the positive integers using the set builder notation:

$$\mathbb{N}^+ = \{x : x \in \mathbb{N} \text{ and } x > 0\}.$$

We can also write

$$\mathbb{N}^+ = \{x \in \mathbb{N} : x > 0\}.$$

This is a matter of taste. In general, use the form that will be easiest for the reader of your work to understand. Often it is the least “cluttered” one.

Ok, now onto the integers:

$$\mathbb{Z} = \{x : x \in \mathbb{N} \text{ or } -x \in \mathbb{N}\}.$$

Hmm, perhaps in this case it is actually better to write

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

Remember, when you write mathematics, you should keep your readers’ perspective in mind. For now, we—the staff of this course—are your readers. In the future it might be your colleagues, supervisors, or the readers of your published work. In addition to being reasonably formal and unambiguous, your mathematical writing should be as clear and understandable to your intended readership as possible.

Here are the *rational numbers*:

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0 \right\}.$$

Instead of  $a \in \mathbb{Z}, b \in \mathbb{Z}$ , you can write  $a, b \in \mathbb{Z}$ , which is more concise and generally more readable. Don’t go overboard, though, with writing something like  $a, b \neq 0 \in \mathbb{Z}$ , this is way too confusing and does not say what you want it to.

Finally, the set of *real numbers* is denoted by  $\mathbb{R}$ . All the reals that are not rational are called *irrational*. These include the familiar  $\pi = 3.1415926\dots$ ,  $e = 2.7182818\dots$ ,  $\sqrt{2}$ , and infinitely many others. (How do we know that these numbers are irrational, do you ask? Actually, we will see a proof of this for  $\sqrt{2}$  shortly. The proofs for  $\pi$  and  $e$  require mathematical analysis and are outside our scope.)

**On being formal.** Were the above definitions formal enough? The answer is: it depends. For example, defining the natural numbers is an important and non-trivial accomplishment of mathematics. After all, what do these symbols “1”, “2”, “3”, actually *mean*? These numbers can be formally defined in terms of sets. Even more involved is the formal definition of the reals, usually covered in a first mathematical analysis course.

Here we cannot afford to cover everything in complete detail, which would have to include, among other things, basic algebra and trigonometry. Furthermore, the vast majority of mathematical works, while considered to be “formal”, gloss over details all the time. For example, you’ll be hard-pressed to find a mathematical paper that goes through the trouble of justifying the equation  $a^2 - b^2 = (a - b)(a + b)$ . In effect, every mathematical paper or lecture assumes a shared knowledge base with its readers or listeners. It is extremely important for an author of mathematics, such as yourself during this course, to estimate this shared knowledge base correctly!

In CS103X we will assume most of high-school mathematics, including perhaps some AP math like single-variable calculus, as our shared knowledge base. Thus notions and techniques from this base will generally not be justified in lecture, and can be used freely in your homework and exams. Furthermore, once we develop certain

notation or prove some theorem in class, you can use these freely in your homework and exams, provided that you clearly cite the appropriate theorems. In writing and speaking mathematics, a delicate balance is maintained between being formal and not getting bogged down in minutia.<sup>1</sup> This balance usually becomes second-nature with experience. You should all get the hang of it by the end of the quarter.

## 1.2 Set operations

$A$  is said to be a subset of  $B$  if and only if every element of  $A$  is also an element of  $B$ , in which case we write  $A \subseteq B$ .  $A$  is a *strict subset* of  $B$  if  $A$  is a subset of  $B$  and  $A$  is not equal to  $B$ , which is denoted by  $A \subset B$ . For example,  $\{4, 23\} \subset \{2, 4, 17, 23\} \subseteq \{2, 4, 17, 23\}$ .

Two sets  $A$  and  $B$  are considered equal if and only if they have the same elements. This is denoted by  $A = B$ . More formally,  $A = B$  if and only if  $A \subseteq B$  and  $B \subseteq A$ .

For two sets  $A$  and  $B$ , the operations of union, intersection, and difference are defined as follows:

$$\begin{aligned} A \cup B &= \{x : x \in A \text{ or } x \in B\} \\ A \cap B &= \{x : x \in A \text{ and } x \in B\} \\ A \setminus B &= \{x : x \in A \text{ and } x \notin B\} \end{aligned}$$

The  $\cup$  and  $\cap$  notation can be extended to the union and intersection of multiple sets. Given  $n$  sets  $A_1, A_2, \dots, A_n$ , we can write

$$\bigcup_{i=1}^n A_i$$

for their union, and

$$\bigcap_{i=1}^n A_i$$

for their intersection. In fact, this notation is pretty flexible and the same union can be written as

$$\bigcup_{i=1}^n A_i = \bigcup_{1 \leq i \leq n} A_i = \bigcup_{i \in \{x : 1 \leq x \leq n\}} A_i.$$

Here is another example:

$$\bigcap_{\substack{i \in \{x : 1 \leq x \leq 10\} \\ i \text{ is prime}}} A_i = A_2 \cap A_3 \cap A_5 \cap A_7.$$

Given a set  $A$ , the *cardinality* of  $A$ , also known as the *size* of  $A$ , is simply the number of elements in  $A$ . The cardinality of  $A$  is denoted by  $|A|$ . For example, if  $A = \{2, 4, 17, 23\}$ , then  $|A| = 4$ .

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<sup>1</sup>Of course, what is considered minutia differs from subfield to subfield, and from classroom to classroom.

## 1.3 More sets

The *empty set* is denoted by  $\emptyset$ . It is the unique set without elements. It holds that  $\emptyset \subseteq A$  for any set  $A$ . Why? By definition, this holds if every element of  $\emptyset$  is also an element of  $A$ . Since  $\emptyset$  has no elements, all possible statements about the elements of  $\emptyset$  are true! In particular, all elements of  $\emptyset$  are also elements of  $A$ . If this is confusing don't worry, we will go into such matters more rigorously when we get to logic. (For now, you can ponder the following: If we know for a fact that there are no unicorns <Gasp!>, then it is definitely true that all unicorns have soft light-blue fur.)

A set can contain sets as its elements. For example,  $\{\{2, 4\}, \{17\}, 23\}$  is a perfectly valid set with three elements, two of them sets. (The second element is a *singleton*, a set with one element.) Note that  $\{2, 4\} \in \{\{2, 4\}, \{17\}, 23\}$ , but  $\{2, 4\} \subseteq \{2, 4, 17, 23\}$ , and that  $17 \notin \{\{2, 4\}, \{17\}, 23\}$ , but  $\{17\} \in \{\{2, 4\}, \{17\}, 23\}$ . Also,  $\{\emptyset\}$  is *not* the empty set. (Think about it.)

The *power set* of a set  $A$  is the set of all subsets of  $A$ , and is denoted by  $2^A$ . That is,

$$2^A = \{S : S \subseteq A\}.$$

For example, for  $A = \{2, 4, 17, 23\}$ , we have

$$2^A = \left\{ \emptyset, \{2\}, \{4\}, \{17\}, \{23\}, \{2, 4\}, \{2, 17\}, \{2, 23\}, \{4, 17\}, \{4, 23\}, \{17, 23\}, \right. \\ \left. \{2, 4, 17\}, \{2, 4, 23\}, \{2, 17, 23\}, \{4, 17, 23\}, \{2, 4, 17, 23\} \right\}.$$

The cardinality of this set is 16, and  $16 = 2^4$ . This is not a coincidence: As we shall see when we get to combinatorics and counting, for a set  $A$  with  $n$  elements, the cardinality of  $2^A$  is  $2^n$ . This is in fact the reason for the power set notation.