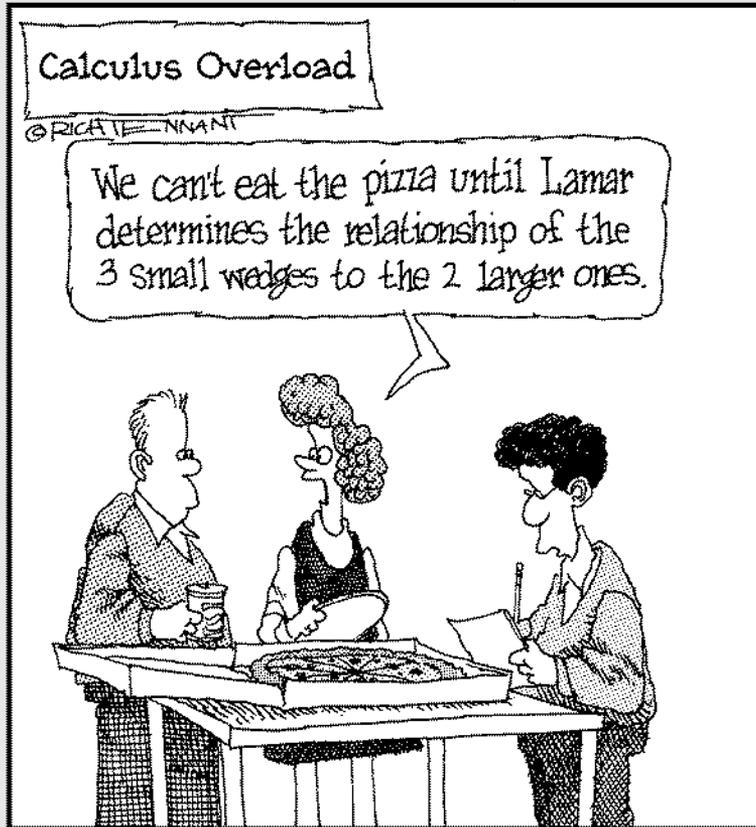


Part I

Introduction to Integration

The 5th Wave

By Rich Tennant



In this part . . .

I give you an overview of Calculus II, plus a review of Pre-Calculus and Calculus I. You discover how to measure the areas of weird shapes by using a new tool: the definite integral. I show you the connection between differentiation, which you know from Calculus I, and integration. And you see how this connection provides a useful way to solve area problems.

Chapter 1

An Aerial View of the Area Problem

In This Chapter

- ▶ Measuring the area of shapes by using classical and analytic geometry
 - ▶ Understanding integration as a solution to the area problem
 - ▶ Building a formula for calculating definite integrals using Riemann sums
 - ▶ Applying integration to the real world
 - ▶ Considering sequences and series
 - ▶ Looking ahead at some advanced math
-

Humans have been measuring the area of shapes for thousands of years. One practical use for this skill is measuring the area of a parcel of land. Measuring the area of a square or a rectangle is simple, so land tends to get divided into these shapes.

Discovering the area of a triangle, circle, or polygon is also easy, but as shapes get more unusual, measuring them gets harder. Although the Greeks were familiar with the conic sections — parabolas, ellipses, and hyperbolas — they couldn't reliably measure shapes with edges based on these figures.

Descartes's invention of analytic geometry — studying lines and curves as equations plotted on a graph — brought great insight into the relationships among the conic sections. But even analytic geometry didn't answer the question of how to measure the area inside a shape that includes a curve.

In this chapter, I show you how *integral calculus* (*integration* for short) developed from attempts to answer this basic question, called the *area problem*. With this introduction to the definite integral, you're ready to look at the practicalities of measuring area. The key to approximating an area that you don't know how to measure is to slice it into shapes that you do know how to measure (for example, rectangles).

Slicing things up is the basis for the *Riemann sum*, which allows you to turn a sequence of closer and closer approximations of a given area into a limit that gives you the exact area that you're seeking. I walk you through a step-by-step process that shows you exactly how the formal definition for the definite integral arises intuitively as you start slicing unruly shapes into nice, crisp rectangles.

Checking out the Area

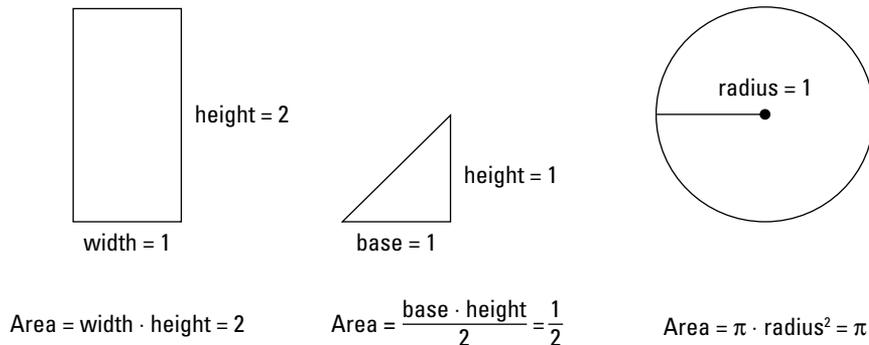
Finding the area of certain basic shapes — squares, rectangles, triangles, and circles — is easy. But a reliable method for finding the area of shapes containing more esoteric curves eluded mathematicians for centuries. In this section, I give you the basics of how this problem, called the area problem, is formulated in terms of a new concept, the definite integral.

The *definite integral* represents the area on a graph bounded by a function, the x -axis, and two vertical lines called the *limits of integration*. Without getting too deep into the computational methods of integration, I give you the basics of how to state the area problem formally in terms of the definite integral.

Comparing classical and analytic geometry

In classical geometry, you discover a variety of simple formulas for finding the area of different shapes. For example, Figure 1-1 shows the formulas for the area of a rectangle, a triangle, and a circle.

Figure 1-1:
Formulas for
the area of a
rectangle, a
triangle, and
a circle.



Wisdom of the ancients

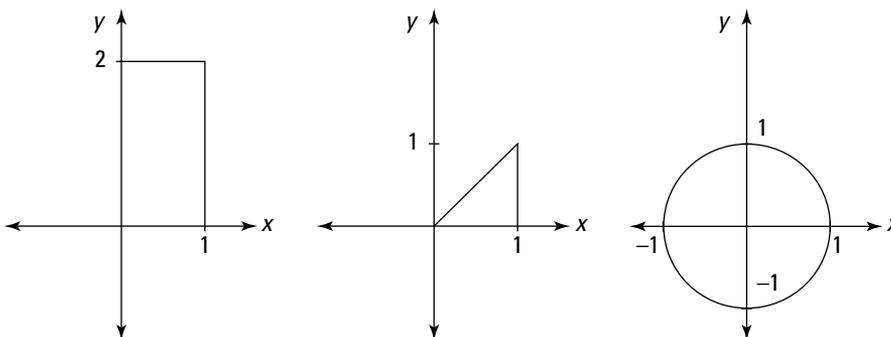
Long before calculus was invented, the ancient Greek mathematician Archimedes used his *method of exhaustion* to calculate the exact area of a segment of a parabola. Indian mathematicians also developed *quadrature* methods

for some difficult shapes before Europeans began their investigations in the 17th century.

These methods anticipated some of the methods of calculus. But before calculus, no single theory could measure the area under arbitrary curves.

When you move on to analytic geometry — geometry on the Cartesian graph — you gain new perspectives on classical geometry. Analytic geometry provides a connection between algebra and classical geometry. You find that circles, squares, and triangles — and many other figures — can be represented by equations or sets of equations, as shown in Figure 1-2.

