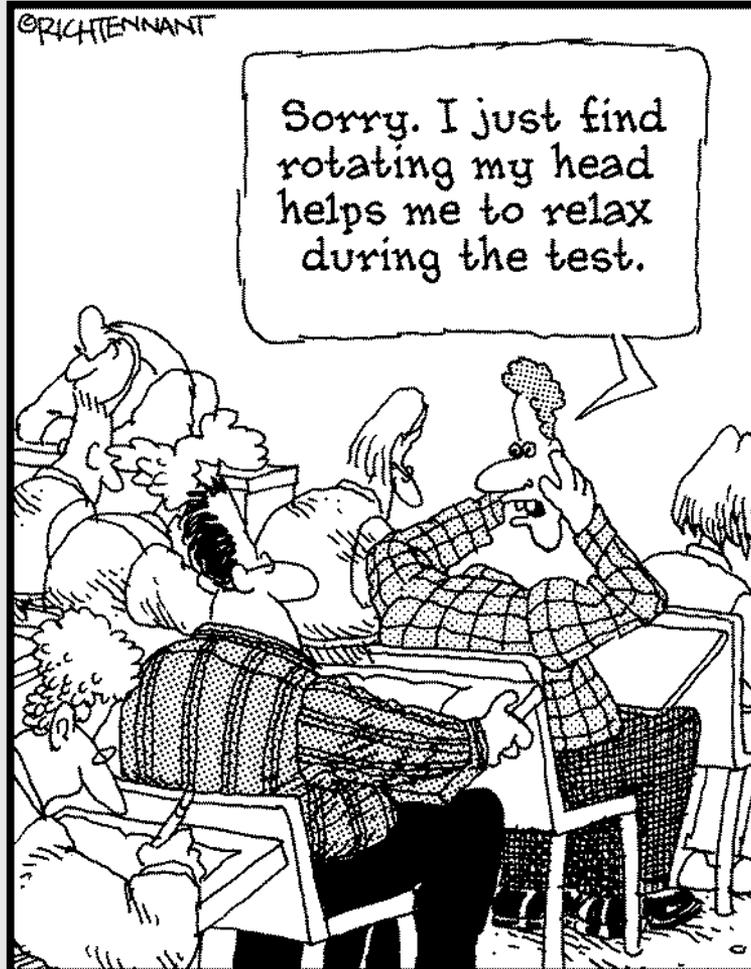


Part V

Advanced Topics

The 5th Wave

By Rich Tennant



In this part . . .



ou get a glimpse of what lies beyond Calculus II. I give you an overview of the next two semesters of math: Calculus III (the study of calculus in three or more dimensions) and Differential Equations (equations with derivatives mixed in as variables).

Chapter 14

Multivariable Calculus

In This Chapter

- ▶ Visualizing vectors
 - ▶ Making the leap from two to three dimensions
 - ▶ Understanding cylindrical and spherical coordinates
 - ▶ Using partial derivatives
 - ▶ Sorting out and solving multiple integrals
-

Space, as Captain Kirk says during the opening credits of the television series *Star Trek*, is the final frontier. Multivariable calculus (also known as Calculus III) focuses on techniques for doing calculus in space — that is, in three dimensions.

Mathematicians have a variety of terms for three dimensions: 3-D, 3-space, and \mathbb{R}^3 are the most common. Whatever you call it, adding a dimension makes multivariable calculus more interesting and useful, but also a bit more tricky than single variable calculus.

In this chapter, I give you a quick introduction to multivariable calculus, touching on the highlights usually taught in a Calculus III class. First, I show you how vectors provide a method for linking a value with a direction. Next, I introduce you to three different 3-D coordinate systems: 3-D Cartesian coordinates, cylindrical coordinates, and spherical coordinates.

Building on this understanding of 3-D, I discuss functions of more than one variable, focusing on the function of two variables $z = f(x, y)$. With an understanding of multivariable functions, I proceed to introduce you to the two most important concepts in multivariable calculus: partial derivatives and multiple integrals.

By the end of this chapter, you'll have a great platform from which to begin Calculus III.

Visualizing Vectors

Vectors are used to link a real number (called a *scalar*) with a direction on the plane or in space. They're useful for navigation, where knowing what direction you're sailing or flying in is important. Vectors also get a lot of play in physics, where forces that push and pull are also directional. And, as you may have guessed, Calculus III is chock full of vectors.

In this section, I introduce you to this important concept. Although I keep this discussion in two dimensions, vectors are commonly used in three dimensions as well.

Understanding vector basics

A simple way to think of a vector is as an arrow that has both length and direction. By convention, a vector starts at the origin of the Cartesian plane (0, 0) and extends a certain length in some direction. Figure 14-1 shows a variety of vectors.

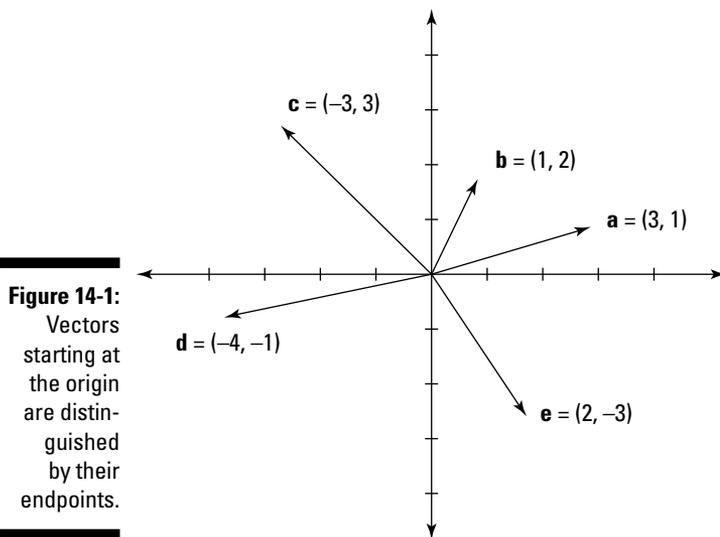


Figure 14-1:
Vectors starting at the origin are distinguished by their endpoints.

As you can see, when a vector begins at the origin, its two *components* (its *x* and *y* values) correspond to the Cartesian coordinates of its endpoint. For example, the vector that begins at (0, 0) and ends at (3, 1) is distinguished as the vector $\langle 3, 1 \rangle$.



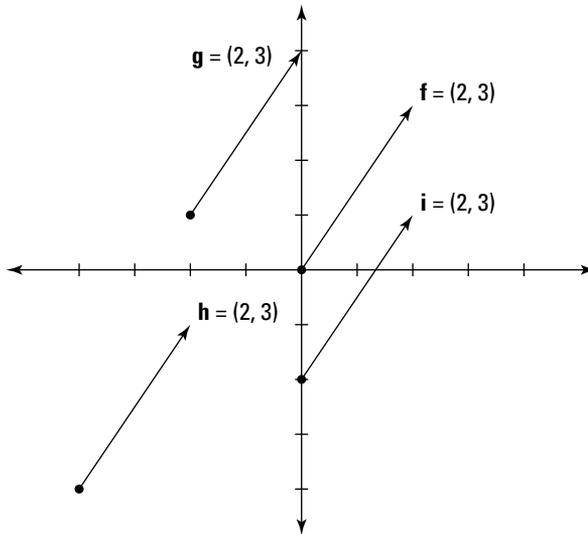
Don't confuse a vector $\langle x, y \rangle$ with its corresponding Cartesian pair (x, y) , which is just a point.



By convention, vectors are labeled in books with boldfaced lowercase letters: **a**, **b**, **c**, and so forth (see Figure 14-1). But when you're working with vectors on paper, most teachers are happy to see you replace the boldface with a little line or arrow over the letter.

Displacing a vector from the origin doesn't change its value. For example, Figure 14-2 shows the vector $\langle 2, 3 \rangle$ with a variety of starting points.

