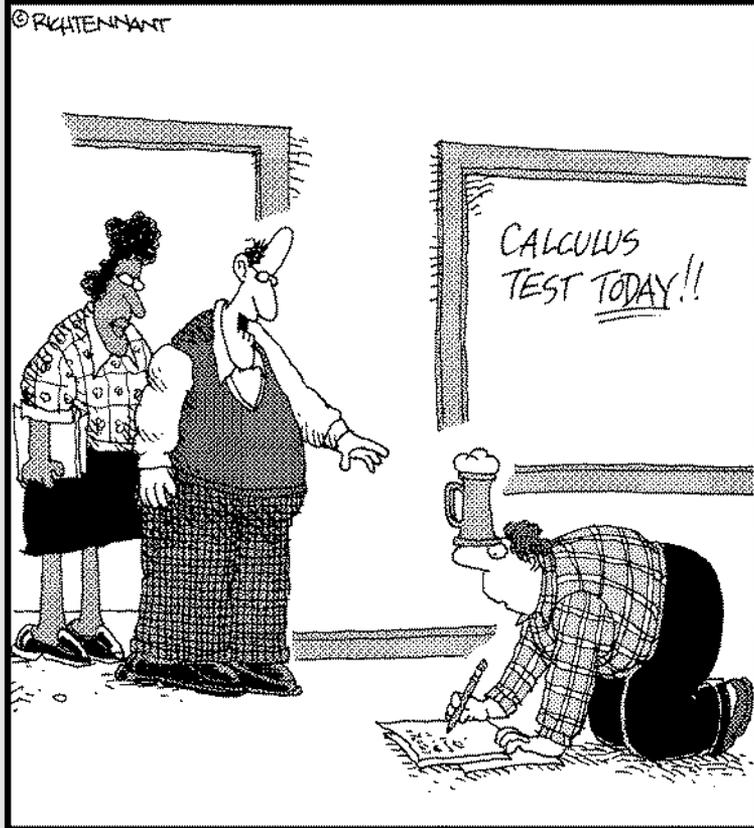


# Part VI

# The Part of Tens

## The 5<sup>th</sup> Wave

By Rich Tennant



"Watch this, Ruth. Steady... steady... calculate the volume of the stein first."

*In this part . . .*

**A**s a special bonus, here are two top-ten lists on calculus-related topics: ten important Calculus II “aha” moments and ten useful test-taking tips.

## Chapter 16

# Ten “Aha!” Insights in Calculus II

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### *In This Chapter*

- ▶ Understanding the key concepts of integration
  - ▶ Distinguishing the definite integral from the indefinite integral
  - ▶ Knowing the basics of infinite series
- 

**O**kay, here you are near the end of the book. You read every single word that I wrote, memorized the key formulas, and worked through all the problems. You’re all set to ace your final exam, and you’ve earned it. Good for you! (Or maybe you just picked up the book and skipped to the end. That’s fine, too! This is a great place to get an overview of what this Calculus II stuff is all about.)

But still, you have this sneaking suspicion that you’re stuck in the middle of the forest and can’t see it because of all those darn trees. Forget the equations for a moment and spend five minutes looking over these top ten “Aha!” insights. When you understand them, you have a solid conceptual framework for Calculus II.

## *Integrating Means Finding the Area*

Finding the area of a polygon or circle is easy. Integration is all about finding the area of shapes with weird edges that are hard to work with. These edges may be the curves that result from polynomials, exponents, logarithms, trig functions, or inverse trig functions, or the products and compositions of these functions.

Integration gives you a concrete way to look at this question, known as the *area problem*. No matter how complicated integration gets, you can always understand what you’re working on in terms of this simple question: “How does what I’m doing help me find an area?”

See Chapter 1 for more about the relationship between integration and area.

## *When You Integrate, Area Means Signed Area*

In the real world, area is always positive. For example, there's no such thing as a piece of land that's  $-4$  square miles in area. This concept of area is called *unsigned area*.

But on the Cartesian graph in the context of integration, area is measured as *signed area*, with area below the  $x$ -axis considered to be *negative area*.

In this context, a  $2 \times 2$ -unit square below the  $x$ -axis is considered to be  $-4$  square units in signed area. Similarly, a  $2 \times 2$ -unit square that's divided in half by the  $x$ -axis is considered to have an area of 0.

The definite integral always produces the signed area between a curve and the  $x$ -axis, within the limits of integration. So if an application calls for the unsigned area, you need to measure the positive area and negative area separately, change the sign of the negative area, and add these two results together.

See Chapter 3 for more about signed area.

## *Integrating Is Just Fancy Addition*

To measure the area of an irregularly shaped polygon, a good first step is to cut it into smaller shapes that you know how to measure — for example, triangles and rectangles — and then add up the areas of these shapes.

Integration works on the same principle. It allows you to slice a shape into smaller shapes that approximate the area that you're trying to measure, and then add up the pieces. In fact, the integral sign  $\int$  itself is simply an elongated  $S$ , which stands for *sum*.

See Chapter 1 for more about how integration relates to addition.