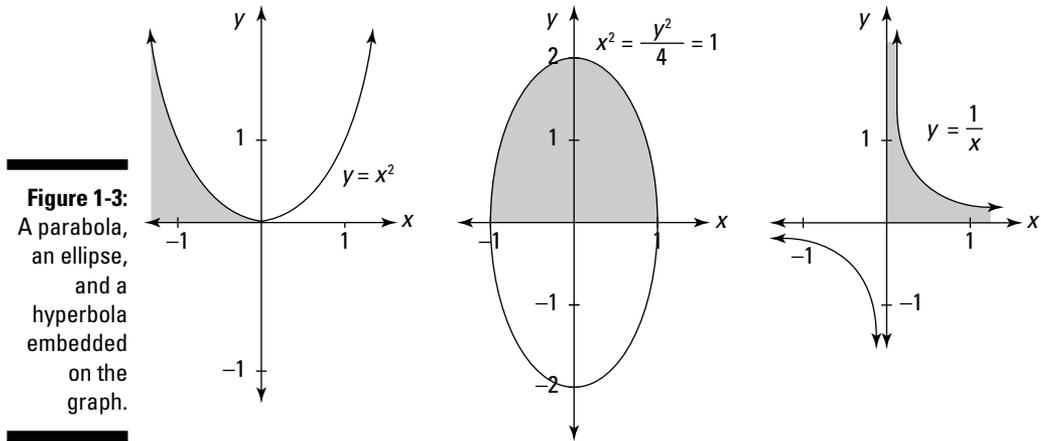
Figure 1-2:
A rectangle,
a triangle,
and a circle
embedded
on the
graph.



You can still use the trusty old methods of classical geometry to find the areas of these figures. But analytic geometry opens up more possibilities — and more problems.

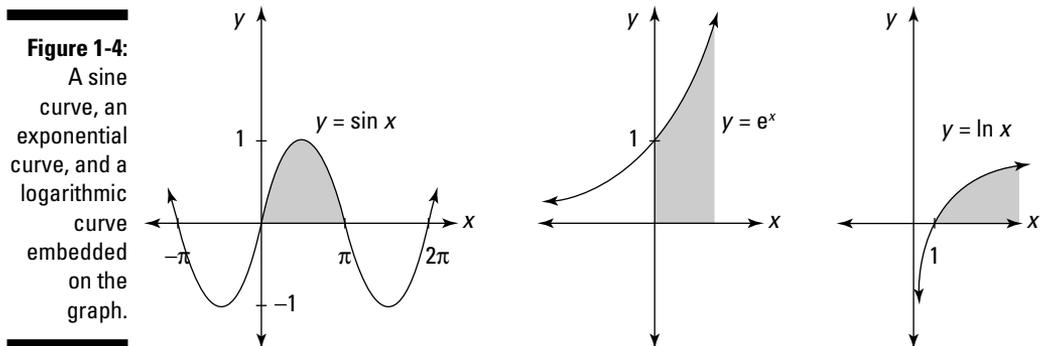
Discovering a new area of study

Figure 1-3 illustrates three curves that are much easier to study with analytic geometry than with classical geometry: a parabola, an ellipse, and a hyperbola.



Analytic geometry gives a very detailed account of the connection between algebraic equations and curves on a graph. But analytic geometry doesn't tell you how to find the shaded areas shown in Figure 1-3.

Similarly, Figure 1-4 shows three more equations placed on the graph: a sine curve, an exponential curve, and a logarithmic curve.



Again, analytic geometry provides a connection between these equations and how they appear as curves on the graph. But it doesn't tell you how to find any of the shaded areas in Figure 1-4.

Generalizing the area problem

Notice that in all the examples in the previous section, I shade each area in a very specific way. Above, the area is bounded by a function. Below, it's bounded by the x -axis. And on the left and right sides, the area is bounded by vertical lines (though in some cases, you may not notice these lines because the function crosses the x -axis at this point).

You can generalize this problem to study any continuous function. To illustrate this, the shaded region in Figure 1-5 shows the area under the function $f(x)$ between the vertical lines $x = a$ and $x = b$.

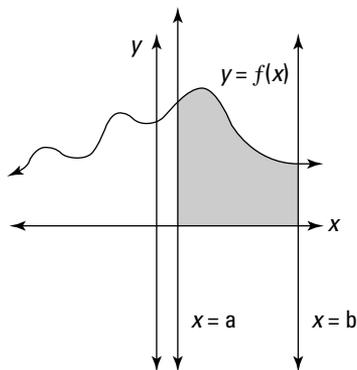


Figure 1-5:
A typical
area
problem.

$$\text{Area} = \int_a^b f(x) dx$$

The area problem is all about finding the area under a continuous function between two constant values of x that are called the *limits of integration*, usually denoted by a and b .



The limits of integration aren't *limits* in the sense that you learned about in Calculus I. They're simply constants that tell you the width of the area that you're attempting to measure.

In a sense, this formula for the shaded area isn't much different from those that I provide earlier in this chapter. It's just a formula, which means that if you plug in the right numbers and calculate, you get the right answer.

The catch, however, is in the word *calculate*. How exactly do you calculate using this new symbol \int ? As you may have figured out, the answer is on the cover of this book: *calculus*. To be more specific, *integral calculus* or *integration*.

Most typical Calculus II courses taught at your friendly neighborhood college or university focus on integration — the study of how to solve the area problem. When Calculus II gets confusing (and to be honest, you probably will get confused somewhere along the way), try to relate what you're doing back to this central question: "How does what I'm working on help me find the area under a function?"

Finding definite answers with the definite integral

You may be surprised to find out that you've known how to integrate some functions for years without even knowing it. (Yes, you can know something without knowing that you know it.)

For example, find the rectangular area under the function $y = 2$ between $x = 1$ and $x = 4$, as shown in Figure 1-6.

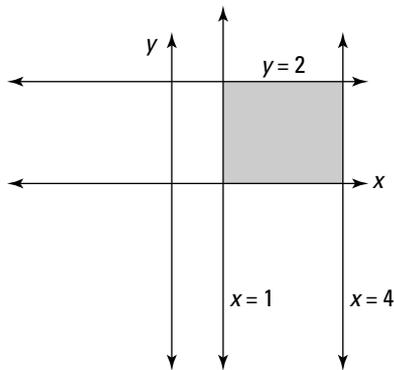


Figure 1-6:
The rectangular area under the function $y = 2$, between $x = 1$ and $x = 4$.

$$Area = \int_1^4 2 \, dx$$

This is just a rectangle with a base of 3 and a height of 2, so its area is obviously 6. But this is also an area problem that can be stated in terms of integration as follows:

$$Area = \int_1^4 2 \, dx = 6$$

As you can see, the function I'm integrating here is $f(x) = 2$. The limits of integration are 1 and 4 (notice that the greater value goes on top). You already

know that the area is 6, so you can solve this calculus problem without resorting to any scary or hairy methods. But, you're still *integrating*, so please pat yourself on the back, because I can't quite reach it from here.

The following expression is called a definite integral:

$$\int_1^4 2 dx$$

For now, don't spend too much time worrying about the deeper meaning behind the \int symbol or the dx (which you may remember from your fond memories of the differentiating that you did in Calculus I). Just think of \int and dx as notation placed around a function — notation that means *area*.

What's so definite about a definite integral? Two things, really:

- ✔ **You definitely know the limits of integration** (in this case, 1 and 4). Their presence distinguishes a definite integral from an *indefinite integral*, which you find out about in Chapter 3. Definite integrals always include the limits of integration; indefinite integrals never include them.
- ✔ **A definite integral definitely equals a number** (assuming that its limits of integration are also numbers). This number may be simple to find or difficult enough to require a room full of math professors scribbling away with #2 pencils. But, at the end of the day, a number is just a number. And, because a definite integral is a measurement of area, you should expect the answer to be a number.