Figure 14-2:
All vectors with the same length and direction are equivalent, regardless of where they start.



Calculate the coordinates of a vector starting at (x_1, y_1) and ending at (x_2, y_2) as $\langle x_2 - x_1, y_2 - y_1 \rangle$. For example, here's how to calculate the coordinates of vectors **g**, **h**, and **i** in Figure 14-2:

$$\mathbf{g} = \langle 0 - (-2), 4 - 1 \rangle = \langle 2, 3 \rangle$$

$$\mathbf{h} = \langle -2 - (-4), -1 - (-4) \rangle = \langle 2, 3 \rangle$$

$$\mathbf{i} = \langle 2 - 0, 1 - (-2) \rangle = \langle 2, 3 \rangle$$

As you can see, regardless of the starting and ending points, these vectors are all equivalent to each other and to **f**.

Distinguishing vectors and scalars

Just as Eskimos have tons of words for snow and Italians have even more words for pasta, mathematicians have bunches of words for real numbers. When you began algebra, you found out quickly that a *constant* or a *coefficient* was just a real number in a specific context.

Similarly, when you're discussing vectors and want to refer to a real number, use the word *scalar*. Only the name is different — deep in its heart, a scalar knows only too well that it's just a good old-fashioned real number.

Some types of vector calculations produce new vectors, while others result in scalars. As I introduce these calculations throughout the next section, I tell you whether to expect a vector or a scalar as a result.

Calculating with vectors

Vectors are commonly used to model forces such as wind, sea current, gravity, and electromagnetism. Vector calculations are essential for all sorts of problems where forces collide. In this section, I give you a taste of how some simple calculations with vectors are accomplished.

Calculating magnitude

The length of a vector is called its *magnitude*. The notation for absolute value ($| \ |$) is also used for the magnitude of a vector. For example, $|\mathbf{v}|$ refers to magnitude of the vector \mathbf{v} . (By the way, some textbooks represent magnitude with double bars $[\ | \]$ instead of single bars. Either way, the meaning is the same.)

What's in a name?

A scalar is just a fancy word for a real number. The name arises because a scalar *scales* a vector — that is, it changes the scale of a vector. For example, the real number 2 scales

the vector \mathbf{v} by a factor of 2, so that $2\mathbf{v}$ is twice as long as \mathbf{v} . You find out more about scalar multiplication in “Calculating with vectors.”

Calculate the magnitude of a vector $\mathbf{v} = \langle x, y \rangle$ by using a variation of the distance formula. This formula is itself a variation of the trusty Pythagorean theorem:

$$|\mathbf{v}| = \sqrt{x^2 + y^2}$$

For example, calculate the magnitude of the vector $\mathbf{n} = \langle 4, -3 \rangle$ as follows:

$$|\mathbf{n}| = \sqrt{4^2 + (-3)^2} = 5$$

As you can see, the use of the absolute value bars for the magnitude of vectors is appropriate: Magnitude, like all other distances, is always measured as a nonnegative value. The magnitude of a vector is the distance from the origin of a graph to its tip, just as the absolute value of a number is the distance from 0 on a number line to that number.



The magnitude of a vector is a *scalar*.

Scalar multiplication

Multiplying a vector by a scalar is called *scalar multiplication*. To perform scalar multiplication, multiply the scalar by each component of the vector. Here's how you multiply the vector $\mathbf{v} = \langle x, y \rangle$ by the scalar k :

$$k\mathbf{v} = k\langle x, y \rangle = \langle kx, ky \rangle$$

For example, here's how you multiply the vector $\mathbf{p} = \langle 3, 5 \rangle$ by the scalars 2, -4, and $\frac{1}{3}$:

$$2\mathbf{p} = 2\langle 3, 5 \rangle = \langle 6, 10 \rangle$$

$$-4\mathbf{p} = -4\langle 3, 5 \rangle = \langle -12, -20 \rangle$$

$$\frac{1}{3}\mathbf{p} = \frac{1}{3}\langle 3, 5 \rangle = \langle 1, \frac{5}{3} \rangle$$



When you multiply a vector by a scalar, the result is a *vector*.

Geometrically speaking, scalar multiplication achieves the following:

- ✓ Scalar multiplication by a positive number other than 1 changes the magnitude of the vector but not its direction.
- ✓ Scalar multiplication by -1 reverses its direction but doesn't change its magnitude.
- ✓ Scalar multiplication by any other negative number both reverses the direction of the vector and changes its magnitude.

Scalar multiplication can change the magnitude of a vector by either increasing it or decreasing it.

- ✓ Scalar multiplication by a number greater than 1 or less than -1 *increases* the magnitude of the vector.
- ✓ Scalar multiplication by a fraction between -1 and 1 *decreases* the magnitude of the vector.

For example, the vector $2\mathbf{p}$ is twice as long as \mathbf{p} , the vector $\frac{1}{2}\mathbf{p}$ is half as long as \mathbf{p} , and the vector $-\mathbf{p}$ is the same length as \mathbf{p} but extends in the opposite direction from the origin (as shown in Figure 14-3).

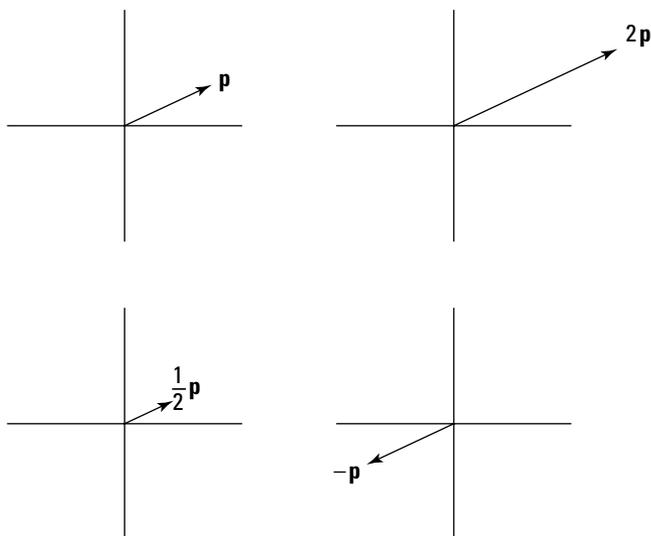


Figure 14-3:
Scalar multiplication of a vector changes its magnitude and/or its direction.

Finding the unit vector

Every vector has a corresponding *unit vector*, which has the same direction as that vector but a magnitude of 1. To find the unit vector \mathbf{u} of the vector $\mathbf{v} = \langle x, y \rangle$, divide that vector by its magnitude as follows:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

Note that this formula uses scalar multiplication, as I show you in the preceding section, because the numerator is a vector and the denominator is a scalar.



As you may guess from its name, the unit vector is a *vector*.