## *Integration Uses Infinitely Many Infinitely Thin Slices*

Here's where integration differs from other methods of measuring area: Integration allows you to slice an area into infinitely many pieces, all of which are infinitely thin, and then add up these pieces to find the total area.

Or, to put a slightly more mathematical spin on it: The definite integral is the limit of the total area of all these slices as the number of slices approaches infinity and the thickness of each slice approaches 0.

This concept is also useful when you’re trying to find volume, as I show you in Chapter 10.

See Chapter 1 for more about how this concept of infinite slicing relates to integration.

## *Integration Contains a Slack Factor*

Math is a harsh mistress. A small error at the beginning of a problem often leads to a big mistake by the end.

So finding out that you can thin-slice an area in a bunch of different ways and still get the correct answer is refreshing. Some of these methods for thin-slicing include left rectangles, right rectangles, and the midpoint rectangles. I cover them all in Chapter 3.

This *slack factor*, as I call it, comes about because integration exploits an infinite sequence of successive approximations. Each approximation brings you closer to the answer that you’re seeking. So, no matter what route you take to get there, an infinite number of such approximations brings you to the answer.

See Chapter 3 for more about the distinction between approximating and evaluating integrals.

## *A Definite Integral Evaluates to a Number*

A definite integral represents the well-defined area of a shape on a graph. You can represent any such area as a number of square units, so the definite integral is a number.

See Chapter 3 for more about the definite integral.

## *An Indefinite Integral Evaluates to a Function*

An indefinite integral is a template that allows you to calculate an infinite number of related definite integrals by plugging in some parameters. In math, such a template is called a *function*.

The input values to an indefinite integral are the two limits of integration. Specifying these two values turns the indefinite integral into a definite integral, which then outputs a number representing an area.

But if you don't specify the limits of integration, you can still evaluate an indefinite integral as a function. The process of finding an indefinite integral turns an input function (for example,  $\cos x$ ) into an output function ( $\sin x + C$ ).

See Chapter 3 for more about the indefinite integral and Part II for a variety of techniques for evaluating indefinite integrals.

## *Integration Is Inverse Differentiation*

Integration and differentiation are inverse operations: Either of these operations undoes the other (up to a constant  $C$ ). Another way to say this is that integration is *anti-differentiation*.

Here's an example of how differentiation undoes integration:

$$\int 5x^3 dx = \frac{5}{4}x^4 + C$$
$$\frac{d}{dx} \frac{5}{4}x^4 + C = 5x^3$$

As you can see, integrating a function and then differentiating the result produces the function that you started with.

Now, here's an example of how integration undoes differentiation:

$$\frac{d}{dx} \sin x = \cos x$$
$$\int \cos x dx = \sin x + C$$

As you can see, differentiating a function and then integrating the result produces the function that you started with, plus a constant  $C$ .

See Part II for more on how this inverse relationship between integration and differentiation provides a variety of clever methods for integrating complicated functions.

## Every Infinite Series Has Two Related Sequences

Every infinite series has two related sequences that are important for understanding how that series works: its defining sequence and its sequence of partial sums.

The *defining sequence* of a series is simply the sequence that defines the series in the first place. For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

has the defining sequence

$$\left\{ \frac{1}{n} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}$$

Notice that the same function — in this case,  $\frac{1}{n}$  — appears in the shorter notation for both the series and its defining sequence.

The *sequence of partial sums* of a series is the sequence that results when you successively add a finite number of terms. For example, the previous series has the following sequence of partial sums:

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \dots$$

Notice that a series may diverge while its defining sequence converges, as in this example. However, a series and its sequence of partial sums always converge or diverge together. In fact, the definition of convergence for a series is based upon the behavior of its sequence of partial sums (see the next section for more on convergence and divergence).

See Part IV for more about infinite series.

## Every Infinite Series Either Converges or Diverges

Every infinite series either converges or diverges, with no exceptions.

A series *converges* when it evaluates to (equals) a real number. For example:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$$

On the other hand, a series *diverges* when it doesn't evaluate to a real number. Divergence can happen in two different ways. The more common type of divergence is when the series explodes to  $\infty$  or  $-\infty$ . For example:

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \dots$$

Clearly, this series doesn't add up to a real number — it just keeps getting bigger and bigger forever.

Another type of divergence occurs when a series bounces forever among two or more values. This happens only when a series is *alternating* (see Chapter 12 for more on alternating series). For example:

$$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - \dots$$

The sequence of partial sums (see the previous section) for this series alternates forever between  $-1$  and  $0$ , never settling in to a single value, so the series diverges.

See Part IV for more about infinite series.

## Chapter 17

# Ten Tips to Take to the Test

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### *In This Chapter*

- ▶ Staying calm when the test is passed out
  - ▶ Remembering those  $dx$ s and  $+Cs$
  - ▶ Getting unstuck
  - ▶ Checking for mistakes
- 

**I**'ve never met anyone who loved taking a math test. The pressure is on, the time is short, and that formula that you can't quite remember is out of reach. Unfortunately, exams are a part of every student's life. Here are my top ten suggestions to make test-taking just a little bit easier.

## *Breathe*

This is always good advice — after all, where would you be if you weren't breathing? Well, not a very nice place at all.