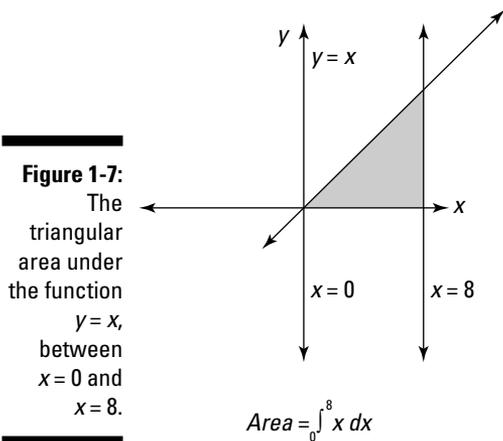


When the limits of integration *aren't* numbers, a definite integral doesn't necessarily equal a number. For example, a definite integral whose limits of integration are k and $2k$ would most likely equal an algebraic expression that includes k . Similarly, a definite integral whose limits of integration are $\sin \theta$ and $2 \sin \theta$ would most likely equal a trig expression that includes θ . To sum up, because a definite integral represents an area, it always equals a number — though you may or may not be able to compute this number.

As another example, find the triangular area under the function $y = x$, between $x = 0$ and $x = 8$, as shown in Figure 1-7.

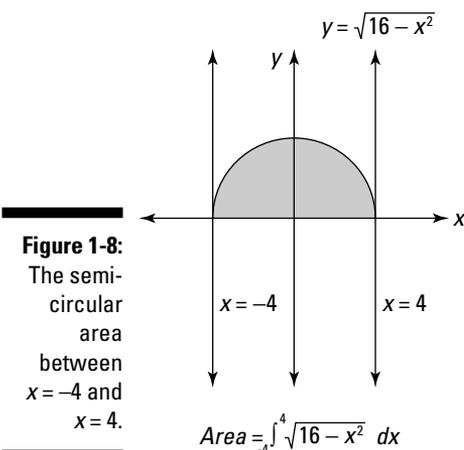
This time, the shape of the shaded area is a triangle with a base of 8 and a height of 8, so its area is 32 (because the area of a triangle is half the base times the height). But again, this is an area problem that can be stated in terms of integration as follows:

$$Area = \int_0^8 x dx = 32$$



The function I'm integrating here is $f(x) = x$ and the limits of integration are 0 and 8. Again, you can evaluate this integral with methods from classical and analytic geometry. And again, the definite integral evaluates to a number, which is the area below the function and above the x -axis between $x = 0$ and $x = 8$.

As a final example, find the semicircular area between $x = -4$ and $x = 4$, as shown in Figure 1-8.



First of all, remember from Pre-Calculus how to express the area of a circle with a radius of 4 units:

$$x^2 + y^2 = 16$$

Next, solve this equation for y :

$$y = \pm \sqrt{16 - x^2}$$

A little basic geometry tells you that the area of the whole circle is 16π , so the area of the shaded semicircle is 8π . Even though a circle isn't a function (and remember that integration deals exclusively with *continuous functions*!), the shaded area in this case is beneath the top portion of the circle. The equation for this curve is the following function:

$$y = \sqrt{16 - x^2}$$

So, you can represent this shaded area as a definite integral:

$$\text{Area} = \int_{-4}^4 \sqrt{16 - x^2} dx = 8\pi$$

Again, the definite integral evaluates to a number, which is the area under the function between the limits of integration.

Slicing Things Up

One good way of approaching a difficult task — from planning a wedding to climbing Mount Everest — is to break it down into smaller and more manageable pieces.

In this section, I show you the basics of how mathematician Bernhard Riemann used this same type of approach to calculate the definite integral, which I introduce in the previous section “Checking out the Area.” Throughout this section I use the example of the area under the function $y = x^2$, between $x = 1$ and $x = 5$. You can find this example in Figure 1-9.

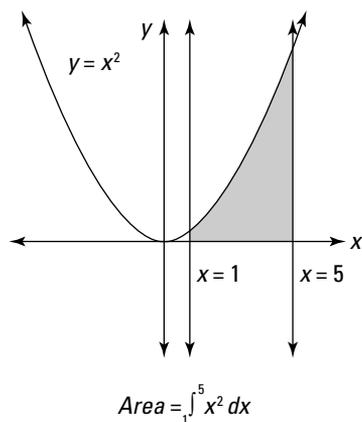


Figure 1-9:
The area
under the
function
 $y = x^2$,
between
 $x = 1$ and
 $x = 5$.

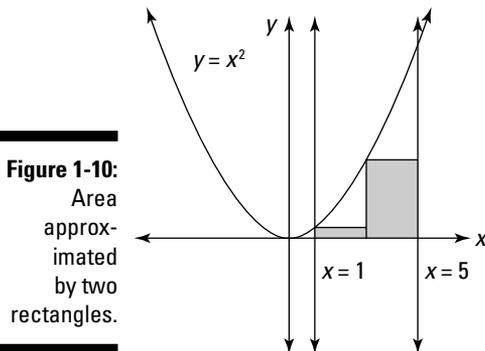
Untangling a hairy problem by using rectangles

The earlier section “Checking out the Area” tells you how to write the definite integral that represents the area of the shaded region in Figure 1-9:

$$\int_1^5 x^2 dx$$

Unfortunately, this definite integral — unlike those earlier in this chapter — doesn’t respond to the methods of classical and analytic geometry that I use to solve the problems earlier in this chapter. (If it did, integrating would be much easier and this book would be a lot thinner!)

Even though you can’t solve this definite integral directly (yet!), you can approximate it by slicing the shaded region into two pieces, as shown in Figure 1-10.



Obviously, the region that’s now shaded — it looks roughly like two steps going up but leading nowhere — is less than the area that you’re trying to find. Fortunately, these steps do lead someplace, because calculating the area under them is fairly easy.

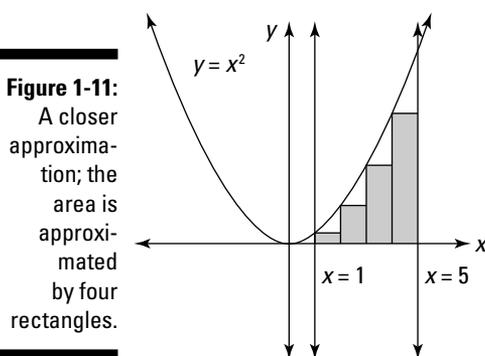
Each rectangle has a width of 2. The tops of the two rectangles cut across where the function x^2 meets $x = 1$ and $x = 3$, so their heights are 1 and 9, respectively. So, the total area of the two rectangles is 20, because

$$2(1) + 2(9) = 2(1 + 9) = 2(10) = 20$$

With this approximation of the area of the original shaded region, here's the conclusion you can draw:

$$\int_1^5 x^2 dx \approx 20$$

Granted, this is a ballpark approximation with a really big ballpark. But, even a lousy approximation is better than none at all. To get a better approximation, try cutting the figure that you're measuring into a few more slices, as shown in Figure 1-11.



Again, this approximation is going to be less than the actual area that you're seeking. This time, each rectangle has a width of 1. And the tops of the four rectangles cut across where the function x^2 meets $x = 1$, $x = 2$, $x = 3$, and $x = 4$, so their heights are 1, 4, 9, and 16, respectively. So the total area of the four rectangles is 30, because

$$1(1) + 1(4) + 1(9) + 1(16) = 1(1 + 4 + 9 + 16) = 1(30) = 30$$

Therefore, here's a second approximation of the shaded area that you're seeking:

$$\int_1^5 x^2 dx \approx 30$$

Your intuition probably tells you that your second approximation is better than your first, because slicing the rectangles more thinly allows them to cut in closer to the function. You can verify this intuition by realizing that both 20 and 30 are *less* than the actual area, so whatever this area turns out to be, 30 must be closer to it.

How high is up?

When you're slicing a weird shape into rectangles, finding the width of each rectangle is easy because they're all the same width. You just divide the total width of the area that you're measuring into equal slices.