For example, to find the unit vector \mathbf{u} of the vector $\mathbf{q} = \langle -2, 1 \rangle$, first calculate its magnitude $|\mathbf{q}|$ as I show you earlier in this section:

$$|\mathbf{q}| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$$

Now, use the previous formula to calculate the unit vector:

$$\mathbf{u} = \frac{\langle -2, 1 \rangle}{\sqrt{5}} = \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

You can check that the magnitude of resulting vector \mathbf{u} really is 1 as follows:

$$\begin{aligned} |\mathbf{u}| &= \sqrt{\left(-\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} \\ &= \sqrt{\frac{4}{5} + \frac{1}{5}} = 1 \end{aligned}$$

Adding and subtracting vectors

Add and subtract vectors component by component, as follows:

$$\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$$

$$\langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle = \langle x_1 - x_2, y_1 - y_2 \rangle$$

For example, if $\mathbf{r} = \langle -1, 3 \rangle$ and $\mathbf{s} = \langle 4, 2 \rangle$, here's how to add and subtract these vectors:

$$\mathbf{r} + \mathbf{s} = \langle -1, 3 \rangle + \langle 4, 2 \rangle = \langle -1 + 4, 3 + 2 \rangle = \langle 3, 5 \rangle$$

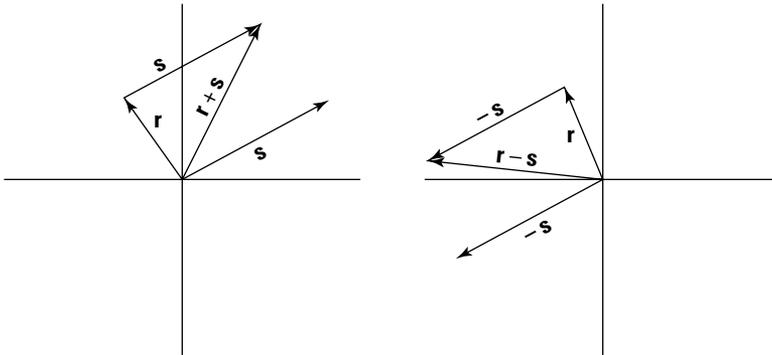
$$\mathbf{r} - \mathbf{s} = \langle -1, 3 \rangle - \langle 4, 2 \rangle = \langle -1 - 4, 3 - 2 \rangle = \langle -5, 1 \rangle$$



When you add or subtract two vectors, the result is a *vector*.

Geometrically speaking, the net effects of vector addition and subtraction are shown in Figure 14-4. In this example, the endpoint of $\mathbf{r} + \mathbf{s}$ is equivalent to the endpoint of \mathbf{s} when \mathbf{s} begins at the endpoint of \mathbf{r} . Similarly, the endpoint of $\mathbf{r} - \mathbf{s}$ is equivalent to the endpoint of $-\mathbf{s}$ — that is, $\langle -4, -2 \rangle$ — when $-\mathbf{s}$ begins at the endpoint of \mathbf{r} .

Figure 14-4:
Add and subtract vectors on the graph by beginning one vector at the endpoint of another vector.



Leaping to Another Dimension

Multivariable calculus is all about three (or more) dimensions. In this section, I show you three important systems for plotting points in 3-D.

Understanding 3-D Cartesian coordinates

The three-dimensional (3-D) Cartesian coordinate system (also called 3-D rectangular coordinates) is the natural extension of the 2-D Cartesian graph. The key difference is the addition of a third axis, the z -axis, extending perpendicularly through the origin.

Drawing a 3-D graph in two dimensions is kind of tricky. To get a better sense about how to think in 3-D, hold up Figure 14-5 where you can compare it with the interior corner of a room (not a round room!). Note the following:

- ✓ The x -axis corresponds to where the left-hand wall meets the floor.
- ✓ The y -axis corresponds to where the right-hand wall meets the floor.
- ✓ The z -axis corresponds to where the two walls meet.

Just as the 2-D Cartesian graph is divided into four quadrants, the 3-D graph is divided into eight *octants*. From your perspective as you look at the graph, you're standing inside the first octant, where all values of x , y , and z are positive.

Figure 14-5:
The first
octant of
the 3-D
Cartesian
coordinate
system.

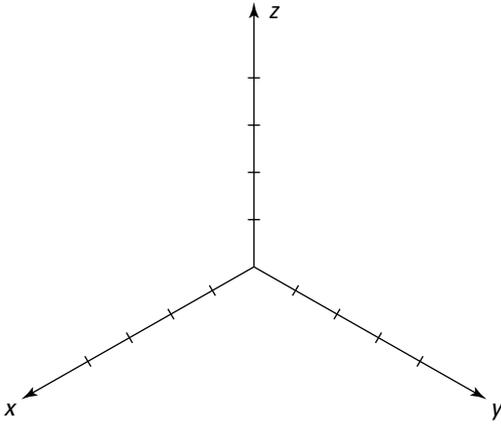
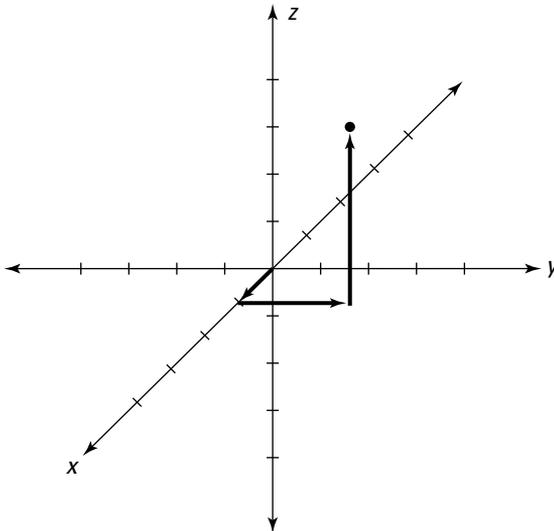


Figure 14-6 shows the complete 3-D Cartesian system with the point $(1, 2, 5)$ plotted. In similarity with regular Cartesian coordinates, you plot this point by counting 1 unit in the positive x direction, and then 2 units in the positive y direction, and finally 5 units in the positive z direction.

Figure 14-6:
Plotting the
point $(1, 2, 5)$
on the 3-D
Cartesian
coordinate
system.



Using alternative 3-D coordinate systems

In Chapter 2, I discuss polar coordinates, an alternative to the Cartesian graph. Polar coordinates are useful because they allow you to express and solve a variety of problems more easily than Cartesian coordinates.

In this section, I show you two alternatives to the 3-D Cartesian coordinate system: cylindrical coordinates and spherical coordinates. As with polar coordinates, both of these systems give you greater flexibility to solve a wider range of problems.

Cylindrical coordinates

You probably remember polar coordinates from Pre-Calculus or maybe even Calculus I. Like the Cartesian coordinate system, the polar coordinate system assigns a pairing of values to every point on the plane. Unlike the Cartesian coordinate system, however, these values aren't dependent upon two perpendicular axes (though these axes are often drawn in to make the graph more readable). The key axis is the *horizontal axis*, which corresponds to the positive x -axis in Cartesian coordinates.

While a Cartesian pair is of the form (x, y) , polar coordinates use (r, θ) . *Cylindrical coordinates* are simply polar coordinates with the addition of a vertical z -axis extending from the origin, as in 3-D Cartesian coordinates (see "Understanding 3-D Cartesian coordinates" earlier in this chapter). Every point in space is assigned a set of cylindrical coordinates of the form (r, θ, z) .

Here's what you need to know:

- ✓ The variable r measures the distance from the z -axis to that point.
- ✓ The variable θ measures angular distance from the horizontal axis. This angle is measured in radians rather than degrees, so that $2\pi = 360^\circ$. (See Chapter 2 for more about radians.)
- ✓ The variable z measures the distance from that point to the xy -plane.



When plotting cylindrical coordinates, plot the first coordinates (r and θ) just as you would for polar coordinates (see Chapter 2). Then plot the z -coordinate as you would for 3-D Cartesian coordinates.

Figure 14-7 shows you how to plot the point $(3, \frac{\pi}{2}, 2)$ in cylindrical coordinates:

1. **Count 3 units to the right of the origin on the horizontal axis (as you would when plotting polar coordinates).**

2. Travel counterclockwise along the arc of a circle until you reach the line drawn at a $\frac{\pi}{2}$ -angle from the horizontal axis (again, as with polar coordinates).
3. Count 2 units above the plane and plot your point there.