A lot of what you may feel when facing a test — for example, butterflies in your stomach, sweaty palms, or trembling — is simply a physical reaction to stress that's caused by adrenalin. Your body is preparing you for a fight-or-flight response, but with a test, you have nothing to fight and nowhere to fly.

A little deep breathing is a simple physical exertion that can help dissipate the adrenalin and calm you down. So, while you're waiting for the professor to arrive and hand out the exams, take a few deep breaths in and out. If you like, picture serenity and deep knowledge of all things mathematical entering your body on the in-breath, and all the bad stuff exiting on the out-breath.

## *Start by Reading through the Exam*

When you receive your exam, take a minute to read through it so that you know what you're up against. This practice starts your brain working (consciously or not) on the problems.

While you're reading, see whether you can find a problem that looks easier to you than the others (see the next section).

## *Solve the Easiest Problem First*

After the initial read-through, turn to the page with the easiest problem and solve it. This warm-up gets your brain working and usually reduces your anxiety.

## *Don't Forget to Write $dx$ and $+ C$*

Remember to include those pesky little  $dx$ s in every integration statement. They need to be there, and some professors take it very personally when you don't include them. You have absolutely no reason to lose points over something so trivial.

And don't forget that the solution to every *indefinite* integral ends with  $+ C$  (or whatever constant you choose). No exceptions! As with the  $dx$ s, omitting this constant can cost you points on an exam, so get in the habit of including it.

## *Take the Easy Way Out Whenever Possible*

In Chapters 4 through 8, I introduce the integration techniques in the order of difficulty. Before you jump in to your calculation, take a moment to walk through all the methods you know, from easiest to hardest.

Always check first to see whether you know a simple formula: For example,

$\int \frac{1}{x\sqrt{x^2-1}} dx$  may cause you to panic until you remember that the answer

is simply  $\operatorname{arcsec} x + C$ . If no formula exists, think through whether a simple variable substitution is possible. What about integration by parts? Your last resorts are always trig substitution and integration with partial fractions.

When you're working on solving area problems, stay open to the possibility that calculus may not be necessary. For example, you don't need calculus to find the area under a straight line or semicircle. So, before you start integrating, step back for a moment to see whether you can spot an easier way.

## *If You Get Stuck, Scribble*

When you look at a problem and you just don't know which way to go, grab a piece of scratch paper and scribble everything you can think of, without trying to make sense of it.

Use algebra, trig identities, and variable substitutions of all kinds. Write series in both sigma notation and expanded notation. Draw pictures and graphs. Write it all down, even the ideas that seem worthless.

You may find that this process jogs your brain. Even copying the problem — equations, graphs, and all — can sometimes help you to notice something important that you missed in your first reading of the question.

## *If You Really Get Stuck, Move On*

I see no sense in beating your head against a brick wall, unless you like getting brick dust in your hair. Likewise, I see no sense in spending the whole exam frozen in front of one problem.

So, after you scribble and scribble some more (see the previous section) and you're still getting nowhere with a problem, move on. You may as well make the most of the time you're given by solving the problems that you can solve. What's more, many problems seem easier on the second try. And working on other areas of the test may remind you of some important information that you'd forgotten.

## *Check Your Answers*

Toward the end of the test, especially if you're stuck, take a moment to check over some of the problems that you already completed. Does what you've written still make sense? If you see any missing  $dx$ s or  $+ C$ s, fill them in. Make sure

you didn't drop any minus signs. Most important, do a reality check of your answer compared with the original problem to see whether it makes sense.

For example, suppose that you're integrating to find an area someplace inside a  $2 \times 2$  region on a graph, and your answer is 7 trillion. Obviously, something went wrong. If you have time to find out what happened, trace back over your steps.

Although fixing a problem on an exam can be tedious, it usually takes less time than starting (and maybe not finishing) a brand-new problem from scratch.

## *If an Answer Doesn't Make Sense, Acknowledge It*

Suppose that you're integrating to find an area someplace inside a  $2 \times 2$  region on a graph, and your answer is 7 trillion. Obviously, something went wrong. If you don't have time to find out what happened, write a note to the professor acknowledging the problem.

Writing such a note lets your professor know that your conceptual understanding of the problem is okay — that is, you get the idea that integration means area. So, if it turns out that your calculation got messed up because of a minor mistake like a lost decimal point, you'll probably lose only a couple of points.

## *Repeat the Mantra "I'm Doing My Best," and Then Do Your Best*

All you can do is your best, and even the best math student occasionally forgets a formula or stares at an exam question and goes "Huh?"

When these moments arrive, and they will, you can do a shame spiral about all the studying you shoulda, coulda, woulda done. But there's no cheese down that tunnel. You can also drop your pencil, leave the room, quit school, fly to Tibet, and join a monastery. This plan of action is also not recommended unless you're fluent in Tibetan (which is way harder than calculus!).

Instead, breathe (see the section on breathing earlier in this chapter) and gently remind yourself "I'm doing my best." And then do your best with what you have. Perfection is not of this world, but if you can cut yourself a bit of slack when you're under pressure, you'll probably end up doing better than you would've otherwise.