Finding the height of each individual rectangle, however, requires a bit more work. Start by drawing the horizontal tops of all the rectangles you'll be using. Then, for each rectangle:

1. **Locate where the top of the rectangle meets the function.**
2. **Find the value of x at that point by looking down at the x -axis directly below this point.**
3. **Get the height of the rectangle by plugging that x -value into the function.**

You might imagine that by slicing the area into more rectangles (say 10, or 100, or 1,000,000), you'd get progressively better estimates. And, again, your intuition would be correct: As the number of slices increases, the result approaches 41.3333....

In fact, you may very well decide to write:

$$\int_1^5 x^2 dx = 41.\overline{33}$$

This, in fact, is the correct answer. But to justify this conclusion, you need a bit more rigor.

Building a formula for finding area

In the previous section, you calculate the areas of two rectangles and four rectangles, respectively, as follows:

$$2(1) + 2(9) = 2(1 + 9) = 20$$

$$1(1) + 1(4) + 1(9) + 1(16) = 1(1 + 4 + 9 + 16) = 30$$

Each time, you divide the area that you're trying to measure into rectangles that all have the same width. Then, you multiply this width by the sum of the heights of *all* the rectangles. The result is the area of the shaded area.

In general, then, the formula for calculating an area sliced into n rectangles is:

$$\text{Area of rectangles} = wh_1 + wh_2 + \dots + wh_n$$

In this formula, w is the width of each rectangle and h_1, h_2, \dots, h_n , and so forth are the various heights of the rectangles. The width of all the rectangles is the same, so you can simplify this formula as follows:

$$\text{Area of rectangles} = w (h_1 + h_2 + \dots + h_n)$$

Remember that as n increases — that is, the more rectangles you draw — the total area of all the rectangles approaches the area of the shape that you're trying to measure.

I hope that you agree that there's nothing terribly tricky about this formula. It's just basic geometry, measuring the area of rectangles by multiplying their width and height. Yet, in the rest of this section, I transform this simple formula into the following formula, called the *Riemann sum formula* for the definite integral:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \left(\frac{b-a}{n} \right)$$

No doubt about it, this formula is eye-glazing. That's why I build it step by step by starting with the simple area formula. This way, you understand completely how all this fancy notation is really just an extension of what you can see for yourself.

If you're sketchy on any of these symbols — such as Σ and the limit — read on, because I explain them as I go along. (For a more thorough review of these symbols, see Chapter 2.)

Approximating the definite integral

Earlier in this chapter I tell you that the definite integral means area. So in transforming the simple formula

$$\text{Area of rectangles} = w (h_1 + h_2 + \dots + h_n)$$

the first step is simply to introduce the definite integral:

$$\int_a^b f(x) dx \approx w (h_1 + h_2 + \dots + h_n)$$

As you can see, the $=$ has been changed to \approx — that is, the equation has been demoted to an approximation. This change is appropriate — the definite integral is the *precise* area inside the specified bounds, which the area of the rectangles merely approximates.

Limiting the margin of error

As n increases — that is, the more rectangles you draw — your approximation gets better and better. In other words, as n approaches infinity, the area

of the rectangles that you're measuring approaches the area that you're trying to find.

So, you may not be surprised to find that when you express this approximation in terms of a limit, you remove the margin of error and restore the approximation to the status of an equation:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} w(h_1 + h_2 + \dots + h_n)$$

This limit simply states mathematically what I say in the previous section: As n approaches infinity, the area of all the rectangles approaches the *exact* area that the definite integral represents.

Widening your understanding of width

The next step is to replace the variable w , which stands for the width of each rectangle, with an expression that's more useful.

Remember that the limits of integration tell you the width of the area that you're trying to measure, with a as the smaller value and b as the greater. So you can write the width of the entire area as $b - a$. And when you divide this area into n rectangles, each rectangle has the following width:

$$w = \frac{b - a}{n}$$

Substituting this expression into the approximation results in the following:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \frac{b - a}{n} (h_1 + h_2 + \dots + h_n)$$

As you can see, all I'm doing here is expressing the variable w in terms of a , b , and n .

Summing things up with sigma notation

You may remember that sigma notation — the Greek symbol Σ used in equations — allows you to streamline equations that have long strings of numbers added together. Chapter 2 gives you a review of sigma notation, so check it out if you need a review.

The expression $h_1 + h_2 + \dots + h_n$ is a great candidate for sigma notation:

$$\sum_{i=1}^n h_i = h_1 + h_2 + \dots + h_n$$

So, in the equation that you're working with, you can make a simple substitution as follows:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \frac{b - a}{n} \sum_{i=1}^n h_i$$

Now, I tweak this equation by placing $\frac{b-a}{n}$ inside the sigma expression (this is a valid rearrangement, as I explain in Chapter 2):

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n h_i \left(\frac{b-a}{n} \right)$$

Heightening the functionality of height

Remember that the variable h_i represents the height of a single rectangle that you're measuring. (The sigma notation takes care of adding up these heights.) The last step is to replace h_i with something more functional. And *functional* is the operative word, because the *function* determines the height of each rectangle.

Here's the short explanation, which I clarify later: The height of each individual rectangle is determined by a value of the function at some value of x lying someplace on that rectangle, so:

$$h_i = f(x_i^*)$$

The notation x_i^* , which I explain further in "Moving left, right, or center," means something like "an appropriate value of x_i ." That is, for each h_i in your sum (h_1, h_2 , and so forth) you can replace the variable h_i in the equation for an appropriate value of the function. Here's how this looks:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \left(\frac{b-a}{n} \right)$$

This is the complete Riemann sum formula for the definite integral, so in a sense I'm done. But I still owe you a complete explanation for this last substitution, and here it comes.

Moving left, right, or center

Go back to the example that I start with, and take another look at the way I slice the shaded area into four rectangles in Figure 1-12.

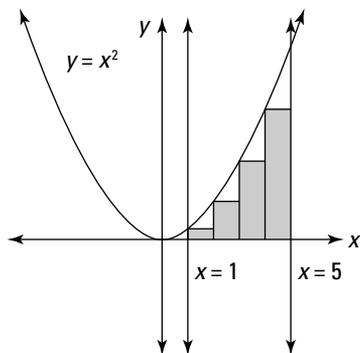
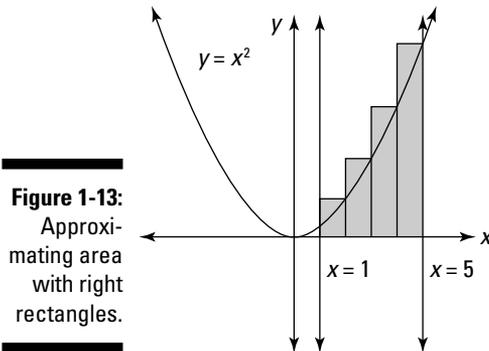


Figure 1-12:
Approximating area with left rectangles.

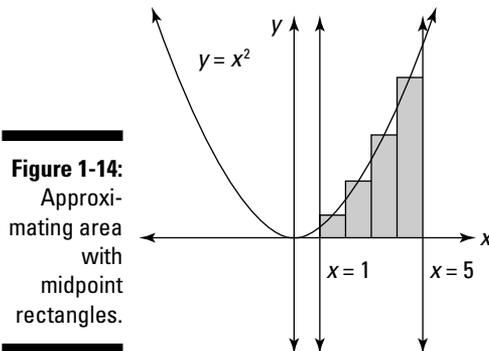
As you can see, the heights of the four rectangles are determined by the value of $f(x)$ when x is equal to 1, 2, 3, and 4, respectively — that is, $f(1)$, $f(2)$, $f(3)$, and $f(4)$. Notice that the upper-left corner of each rectangle touches the function and determines the height of each rectangle.