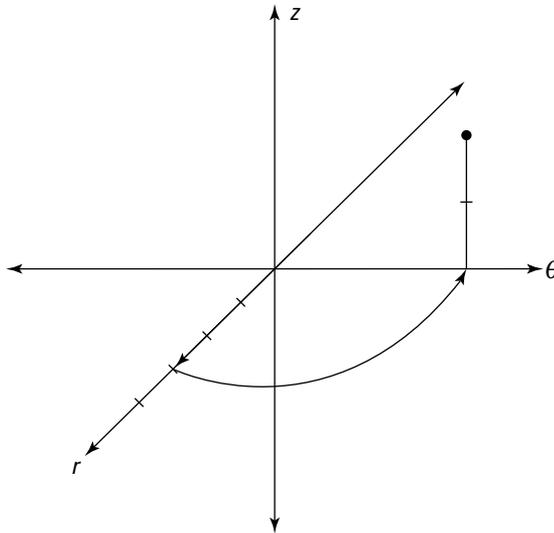


Figure 14-7:
Plotting
the point
 $(3, \frac{\pi}{2}, 2)$ in
cylindrical
coordinates.

Spherical coordinates

Spherical coordinates are used — with slight variation — to measure latitude, longitude, and altitude on the most important sphere of them all, the planet Earth.

Every point in space is assigned a set of spherical coordinates of the form (ρ, θ, ϕ) . In case you're not in a sorority or fraternity, ρ is the lowercase Greek letter *rho*, θ is the lowercase Greek letter *theta* (commonly used in math to represent an angle), ϕ is the lower-case Greek letter *phi*, which is commonly pronounced either “fee” or “fye” (but never “foe” or “fum”).

The coordinate ρ corresponds to altitude. On the Earth, altitude is measured as the distance above or below sea level. In spherical coordinates, however, altitude indicates how far in space a point is from the origin.

The coordinate θ corresponds to longitude: A measurement of angular distance from the horizontal axis.

The coordinate ϕ corresponds to latitude. On the Earth, latitude is measured as angular distance from the equator. In spherical coordinates, however, latitude is measured as the angular distance from the north pole.



Plotting ϕ can be tricky at first. To get a feel for it, picture a globe and imagine traveling up and down along a single longitude line. Notice that as you travel, your latitude keeps changing, so

- ✓ At the north pole, $\phi = 0$
- ✓ At the equator, $\phi = \frac{\pi}{2}$
- ✓ At the south pole, $\phi = \pi$



Some textbooks substitute the Greek letter ρ (*rho*) for r . Either way, the coordinate means the same thing: altitude, which is the distance of a point from the origin. In other textbooks, the order of the last two coordinates is changed around. Make sure that you know which convention your book uses.

Figure 14-8 shows you how to plot a point in spherical coordinates. For example, suppose that you want to plot the point $(4, \frac{\pi}{2}, \frac{3\pi}{4})$. Follow these steps to do that:

1. Count 4 units to the right of the origin on the horizontal axis.
2. Travel counterclockwise along the arc of a circle until you reach the line drawn at a $\frac{\pi}{2}$ -angle from the horizontal axis (again, as with polar coordinates).
3. Imagine a single longitude line arcing from the north pole of a sphere through the point on the equator where you are right now and onward to the south pole.
4. Travel down to the line of latitude at an angular distance of $\frac{3\pi}{4}$ from the north pole — that is, halfway between the equator and the south pole — and plot your point there.

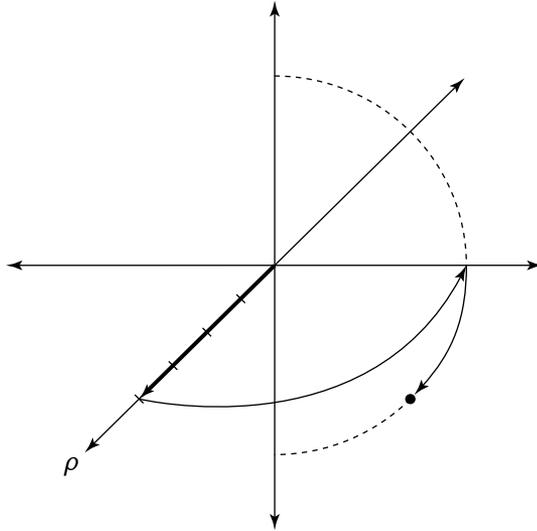


Figure 14-8:
Plotting
the point
 $(4, \frac{\pi}{2}, \frac{3\pi}{4})$
in spherical
coordinates.

Functions of Several Variables

You know from algebra that a function $y = f(x)$ is basically a mathematical machine for turning one number into another. The variable x is the input variable and y is the output variable so that every value of x gives you no more than one y value.

When you graph a curve, you can use the vertical-line test to make sure that it's a function: Any vertical line that intersects a function intersects it at exactly one point, as illustrated in Figure 14-9.

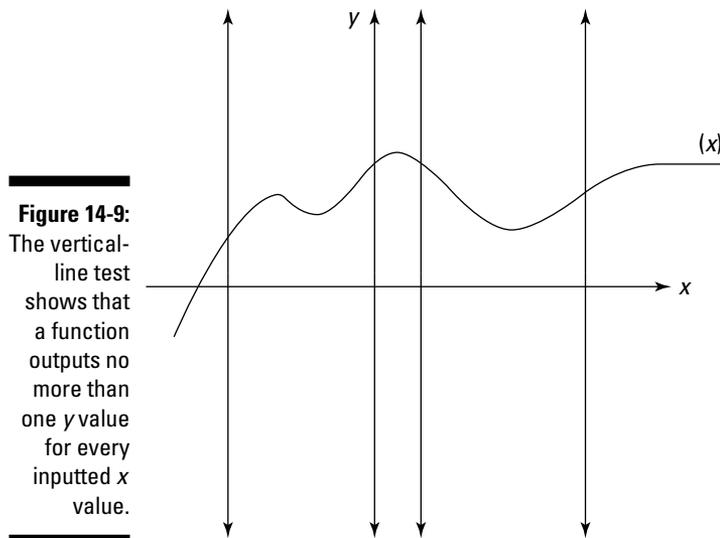
These concepts related to functions also carry over into functions of more than one variable. For example, here are some functions of two variables:

$$z = 2x + y + 5$$

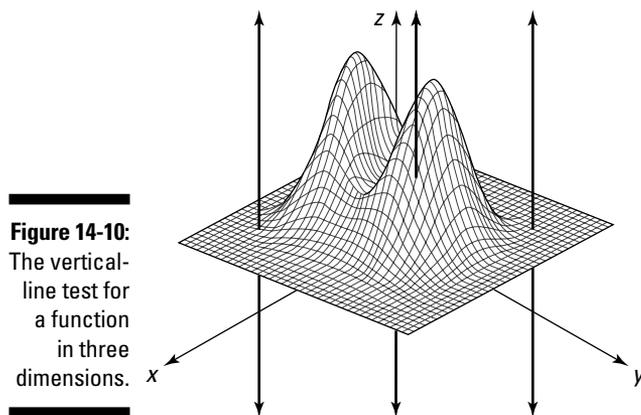
$$z = \sin x + y + \sqrt{y}$$

$$z = e^{xy} - \ln(1 + x^2y^2)$$

The general form for a function of two variables is $z = f(x, y)$. Every function of two variables takes a Cartesian pair (x, y) as its input and in turn outputs a z value. Looking at the three previous examples, plugging in the input $(0, 1)$ gives you an output of 6 for the first function, 2 for the second function, and 1 for the third.



A good way to visualize a function of two variables is as a surface floating over the xy -plane in a 3-D Cartesian graph. (True, this surface can cross the plane and continue below it — just as a function can cross the x -axis — but for now, just picture it floating.) Every point on this surface looms directly above exactly one point on the plane. That is, if you pass a vertical line through any point on the plane, it crosses the function at no more than one point. Figure 14-10 illustrates this concept.





The concept of a function can be extended to higher dimensions. For example, $w = f(x, y, z)$ is the basic form for a function of three variables. Because this function (as well as other functions of more than two variables) exists in more than three dimensions, it's a lot harder to picture. For now, just concentrate on making the leap from two to three dimensions — that is, from functions of one variable to functions of two variables. Most of the multivariable calculus that you study in Calculus III is in three dimensions. (Maybe it should be called Calculus 3-D.)

Partial Derivatives

Partial derivatives are the higher-dimensional equivalent of the derivatives that you know from Calculus I. Just as a derivative represents the slope of a function on the Cartesian plane, a partial derivative represents a similar concept of slope in higher dimensions. In this section, I clarify this notion of slope in three dimensions. I also show you how to calculate the partial derivatives of functions of two variables.

Measuring slope in three dimensions

In the earlier section “Functions of Several Variables,” I recommend that you visualize a function of two variables $z = f(x, y)$ as a surface floating over the xy -plane of a 3-D Cartesian graph. (See Figure 14-9 for a picture of a sample function.)

For example, take the function $z = y$, as shown in Figure 14-11. As you can see, this function looks a lot like the sloped roof of a house. Imagine yourself standing on this surface. When you walk parallel with the y -axis, your altitude either rises or falls. In other words, as the value of y changes, so does the value of z . But when you walk parallel with the x -axis, your altitude remains the same; changing the value of x has no effect on z .

So intuitively, you expect that the partial derivative $\frac{\partial z}{\partial y}$ — the slope in the direction of the y -axis — is 1. You also expect that the partial derivative $\frac{\partial z}{\partial x}$ — the slope in the direction of the x -axis — is 0. In the next section, I show you how to calculate partial derivatives to verify this result.

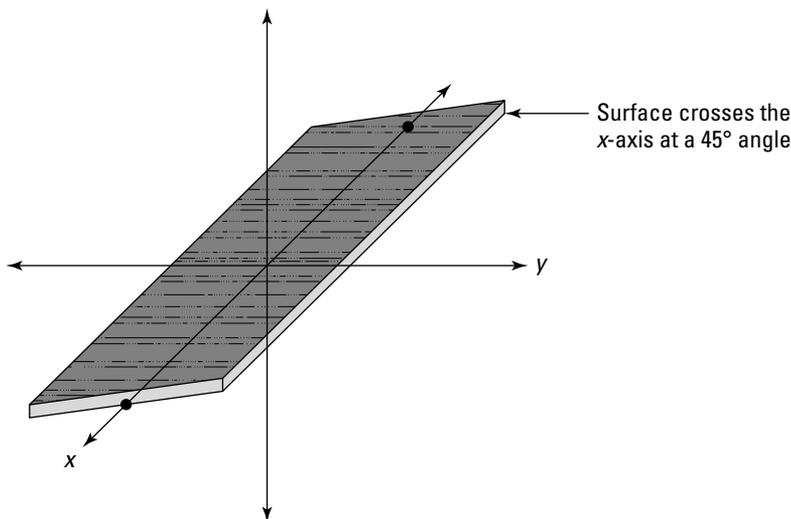


Figure 14-11:
The function
 $z = y$.

Evaluating partial derivatives

Evaluating partial derivatives isn't much more difficult than evaluating regular derivatives. Given a function $z(x, y)$, the two partial derivatives are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Here's how you calculate them:

- ✓ To calculate $\frac{\partial z}{\partial x}$, treat y as a constant and use x as your differentiation variable.
- ✓ To calculate $\frac{\partial z}{\partial y}$, treat x as a constant and use y as your differentiation variable.

For example, suppose that you're given the equation $z = 5x^2y^3$. To find $\frac{\partial z}{\partial x}$, treat y as if it were a constant — that is, treat the entire factor $5y^3$ as if it's one big coefficient — and differentiate x^2 :

$$\frac{\partial z}{\partial x} = 5y^3(2x) = 10xy^3$$

To find $\frac{\partial z}{\partial y}$, treat x as if it were a constant — that is, treat $5x^2$ as if it's the coefficient — and differentiate y^3 :

$$\frac{\partial z}{\partial y} = 5x^2(3y^2) = 15x^2y^2$$

As another example, suppose that you're given the equation $z = 2e^x \sin y + \ln x$. To find $\frac{\partial z}{\partial x}$, treat y as if it were a constant and differentiate by the variable x :

$$\frac{\partial z}{\partial x} = 2 \sin y (e^x) + \frac{1}{x} = 2e^x \sin y + \frac{1}{x}$$

To find $\frac{\partial z}{\partial y}$, treat x as if it were a constant and differentiate by the variable y :

$$\frac{\partial z}{\partial y} = 2 e^x \cos y$$

As you can see, when differentiating by y , the $\ln x$ term is treated as a constant and drops away completely.

Returning to the example from the previous section — the “sloped-roof” function $z = y$ — here are both partial derivatives of this function:

$$\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial y} = 1$$

As you can see, this calculation produces the predicted results.

Multiple Integrals

You already know that an integral allows you to measure area in two dimensions (see Chapter 1 if this concept is unclear). And as you probably know from solid geometry, the analog of area in three dimensions is volume.

Multiple integrals are the higher-dimensional equivalent of the good old-fashioned integrals that you discover in Calculus II. They allow you to measure volume in three dimensions (or more).

Most of the multiple integrals that you’ll ever have to solve come in two varieties: double integrals and triple integrals. In this section, I show you how to understand and calculate both of these types of multiple integrals.

Measuring volume under a surface

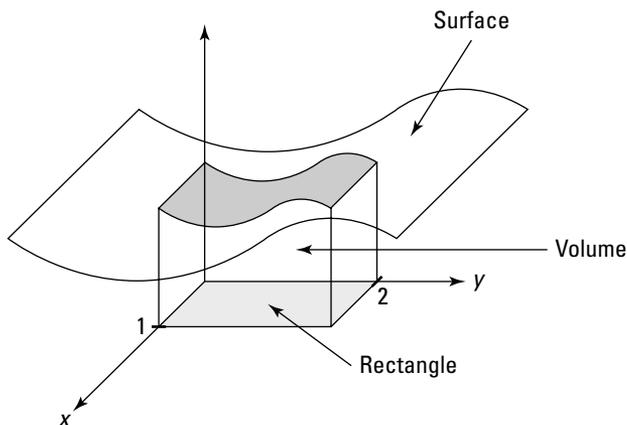
Definite integrals provide a reliable way to measure the signed area between a function and the x -axis as bounded by any two values of x . (I cover this in detail in Chapters 1 and 3.) Similarly, a *double integral* allows you to measure the signed volume between a function $z = f(x, y)$ and the xy -plane as bounded by any two values of x and any two values of y .

$$\int_0^2 \int_0^1 f(x, y) \, dx \, dy$$

To get a picture of this volume, look at Figure 14-12. The double integral measures the volume between $f(x, y)$ and the xy -plane as bounded by a rectangle. In this case, the rectangle is described by the four lines $x = 0$, $x = 1$, $y = 0$, and $y = 2$.

Figure 14-12:

A double integral allows you to measure the area under a surface as bounded by a rectangle.



A double integral is really an integral inside another integral. To help you see this, I bracket off the inner integral in the previous example:

$$\int_0^2 \left(\int_0^1 f(x, y) dx \right) dy$$

When you focus on the integral inside the brackets, you can see that the limits of integration 0 and 1 correspond with the dx — that is, $x = 0$ and $x = 1$. Similarly, the limits of integration 0 and 2 correspond with the dy — that is, $y = 0$ and $y = 2$.

Evaluating multiple integrals

Multiple integrals (double integrals, triple integrals, and so forth) are usually definite integrals, so evaluating them results in a real number. Evaluating multiple integrals is similar to evaluating nested functions: Work from the inside out.