However, suppose that I draw the rectangles as shown in Figure 1-13.



In this case, the upper-right corner touches the function, so the heights of the four rectangles are $f(2)$, $f(3)$, $f(4)$, and $f(5)$.

Now, suppose that I draw the rectangles as shown in Figure 1-14.



This time, the midpoint of the top edge of each rectangle touches the function, so the heights of the rectangles are $f(1.5)$, $f(2.5)$, $f(3.5)$, and $f(4.5)$.

It seems that I can draw rectangles at least three different ways to approximate the area that I'm attempting to measure. They all lead to different approximations, so which one leads to the correct answer? The answer is *all of them*.

This surprising answer results from the fact that the equation for the definite integral includes a limit. No matter how you draw the rectangles, as long as the top of each rectangle coincides with the function at one point (at least), the limit smoothes over any discrepancies as n approaches infinity. This slack in the equation shows up as the $*$ in the expression $f(x_i^*)$.

For example, in the example that uses four rectangles, the first rectangle is located from $x = 1$ to $x = 2$, so

$$1 \leq x_1^* \leq 2 \quad \text{therefore} \quad 1 \leq f(x_1^*) \leq 4$$

Table 1-2 shows you the range of allowable values for x_i when approximating this area with four rectangles. In each case, you can draw the height of the rectangle on a range of different values of x .

Value of i	Location of Rectangle	Allowable Value of x_i^*	Lowest Value of $f(x_i^*)$	Highest Value of $f(x_i^*)$
$i = 1$	$x = 1$ to $x = 2$	$1 \leq x_1^* \leq 2$	$f(1) = 1$	$f(2) = 4$
$i = 2$	$x = 2$ to $x = 3$	$2 \leq x_2^* \leq 3$	$f(2) = 4$	$f(3) = 9$
$i = 3$	$x = 3$ to $x = 4$	$3 \leq x_3^* \leq 4$	$f(3) = 9$	$f(4) = 16$
$i = 4$	$x = 4$ to $x = 5$	$4 \leq x_4^* \leq 5$	$f(4) = 16$	$f(5) = 25$

In Chapter 3, I discuss this idea — plus a lot more about the fine points of the formula for the definite integral — in greater detail.

Defining the Indefinite

The Riemann sum formula for the definite integral, which I discuss in the previous section, allows you to calculate areas that you can't calculate by using classical or analytic geometry. The downside of this formula is that it's quite a hairy beast. In Chapter 3, I show you how to use it to calculate area, but most students throw their hands up at this point and say, "There has to be a better way!"

The better way is called the *indefinite integral*. The indefinite integral looks a lot like the definite integral. Compare for yourself:

Definite Integrals

$$\int_1^5 x^2 dx$$

$$\int_0^\pi \sin x dx$$

$$\int_{-1}^1 e^x dx$$

Indefinite Integrals

$$\int x^2 dx$$

$$\int \sin x dx$$

$$\int e^x dx$$

Like the definite integral, the indefinite integral is a tool for measuring the area under a function. Unlike it, however, the indefinite integral has no limits of integration, so evaluating it doesn't give you a number. Instead, when you evaluate an indefinite integral, the result is a *function* that you can use to obtain all related definite integrals. Chapter 3 gives you the details of how definite and indefinite integrals are related.

Indefinite integrals provide a convenient way to calculate definite integrals. In fact, the indefinite integral is the *inverse* of the derivative, which you know from Calculus I. (Don't worry if you don't remember all about the derivative — Chapter 2 gives you a thorough review.) By inverse, I mean that the indefinite integral of a function is really the *anti-derivative* of that function. This connection between integration and differentiation is more than just an odd little fact: It's known as the *Fundamental Theorem of Calculus* (FTC).

For example, you know that the derivative of x^2 is $2x$. So, you expect that the anti-derivative — that is, the indefinite integral — of $2x$ is x^2 . This is fundamentally correct with one small tweak, as I explain in Chapter 3.

Seeing integration as anti-differentiation allows you to solve tons of integrals without resorting to the Riemann sum formula (I tell you about this in Chapter 4). But integration can still be sticky depending on the function that you're trying to integrate. Mathematicians have developed a wide variety of techniques for evaluating integrals. Some of these methods are variable substitution (see Chapter 5), integration by parts (see Chapter 6), trig substitution (see Chapter 7), and integration by partial fractions (see Chapter 8).

Solving Problems with Integration

After you understand how to describe an area problem by using the definite integral (Part I), and how to calculate integrals (Part II), you're ready to get into action solving a wide range of problems.

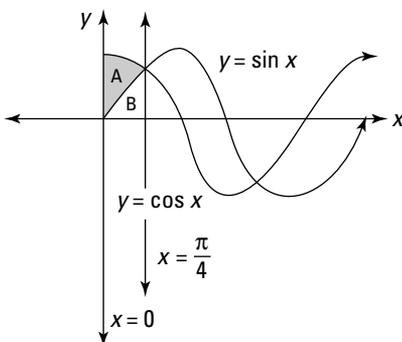
Some of these problems know their place and stay in two dimensions. Others rise up and create a revolution in three dimensions. In this section, I give you a taste of these types of problems, with an invitation to check out Part III of this book for a deeper look.

Three types of problems that you're almost sure to find on an exam involve finding the area between curves, the length of a curve, and volume of revolution. I focus on these types of problems and many others in Chapters 9 and 10.

We can work it out: Finding the area between curves

When you know how the definite integral represents the area under a curve, finding the area between curves isn't too difficult. Just figure out how to break the problem into several smaller versions of the basic area problem. For example, suppose that you want to find the area between the function $y = \sin x$ and $y = \cos x$, from $x = 0$ to $x = \frac{\pi}{4}$ — that is, the shaded area A in Figure 1-15.

Figure 1-15:
The area between the function $y = \sin x$ and $y = \cos x$, from $x = 0$ to $x = \frac{\pi}{4}$.



In this case, integrating $y = \cos x$ allows you to find the total area $A + B$. And integrating $y = \sin x$ gives you the area of B . So, you can subtract $A + B - B$ to find the area of A .

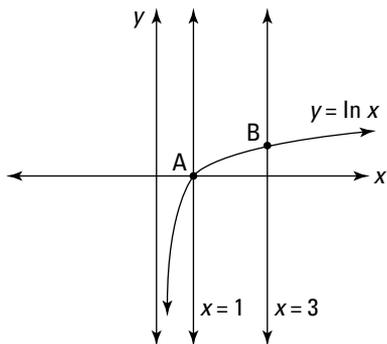
For more on how to find an area between curves, flip to Chapter 9.

Walking the long and winding road

Measuring a segment of a straight line or a section of a circle is simple when you're using classical and analytic geometry. But how do you measure a length along an unusual curve produced by a polynomial, exponential, or trig equation?

For example, what's the distance from point A to point B along the curve shown in Figure 1-16?

Figure 1-16:
The distance from point A to point B along the function $y = \ln x$.



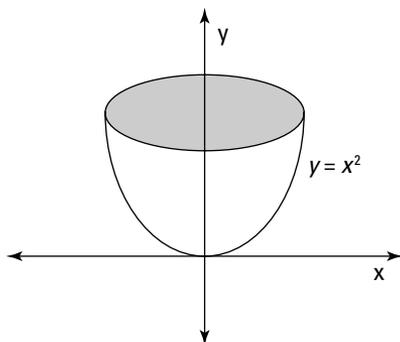
Once again, integration is your friend. In Chapter 9, I show you how to use integration provides a formula that allows you to measure arc length.

You say you want a revolution

Calculus also allows you to find the volume of unusual shapes. In most cases, calculating volume involves a dimensional leap into *multivariable calculus*, the topic of Calculus III, which I touch upon in Chapter 14. But in a few situations, setting up an integral just right allows you to calculate volume by integrating over a single variable — that is, by using the methods you discover in Calculus II.