Solving double integrals

Solve double integrals in two steps: First evaluate the inner integral, and then plug this solution into the outer integral and solve that. For example, suppose that you want to integrate the following double integral:

$$\int_1^3 \int_{4x}^{5x} \frac{y^2}{x} dy dx$$

To start out, place the inner integral in parentheses so that you can better see what you're working with:

$$\int_1^3 \left(\int_{4x}^{5x} \frac{y^2}{x} dy \right) dx$$

Now, focus on what's inside the parentheses. For the moment, you can ignore the rest. Your integration variable is y , so treat the variable x as if it were a constant, moving it outside the integral:

$$\frac{1}{x} \int_{4x}^{5x} y^2 dy$$

Notice that the limits of integration in this integral are functions of x . So the result of this definite integral will also be a function of x :

$$\begin{aligned} &= \frac{y^3}{3x} \Big|_{y=4x}^{y=5x} \\ &= \frac{5x^3}{3x} - \frac{4x^3}{3x} = \frac{125x^3}{3x} - \frac{64x^3}{3x} = \frac{61}{3} x^2 \end{aligned}$$

Now, plug this expression into the outer integral. In other words, substitute it for what's inside the parentheses:

$$\int_1^3 \frac{61}{3} x^2 dx$$

Evaluate this integral as usual:

$$\begin{aligned} &= \frac{61}{9} x^3 \Big|_{x=1}^{x=3} \\ &= \frac{61}{9} (3)^3 - \frac{61}{9} (1)^3 \\ &= 61 - \frac{61}{9} = \frac{488}{9} \end{aligned}$$

Making sense of triple integrals

Triple integrals look scary, but if you take them step by step, they're no more difficult than regular integrals. As with double integrals, start in the center and work your way out. For example:

$$\int_1^2 \int_z^{4z} \int_{\frac{z}{y}}^{\frac{2z}{y}} x^3 y^5 z dx dy dz$$

Begin by separating the two inner integrals:

$$\int_1^2 \left[\int_z^{4z} \left(\int_{\frac{z}{y}}^{\frac{2z}{y}} x^3 y^5 z dx \right) dy \right] dz$$

Your plan of attack is to evaluate the integral in the brackets first, and then the integral in the braces, and finally the outer integral. First things first:

$$\begin{aligned} & \int_{\frac{z}{y}}^{\frac{2z}{y}} x^3 y^5 z \, dx \\ &= \frac{1}{4} x^4 y^5 z \Big|_{x=z/y}^{x=2z/y} \\ &= \frac{1}{4} \left(\frac{2z}{y}\right)^4 y^5 z - \frac{1}{4} \left(\frac{z}{y}\right)^4 y^5 z \end{aligned}$$

Notice that plugging in values for x results in an expression in terms of y and z . Now simplify this expression to make it easier to work with:

$$= 4yz^5 - \frac{1}{4} yz^5 = \frac{15}{4} yz^5$$

Plug this solution back in to replace the bracketed integral:

$$\int_1^2 \left[\int_z^{4z} \frac{15}{4} yz^5 \, dy \right] dz$$

One integral down, two to go. This time, focus on the integral inside the braces. This time, the integration variable is y :

$$\begin{aligned} & \int_z^{4z} \frac{15}{4} yz^5 \, dy \\ &= \frac{15}{8} y^2 z^5 \Big|_{y=z}^{y=4z} \\ &= \frac{15}{8} (4z)^2 z^5 - \frac{15}{8} z^2 z^5 \end{aligned}$$

Again, simplify before proceeding:

$$= 30z^7 - \frac{15}{8} z^7 = \frac{225}{8} z^7$$

Plug this result back into the integral as follows:

$$\int_1^2 \frac{225}{8} z^7 \, dz$$

Now evaluate this integral:

$$\begin{aligned} &= \frac{225}{64} z^8 \Big|_{z=1}^{z=2} \\ &= \frac{225}{64} 2^8 - \frac{225}{64} 1^8 \\ &= \frac{57,600}{64} - \frac{225}{64} = \frac{57,375}{64} \end{aligned}$$

Chapter 15

What's so Different about Differential Equations?

In This Chapter

- ▶ Classifying different types of differential equations (DEs)
 - ▶ Understanding the connection between DEs and integrals
 - ▶ Checking a proposed solution to a DE
 - ▶ Seeing how DEs arise in the physical world
 - ▶ Using a variety of methods to solve DEs
-

The very mention of differential equations (DEs for short) strikes a spicy combination of awe, horror, and utter confusion into nonmathematical minds. Even intrepid calculus students have been known to consider a career in art history when these untamed beasts come into focus on the radar screen. Just what are differential equations? Where do they come from? Why are they necessary? And how in the world do you solve them?

In this chapter, I answer these questions and give you some familiarity with DEs. I show you how to identify the basic types of DEs so that if you're ever at a math department cocktail party (lucky you!), you won't feel completely adrift. I relate DEs to the integrals that you now understand so well. I show you how to build your own DEs so that you'll always have a hobby to pass the time, and I also show you how to check DE solutions. In addition, you discover how DEs arise in physics. Finally, I show you a few simple methods for solving some basic differential equations.

Basics of Differential Equations

In a nutshell, a *differential equation* is any equation that includes at least one derivative. For example:

$$\frac{dy}{dx} = \sin x e^y$$

$$\frac{d^2 y}{dx^2} + 10 \frac{dy}{dx} + 9y = 0$$

$$\frac{d^4 y}{dx^4} + \frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = \cos x$$

Solving a differential equation means finding the value of the dependent variable in terms of the independent variable. Throughout this chapter, I use y as the dependent variable, so the goal in each problem is to solve for y in terms of x .

In this section, I show you how to classify DEs. I also show you how to build DEs and check the solution to a DE.

Classifying DEs

As with other equations that you've encountered, differential equations come in many varieties. And different varieties of DEs can be solved by using different methods. In this section, I show you some important ways to classify DEs.

Ordinary and partial differential equations

An *ordinary differential equation* (ODE) has only derivatives of one variable — that is, it has no partial derivatives (flip to Chapter 14 for more on partial differentiation). Here are a few examples of ODEs:

$$\frac{dy}{dx} = x \sin(x^2) \cos y$$

$$\frac{dy}{dx} = y \csc x + e^x$$

$$\frac{d^2 y}{dx^2} + 4xy \frac{dy}{dx} + 5y = 0$$

In contrast, a *partial differential equation* (PDE) has at least one partial derivative. Here are a few examples of PDEs:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = e^{x-y}$$

$$3 \frac{\partial u}{\partial x^2} + 7 \frac{\partial u}{\partial x \partial y} + 6 \frac{\partial u}{\partial y^2} = 0$$

$$\frac{\partial v}{\partial t} - k \frac{\partial v}{\partial x^2}$$

Ordinary differential equations are usually the topic of a typical Differential Equations class in college. They're a step or two beyond what you're used to working with, but many students actually find Differential Equations an easier course than Calculus II (generally considered the most difficult class in the calculus series). However, ODEs are limited in how well they can actually express physical reality.

The real quarry is partial differential equations. A lot of physics gets done with these little gems. Unfortunately, solving PDEs is one giant leap forward in math from what the average calculus student is used to. Delving into the kind of math that makes PDEs come alive is typically reserved for graduate school.

Order of DEs

Differential equations are further classified according to their *order*. This classification is similar to the classification of polynomial equations by degree (see Chapter 2 for more on polynomials).

First-order ODEs contain only first derivatives. For example:

$$\begin{aligned}\frac{dy}{dx} &= ye^x \\ 3\frac{dy}{dx} &= \sin y + 2e^{2x} \\ \ln xy \frac{dy}{dx} &= 2x^2 + y - \tan x\end{aligned}$$

Higher-order ODEs are classified, as polynomials are, by the *greatest* order of their derivatives. Here are examples of second-, third-, and fourth-order ODEs:

Second-order ODE: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 10y = e^x$

Third-order ODE: $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + y = 0$

Fourth-order ODE: $x^2\frac{d^4y}{dx^4} + \cos y = x$

As with polynomials, generally speaking, a higher-order DE is more difficult to solve than one of lower order.

Linear DEs

What constitutes a linear differential equation depends slightly on who you ask. For practical purposes, a linear first-order DE fits into the following form:

$$\frac{dy}{dx} + a(x)y = b(x)$$

where $a(x)$ and $b(x)$ are functions of x . Here are a few examples of linear first-order DEs:

$$\frac{dy}{dx} + y = x$$

$$\frac{dy}{dx} + 4xy = -\ln x$$

$$\frac{dy}{dx} - y \sin x = e^x$$

Linear DEs can often be solved, or at least simplified, using an *integrating factor*. I show you how to do this later in this chapter.

A linear second-degree DE fits into the following form:

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

where a , b , and c are all constants. Here are some examples:

$$\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 4y = 0$$

$$2 \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6y = 0$$

$$\frac{d^2 y}{dx^2} = 0$$

Note that the constant a can always be reduced to 1, resulting in adjustments to the other two coefficients. Linear second-degree DEs are usually an important topic in a college-level course in differential equations. Solving them requires knowledge of matrices and complex numbers that is beyond the scope of this book.

Looking more closely at DEs

You don't have to play professional baseball to enjoy baseball. Instead, you can enjoy the game from the bleachers or, if you prefer, from a nice cushy chair in front of the TV. Similarly, you don't have to get too deep into differential equations to gain a general understanding of how they work. In this section, I give you front-row seats to the game of differential equations.

How every integral is a DE

The integral is a particular example of a more general type of equation — the differential equation. To see how this is so, suppose that you're working with this nice little integral:

$$y = \int \cos x \, dx$$

Differentiating both sides turns it into a DE:

$$\frac{dy}{dx} = \cos x$$

Of course, you know how to solve this DE by thinking of it as an integral:

$$y = \sin x + C$$

So in general, when a DE is of the form

$$\frac{dy}{dx} = f(x)$$

with $f(x)$ an arbitrary function of x , you can express that DE as an integral and solve it by integrating.

Why building DEs is easier than solving them

The reason that the DE in the last section is so simple to solve is that the derivative is isolated on one side of the equation. DEs attain a new level of difficulty when the derivative isn't isolated.

A good analogy can be made in lower math, when you make the jump from arithmetic to algebra. For example, here's an arithmetic problem:

$$x = 20 - (4^2 + 3)$$

Even though this is technically an algebra problem, you can solve it without algebra because x is isolated at the start of the problem. However, the ball-game changes when x becomes more enmeshed in the equation. For example:

$$2x^3 - x^2 + 5x - 17 = 0$$

Arithmetic isn't strong enough for this problem, so algebra takes over. Similarly, when derivatives get entangled into the fabric of an equation — as in most of the DEs I show you earlier in “Classifying DEs” — integrating is no longer effective and the search for new methods begins.

Although solving DEs is often tricky, building them is easy. For example, suppose that you start with this simple quadratic equation:

$$y = 3x^2 + 4x - 5$$

Now, find the first and second derivatives:

$$\frac{dy}{dx} = 6x + 4$$

$$\frac{d^2y}{dx^2} = 6$$

Adding up the left and right sides of all three equations gives you the following differential equation:

$$y + \frac{dy}{dx} + \frac{d^2y}{dx^2} = 3x^2 + 10x + 5$$

Because you built the equation yourself, you know what y equals. But if you hand this equation off to some other students, they probably wouldn't be able to guess how you built it, so they would have to do some work to solve it for y . For example, here's another DE:

$$y + \frac{dy}{dx} + \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} = 0$$

This equation probably looks difficult because you don't have much information. And yet, after I tell you the solution, it appears simple:

$$\begin{aligned} y &= \sin x & \frac{dy}{dx} &= \cos x \\ \frac{d^2y}{dx^2} &= -\sin x & \frac{d^3y}{dx^3} &= -\cos x \end{aligned}$$

But even after you have the solution, how do you know whether it's the *only* solution? For starters, $y = -\sin x$, $y = \cos x$, and $y = -\cos x$ are all solutions. Do other solutions exist? How do you find them? And how do you know when you have them all?

Another difficulty arises when y itself becomes tangled up in the equation. For example, how do you solve this equation for y ?

$$\frac{dy}{dx} = \frac{\sin x}{e^y}$$

As you can see, differential equations contain treacherous subtleties that you don't find in basic calculus.

Checking DE solutions

Even if you don't know how to find a solution to a differential equation, you can always check whether a proposed solution works. This is simply a matter of plugging the proposed value of the dependent variable — I use y throughout this chapter — into both sides of the equation to see whether equality is maintained.

For example, here's a DE:

$$\frac{dy}{dx} = 3y + 4e^{3x} \cos x$$

You may not have a clue how to begin solving this DE, but imagine that an angel lands on your pen and offers you this solution:

$$y = 4e^{3x} \sin x$$

You can check to see whether this angel really knows math by plugging in this value of y as follows:

$$\begin{aligned}\frac{dy}{dx} &= 3y + 4e^{3x} \cos x \\ \frac{d}{dx} 4e^{3x} \sin x &= 3(4e^{3x} \sin x) + 4e^{3x} \cos x \\ 4(3e^{3x} \sin x + e^{3x} \cos x) &= 12 e^{3x} \sin x + 4e^{3x} \cos x \\ = 12 e^{3x} \sin x + 4e^{3x} \cos x &= 12 e^{3x} \sin x + 4e^{3x} \cos x\end{aligned}$$

Because the left and right sides of the equation are equal, the angel's solution checks out.

Solving Differential Equations

In this section, I show you how to solve a few types of DEs. First, you solve everybody's favorite DE, the *separable equation*. Next, you put this understanding to work to solve an *initial-value problem* (IVP). Finally, I show you how to solve a linear first-order DE by using an integrating factor.

Solving separable equations

Differential equations become harder to solve the more entangled they become. In certain cases, however, an equation that looks all tangled up is actually easy to tease apart. Equations of this kind are called *separable equations* (or *autonomous equations*), and they fit into the following form:

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

Separable equations are relatively easy to solve. For example, suppose that you want to solve the following problem:

$$\frac{dy}{dx} = \frac{\sin x}{e^y}$$

You can think of the symbol $\frac{dy}{dx}$ as a fraction and isolate the x and y terms of this equation on opposite sides of the equal sign:

$$e^y dy = \sin x dx$$

Now, integrate both sides:

$$\begin{aligned}\int e^y dy &= \int \sin x dx \\ e^y + C_1 &= -\cos x + C_2\end{aligned}$$



In an important sense, the previous step is questionable because the variable of integration is different on each side of the equal sign. You may think “No problem, it’s all integration!” But imagine if you tried to divide one side of an equation by 2 and the other by 3, and then laughed it off with “It’s all division!” Clearly, you’d have a problem. The good news, however, is that for technical reasons beyond the scope of this book, integrating both sides by different variables actually produces the correct answer.

C_1 and C_2 are both constants, so you can use the equation $C = C_2 - C_1$ to simplify the equation:

$$e^y = -\cos x + C$$

Next, use a natural log to undo the exponent, and then simplify:

$$\ln e^y = \ln(-\cos x + C)$$

$$y = \ln(-\cos x + C)$$

To check this solution, substitute this value for y into both sides of the original equation:

$$\frac{dy}{dx} = \frac{\sin x}{e^y}$$

$$\frac{d}{dx} \ln(-\cos x + C) = \frac{\sin x}{e^{\ln(-\cos x + C)}}$$

$$\frac{d}{dx} \ln(-\cos x + C) = \frac{\sin x}{-\cos x + C}$$

$$\frac{1}{-\cos x + C} \cdot \sin x = \frac{\sin x}{-\cos x + C}$$

$$\frac{\sin x}{-\cos x + C} = \frac{\sin x}{-\cos x + C}$$

Solving initial-value problems (IVPs)

In Chapter 3, I show you that the definite integral is a particular example of a whole family of indefinite integrals. In a similar way, an *initial-value problem* (IVP) is a particular example of a solution to a differential equation. Every IVP gives you extra information — called an *initial value* — that allows you to use the *general solution* to a DE to obtain a *particular solution*.