Among the trickiest of these problems involves the *solid of revolution* of a curve. In such problems, you're presented with a region under a curve. Then, you imagine the solid that results when you spin this region around the axis, and then you calculate the volume of this solid as seen in Figure 1-17.

Figure 1-17:
A solid of revolution produced by spinning the function $y = x^2$ around the axis $x = 0$.



Clearly, you need calculus to find the area of this region. Then you need more calculus and a clear plan of attack to find the volume. I give you all this and more in Chapter 10.

Understanding Infinite Series

The last third of a typical Calculus II course — roughly five weeks — usually focuses on the topic of infinite series. I cover this topic in detail in Part IV. Here's an overview of some of the ideas you find out about there.

Distinguishing sequences and series

A *sequence* is a string of numbers in a determined order. For example:

$$2, 4, 6, 8, 10, \dots$$

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$$

Sequences can be finite or infinite, but calculus deals well with the infinite, so it should come as no surprise that calculus concerns itself only with *infinite sequences*.

You can turn an infinite sequence into an *infinite series* by changing the commas into plus signs:

$$2 + 4 + 6 + 8 + 10 + \dots$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

Sigma notation, which I discuss further in Chapter 2, is useful for expressing infinite series more succinctly:

$$\sum_{n=1}^{\infty} 2n$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Evaluating series

Evaluating an infinite series is often possible. That is, you can find out what all those numbers add up to. For example, here's a solution that should come as no surprise:

$$\sum_{n=1}^{\infty} 2n = 2 + 4 + 6 + 8 + 10 + \dots = \infty$$

A helpful way to get a handle on some series is to create a related *sequence of partial sums* — that is, a sequence that includes the first term, the sum of the first two terms, the sum of the first three terms, and so forth. For example, here's a sequence of partial sums for the second series shown earlier:

$$1 = 1$$

$$1 + \frac{1}{2} = 1\frac{1}{2}$$

$$1 + \frac{1}{2} + \frac{1}{4} = 1\frac{3}{4}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1\frac{7}{8}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1\frac{15}{16}$$

The resulting sequence of partial sums provides strong evidence of this conclusion:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$$

Identifying convergent and divergent series

When a series evaluates to a number — as does $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ — it's called a *convergent series*. However, when a series evaluates to infinity — like $\sum_{n=1}^{\infty} 2n$ — it's called a *divergent series*.

Identifying whether a series is convergent or divergent isn't always simple. For example, take another look at the third series I introduce earlier in this section:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots = ?$$

This is called the *harmonic series*, but can you guess by looking at it whether it converges or diverges? (Before you begin adding fractions, let me warn you that the partial sum of the first 10,000 numbers is less than 10.)

An ongoing problem as you study infinite series is deciding whether a given series is convergent or divergent. Chapter 13 gives you a slew of tests to help you find out.

Advancing Forward into Advanced Math

Although it's further along in math than many people dream of going, calculus isn't the end but a beginning. Whether you're enrolled in a Calculus II class or reading on your own, here's a brief overview of some areas of math that lie beyond integration.

Multivariable calculus

Multivariable calculus generalizes differentiation and integration to three dimensions and beyond. Differentiation in more than two dimensions requires *partial derivatives*. Integration in more than two dimensions utilizes *multiple integrals*.

In practice, multivariable calculus as taught in most Calculus III classes is restricted to three dimensions, using three sets of axes and the three variables x , y , and z . I discuss multivariable calculus in more detail in Chapter 14.

Partial derivatives

As you know from Calculus I, a *derivative* is the slope of a curve at a given point on the graph. When you extend the idea of slope to three dimensions, a new set of issues that need to be resolved arises.

For example, suppose that you're standing on the side of a hill that slopes upward. If you draw a line up and down the hill through the point you're standing on, the slope of this line will be steep. But if you draw a line across the hill through the same point, the line will have little or no slope at all. (For this reason, mountain roads tend to cut sideways, winding their way up slowly, rather than going straight up and down.)

So, when you measure slope on a curved surface in three dimensions, you need to take into account not only the *point* where you're measuring the slope but the *direction* in which you're measuring it. Partial derivatives allow you to incorporate this additional information.

Multiple integrals

Earlier in this chapter, you discover that integration allows you to measure the area under a curve. In three dimensions, the analog becomes finding the volume under a curved surface. *Multiple integrals* (integrals nested inside other integrals) allow you to compute such volume.

Differential equations

After multivariable calculus, the next topic most students learn on their precipitous math journey is *differential equations*.

Differential equations arise in many branches of science, including physics, where key concepts such as velocity and acceleration of an object are computed as first and second derivatives. The resulting equations contain hairy combinations of derivatives that are confusing and tricky to solve. For example:

$$F = m \frac{d^2 s}{dt^2}$$

Beyond ordinary differential equations, which include only ordinary derivatives, *partial differential equations* — such as the heat equation or the Laplace equation — include partial derivatives. For example:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

I provide a look at ordinary and partial differential equations in Chapter 15.

Fourier analysis

So much of physics expresses itself in differential equations that finding reliable methods of solving these equations became a pressing need for 19th-century scientists. Mathematician Joseph Fourier met with the greatest success.

Fourier developed a method for expressing every function as the function of an infinite series of sines and cosines. Because trig functions are continuous and infinitely differentiable, Fourier analysis provided a unified approach to solving huge families of differential equations that were previously incalculable.

Numerical analysis

A lot of math is theoretical and ideal: the search for exact answers without regard to practical considerations such as “How long will this problem take to solve?” (If you’ve ever run out of time on a math exam, you probably know what I’m talking about!)

In contrast, *numerical analysis* is the search for a close-enough answer in a reasonable amount of time.

For example, here's an integral that can't be evaluated:

$$\int e^{x^2} dx$$

But even though you can't solve this integral, you can *approximate* its solution to any degree of accuracy that you desire. And for real-world applications, a good approximation is often acceptable as long as you (or, more likely, a computer) can calculate it in a reasonable amount of time. Such a procedure for approximating the solution to a problem is called an *algorithm*.

Numerical analysis examines algorithms for qualities such as *precision* (the margin of error for an approximation) and *tractability* (how long the calculation takes for a particular level of precision).

Chapter 2

Dispelling Ghosts from the Past: A Review of Pre-Calculus and Calculus I

In This Chapter

- ▶ Making sense of exponents of 0, negative numbers, and fractions
 - ▶ Graphing common continuous functions and their transformations
 - ▶ Remembering trig identities and sigma notation
 - ▶ Understanding and evaluating limits
 - ▶ Differentiating by using all your favorite rules
 - ▶ Evaluating indeterminate forms of limits with L'Hospital's Rule
-

Remember Charles Dickens's *A Christmas Carol*? You know, Scrooge and those ghosts from the past. Math can be just like that: All the stuff you thought was dead and buried for years suddenly pays a spooky visit when you least expect it.

This quick review is here to save you from any unnecessary sleepless nights. Before you proceed any further on your calculus quest, make sure that you're on good terms with the information in this chapter.

First I cover all the Pre-Calculus you forgot to remember: polynomials, exponents, graphing functions and their transformations, trig identities, and sigma notation. Then I give you a brief review of Calculus I, focusing on limits and derivatives. I close the chapter with a topic that you may or may not know from Calculus I: L'Hospital's Rule for evaluating indeterminate forms of limits.

If you still feel stumped after you finish this chapter, I recommend that you pick up a copy of *Pre-Calculus For Dummies* by Deborah Rumsey, PhD, or *Calculus For Dummies* by Mark Ryan (both published by Wiley), for a more in-depth review.

Forgotten but Not Gone: A Review of Pre-Calculus

Here's a true story: When I returned to college to study math, my first degree having been in English, it had been a lot of years since I'd taken a math course. I won't mention how many years, but when I confided this number to my first Calculus teacher, she swooned and was revived with smelling salts (okay, I'm exaggerating a little), and then she asked with a concerned look on her face, "Are you sure you're up for this?"