For example, here's an initial-value problem:

$$\frac{dy}{dx} = y \sec^2 x \quad y(0) = 5$$

This problem includes not only a DE, but also an additional equation. To understand what this equation tells you, remember that y is a dependent variable, a function of x . So, the notation $y(0) = 5$ means “when $x = 0$, $y = 5$.” You see how this information comes into play as I continue with this example.

To solve an IVP, you first have to solve the DE. Do this by finding its general solution without worrying about the initial value. Fortunately, this DE is a separable equation, which you know how to solve from the last section:

$$\frac{1}{y} dy = \sec^2 x dx$$

Integrate both sides:

$$\int \frac{1}{y} dy = \int \sec^2 x dx$$

$$\ln y = \tan x + C$$

In this last step, I use C to consolidate the constants of integration from both sides of the equation into a single constant C . (If this doesn't make sense, I explain why in “Solving separable equations” earlier in this chapter.) Next, I undo the natural log by using e :

$$e^{\ln y} = e^{\tan x + C}$$

$$y = e^{\tan x} \cdot e^C$$

Because e^C is a constant, this equation can be further simplified by using the substitution $D = e^C$:

$$y = De^{\tan x}$$

Before moving on, check to make sure that this solution is correct by substituting this value of y into both sides of the original equation:

$$\frac{dy}{dx} = y \sec^2 x$$

$$\frac{d}{dx} De^{\tan x} = De^{\tan x} \sec^2 x$$

$$De^{\tan x} \sec^2 x = De^{\tan x} \sec^2 x$$

This checks out, so $y = De^{\tan x}$ is, indeed, the *general solution* to the DE. To solve the initial-value problem, however, I need to find the specific value of the variable D by using the additional information I have: When $x = 0$, $y = 5$. Plugging both of these values into the equation makes it possible to solve for D :

$$5 = De^{\tan 0}$$

$$5 = De^0$$

$$5 = D$$

Now, plug this value of D back into the general solution of the problem to get the IVP solution:

$$y = 5e^{\tan x}$$

This solution satisfies not only the differential equation $\frac{dy}{dx} = y \sec^2 x$ but also the initial value $y(0) = 5$.

Using an integrating factor

As I mention earlier in this chapter, in “Classifying DEs,” a linear first-order equation takes the following form:

$$\frac{dy}{dx} + a(x)y = b(x)$$

A clever method for solving DEs in this form involves multiplying the entire equation by an *integrating factor*. Follow these steps:

- 1. Calculate the integrating factor.**
- 2. Multiply the DE by this integrating factor.**
- 3. Restate the left side of the equation as a single derivative.**
- 4. Integrate both sides of the equation and solve for y .**

Don't worry if these steps don't mean much to you. In the upcoming sections, I show you what an integrating factor is and how to use it to solve linear first-order DEs.

Getting very lucky

To help you understand how multiplying by an integrating factor works, I set up an equation that practically solves itself — that is, if you know what to do:

$$\frac{dy}{dx} + \frac{2y}{x} = 0$$

Notice that this is a linear first-degree DE, with $a(x) = \frac{2}{x}$ and $b(x) = 0$. I now tweak this equation by multiplying every term by x^2 (you see why shortly):

$$\frac{dy}{dx} \cdot x^2 + \frac{2y}{x} \cdot x^2 = 0 \cdot x^2$$

Next, I use algebra to do a little simplifying and rearranging:

$$\frac{dy}{dx} \cdot x^2 + 2x \cdot y = 0$$

Now, here's where I appear to get extremely lucky: The two terms on the left side of the equation just happen to be the *result* of the application of the Product Rule to the expression $y \cdot x^2$ (for more on the Product Rule, see Chapter 2):

$$\frac{d}{dx} (y \cdot x^2) = \frac{dy}{dx} \cdot x^2 + 2x \cdot y$$

Notice that the right side of this equation is exactly the same as the left side of the previous equation. So I can make the following substitution:

$$\frac{d}{dx} (y \cdot x^2) = 0$$

Now, to undo the derivative on the left side, I integrate both sides, and then I solve for y :

$$\int \left[\frac{d}{dx} (y \cdot x^2) \right] dx = \int 0 \, dx$$

$$y \cdot x^2 = C$$

$$y = \frac{C}{x^2}$$

To check this solution, I plug this value of y back into the original equation:

$$\frac{dy}{dx} + \frac{2y}{x} = 0$$

$$\frac{d}{dx} \frac{C}{x^2} + \frac{2}{x} \cdot \frac{C}{x^2} = 0$$

$$\frac{-2C}{x^3} + \frac{2C}{x^3} = 0$$

$$0 = 0$$

Making your luck

The previous example works because I found a way to multiply the entire equation by a factor that made the left side of the equation look like a derivative resulting from the Product Rule. Although this looked lucky, if you know what to multiply by, *every* linear first-order DE can be transformed in this way. Recall that the form of a linear first-order DE is as follows:

$$\frac{dy}{dx} + a(x)y = b(x)$$

The trick is to multiply the DE by an *integrating factor* based on $a(x)$. Here's the integrating factor:

$$e^{\int a(x) dx}$$



How the integrating factor is originally derived is beyond the scope of this book. All you need to know here is that it works.

For example, in the previous problem, you know that $a(x) = \frac{2}{x}$. So here's how to find the integrating factor:

$$e^{\int a(x) dx} = e^{\int \frac{2}{x} dx} = e^{2 \ln x}$$

Remember that $2 \ln x = \ln x^2$, so:

$$\begin{aligned} e^{\ln x^2} \\ = x^2 \end{aligned}$$

As you can see, the integrating factor x^2 is the exact value that I multiplied by to solve the problem. To see how this process works now that you know the trick, here's another DE to solve:

$$\frac{dy}{dx} + 3y = e^x$$

In this case, $a(x) = 3$, so compute the integrating factor as follows:

$$e^{\int a(x) dx} = e^{\int 3 dx} = e^{3x}$$

Now, multiply every term in the equation by this factor:

$$\frac{dy}{dx} \cdot e^{3x} + 3y \cdot e^{3x} = e^x \cdot e^{3x}$$

If you like, use algebra to simplify the right side and rearrange the left side:

$$\frac{dy}{dx} \cdot e^{3x} + 3e^{3x} \cdot y = e^{4x}$$

Now, you can see how the left side of this equation looks like the *result* of the Product Rule applied to evaluate the following derivative:

$$\frac{d}{dx} (y \cdot e^{3x}) = \frac{dy}{dx} \cdot e^{3x} + 3e^{3x} \cdot y$$

Because the right side of this equation is the same as the left side of the previous equation, I can make the following substitution:

$$\frac{d}{dx} (y \cdot e^{3x}) = e^{4x}$$

Notice that I change the left side of the equation by using the Product Rule *in reverse*. That is, I'm expressing the whole left side as a single derivative. Now, I can integrate both sides to undo this derivative:

$$\int \left[\frac{d}{dx} (y \cdot e^{3x}) \right] dx = \int e^{4x} dx$$

$$y \cdot e^{3x} = \frac{e^{4x}}{4} + C$$

Now, solve for y and simplify:

$$y = \frac{e^{4x}}{4e^{3x}} + \frac{C}{e^{3x}} = \frac{e^x}{4} + Ce^{-3x}$$

To check this answer, substitute this value of y back into the original DE:

$$\frac{dy}{dx} + 3y = e^x$$

$$\frac{d}{dx} \left(\frac{e^x}{4} + Ce^{-3x} \right) + 3 \left(\frac{e^x}{4} + Ce^{-3x} \right) = e^x$$

$$\frac{e^x}{4} - 3Ce^{-3x} + \frac{3e^x}{4} + 3Ce^{-3x} = e^x$$

$$\frac{e^x}{4} + \frac{3e^x}{4} = e^x$$

$$e^x = e^x$$

As if by magic, this answer checks out, so the solution is valid.