I wasn't sure at all, but I hung in there. Along the way, I kept refining a stack of notes labeled "Brute Memorization" — basically, what you find in this section. Here's what I learned that semester: Whether it's been one year or 20 since you took Pre-Calculus, make sure that you're comfy with this information.

Knowing the facts on factorials

The *factorial* of a positive integer, represented by the symbol $!$, is that number multiplied by every positive integer less than itself. For example:

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

Notice that the factorial of every positive number equals that number multiplied by the next-lowest factorial. For example:

$$6! = 6 (5!)$$

Generally speaking, then, the following equality is true:

$$(x + 1)! = (x + 1) x!$$

This equality provides the rationale for the odd-looking convention that $0! = 1$:

$$(0 + 1)! = (0 + 1) 0!$$

$$1! = (1) 0!$$

$$1 = 0!$$

When factorials show up in fractions (as they do in Chapters 12 and 13), you can usually do a lot of cancellation that makes them simpler to work with. For example:

$$\frac{3!}{5!} = \frac{(3 \cdot 2 \cdot 1)}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} = \frac{1}{(5 \cdot 4)} = \frac{1}{20}$$

Even when a fraction includes factorials with variables, you can usually simplify it. For example:

$$\frac{(x+1)!}{x!} = \frac{(x+1)x!}{x!} = x+1$$

Polishing off polynomials

A polynomial is any function of the following form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$$

Note that every term in a polynomial is x raised to the power of a nonnegative integer, multiplied by a real-number coefficient. Here are a few examples of polynomials:

$$f(x) = x^3 - 4x^2 + 2x - 5$$

$$f(x) = x^{12} - \frac{3}{4}x^7 + 100x - \pi$$

$$f(x) = (x^2 + 8)(x - 6)^3$$

Note that in the last example, multiplying the right side of the equation will change the polynomial to a more recognizable form.

Polynomials enjoy a special status in math because they're particularly easy to work with. For example, you can find the value of $f(x)$ for any x value by plugging this value into the polynomial. Furthermore, polynomials are also easy to differentiate and integrate. Knowing how to recognize polynomials when you see them will make your life in any math course a whole lot easier.

Powering through powers (exponents)

Remember when you found out that any number (except 0) raised to the power of 0 equals 1? That is:

$$n^0 = 1 \text{ (for all } n \neq 0\text{)}$$

It just seemed weird, didn't it? But when you asked your teacher why, I suspect you got an answer that sounded something like "That's just how mathematicians define it." Not a very satisfying answer, is it?

However if you're absolutely dying to know why (or if you're even mildly curious about it), the answer lies in number patterns.

For starters, suppose that $n = 2$. Table 2-1 is a simple chart that encapsulates information you already know.

Table 2-1		Positive Integer Exponents of 2							
x	1	2	3	4	5	6	7	8	
2^x	2	4	8	16	32	64	128	256	

As you can see, as x increases by 1, 2^x doubles. So, as x decreases by 1, 2^x is halved. You don't need rocket science to figure out what happens when $x = 0$. Table 2-2 shows you what happens.

Table 2-2		Nonnegative Integer Exponents of 2							
x	0	1	2	3	4	5	6	7	8
2^x	1	2	4	8	16	32	64	128	256

This chart provides a simple rationale of why $2^0 = 1$. The same reasoning works for all other real values of n (except 0). Furthermore, Table 2-3 shows you what happens when you continue the pattern into negative values of x .

Table 2-3		Positive and Negative Integer Exponents of 2							
x	-4	-3	-2	-1	0	1	2	3	4
n^x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

As the table shows, $2^{-x} = \frac{1}{2^x}$. This pattern also holds for all real, nonzero values of n , so

$$n^{-x} = \frac{1}{n^x}$$

Notice from this table that the following rule holds:

$$n^a n^b = n^{a+b}$$

For example:

$$2^3 \cdot 2^4 = 2^{3+4} = 2^7 = 128$$

This rule allows you to evaluate fractional exponents as roots. For example:

$$2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2^{\left(\frac{1}{2} + \frac{1}{2}\right)} = 2^1 = 2 \quad \text{so } 2^{\frac{1}{2}} = \sqrt{2}$$

You can generalize this rule for all bases and fractional exponents as follows:

$$n^{\frac{a}{b}} = \sqrt[b]{n^a}$$

Plotting these values for x and $f(x) = 2^x$ onto a graph provides an even deeper understanding (check out Figure 2-1):

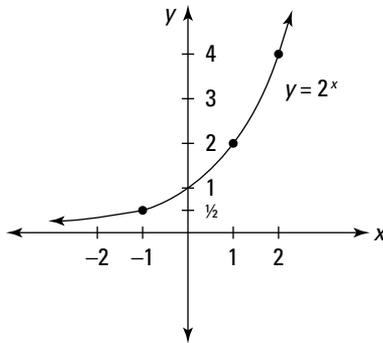


Figure 2-1:
Graph of the
function
 $y = 2^x$.

In fact, assuming the continuity of the exponential curve even provides a rationale (or, I suppose, an *irrational*) for calculating a number raised to an irrational exponent. This calculation is beyond the scope of this book, but it's a problem in numerical analysis, a topic that I discuss briefly in Chapter 1.

Noting trig notation

Trigonometry is a big and important subject in Calculus II. I can't cover everything you need to know about trig here. For more detailed information on trig, see *Trigonometry For Dummies* by Mary Jane Sterling (Wiley). But I do want to spend a moment on one aspect of trig notation to clear up any confusion you may have.

When you see the notation

$$2 \cos x$$

remember that this means $2(\cos x)$. So, to evaluate this function for $x = \pi$, evaluate the inner function $\cos x$ first, and then multiply the result by 2:

$$2 \cos \pi = 2 \cdot -1 = -2$$

On the other hand, the notation

$$\cos 2x$$

means $\cos(2x)$. For example, to evaluate this function for $x = 0$, evaluate the inner function $2x$ first, and then take the cosine of the result:

$$\cos(2 \cdot 0) = \cos 0 = 1$$

Finally (and make sure that you understand this one!), the notation

$$\cos^2 x$$

means $(\cos x)^2$. In other words, to evaluate this function for $x = \pi$, evaluate the inner function $\cos x$ first, and then take the square of the result:

$$\cos^2 \pi = (\cos \pi)^2 = (-1)^2 = 1$$

Getting clear on how to evaluate trig functions really pays off when you're applying the Chain Rule (which I discuss later in this chapter) and when integrating trig functions (which I focus on in Chapter 7).

Figuring the angles with radians

When you first discovered trigonometry, you probably used degrees because they were familiar from geometry. Along the way, you were introduced to radians, and forced to do a bunch of conversions between degrees and radians, and then in the next chapter you went back to using degrees.

Degrees are great for certain trig applications, such as land surveying. But for math, radians are the right tool for the job. In contrast, degrees are awkward to work with.

For example, consider the expression $\sin 1,260^\circ$. You probably can't tell just from looking at this expression that it evaluates to 0, because $1,260^\circ$ is a multiple of 180° .

In contrast, you can tell immediately that the equivalent expression $\sin 7\pi$ is a multiple of π . And as an added bonus, when you work with radians, the numbers tend to be smaller and you don't have to add the degree symbol ($^\circ$).

You don't need to worry about calculating conversions between degrees and radians. Just make sure that you know the most common angles in both degrees and radians. Figure 2-2 shows you some common angles.

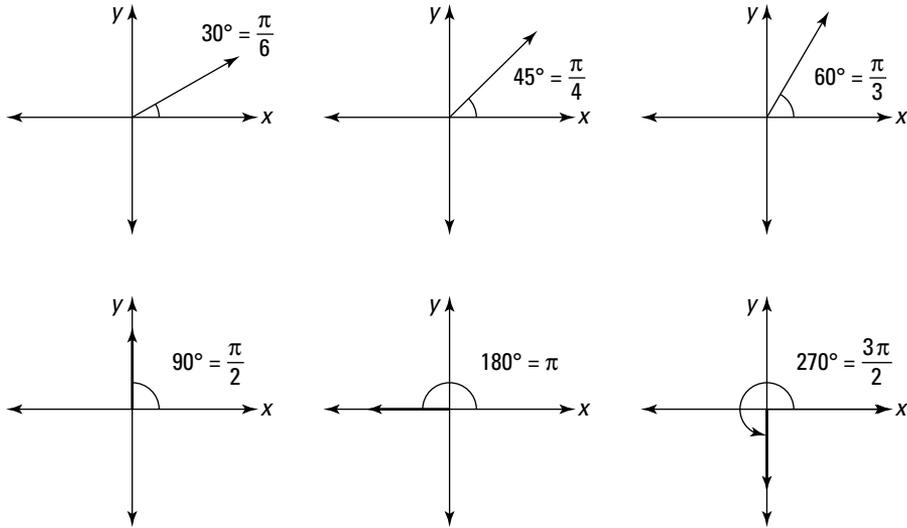


Figure 2-2:
Some common angles in degrees and radians.

Radians are the basis of polar coordinates, which I discuss later in this section.

Graphing common functions

You should be familiar with how certain common functions look when drawn on a graph. In this section, I show you the most common graphs of functions. These functions are all continuous, so they're integrative at all real values of x .

Linear and polynomial functions

Figure 2-3 shows three simple functions.

Figure 2-3:
Graphs of two linear functions $y = n$ and $y = x$ and the absolute value function $y = |x|$.

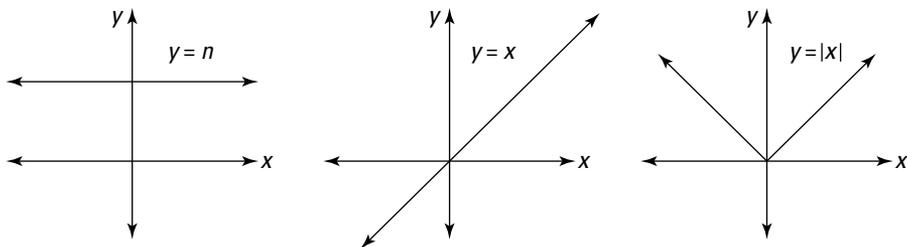
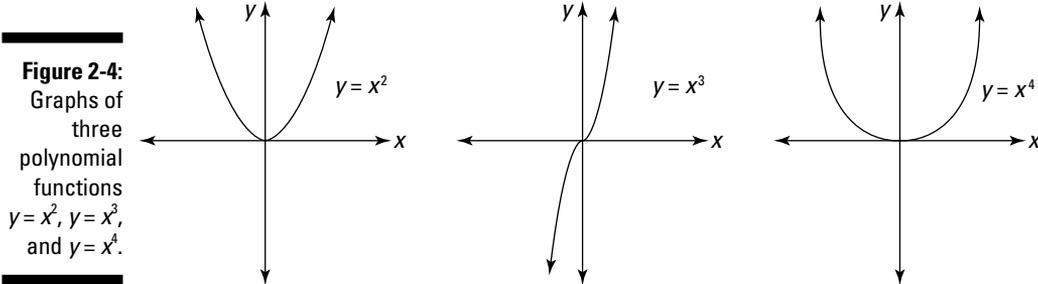


Figure 2-4 includes a few basic polynomial functions.



Exponential and logarithmic functions

Here are some *exponential functions* with whole number bases:

$$y = 2^x$$

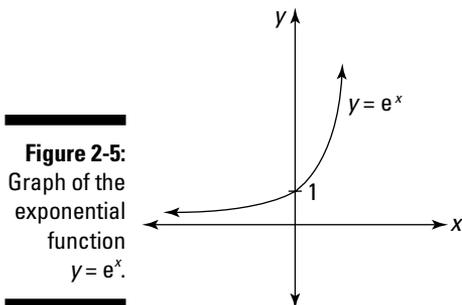
$$y = 3^x$$

$$y = 10^x$$

Notice that for every positive base, the exponential function

- ✓ Crosses the y-axis at $x = 1$
- ✓ Explodes to infinity as x increases (that is, it has an unbounded y value)
- ✓ Approaches $y = 0$ as x decreases (that is, in the negative direction the x -axis is an asymptote)

The most important exponential function is e^x . See Figure 2-5 for a graph of this function.



The unique feature of this exponential function is that at every value of x , its slope is e^x . That is, this function is its own derivative (see “Recent Memories: A Review of Calculus I” later in this chapter for more on derivatives).

Another important function is the *logarithmic function* (also called the *natural log function*). Figure 2-6 is a graph of the logarithmic function $y = \ln x$.

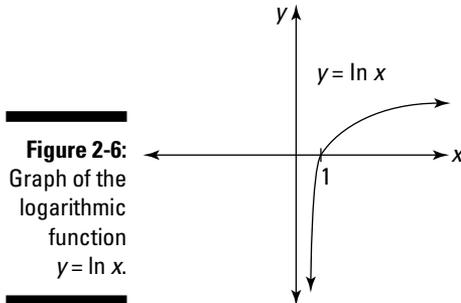


Figure 2-6:
Graph of the
logarithmic
function
 $y = \ln x$.

Notice that this function is the reflection of e^x along the diagonal line $y = x$. So the log function does the following:

- ✓ Crosses the x -axis at $x = 1$
- ✓ Explodes to infinity as x increases (that is, it has an unbounded y value), though more slowly than any exponential function
- ✓ Produces a y value that approaches $-\infty$ as x approaches 0 from the right

Furthermore, the domain of the log functions includes only positive values. That is, inputting a nonpositive value to the log function is a big no-no, on par with placing 0 in the denominator of a fraction or a negative value inside a square root.

For this reason, functions placed inside the log function often get “pretreated” with the absolute value operator. For example:

$$y = \ln |x^3|$$

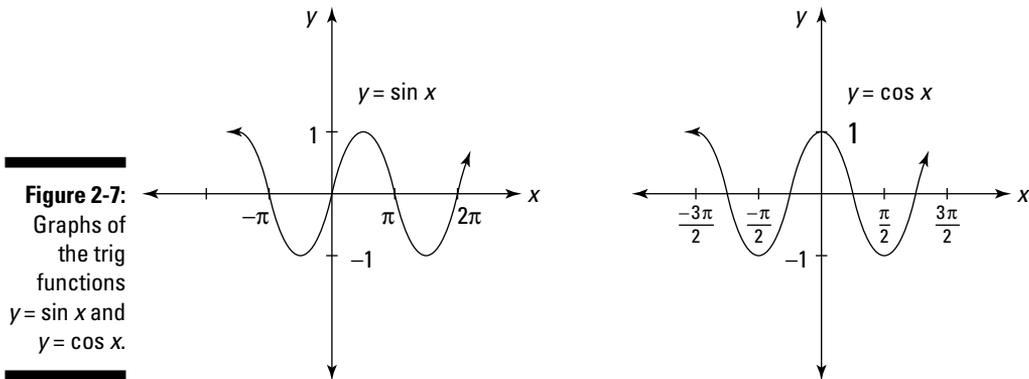


You can bring an exponent outside of a natural log and make it a coefficient, as follows:

$$\ln(a^b) = b \ln a$$

Trigonometric functions

The two most important graphs of trig functions are the sine and cosine. See Figure 2-7 for graphs of these functions.



Note that the x values of these two graphs are typically marked off in multiples of π . Each of these functions has a period of 2π . In other words, it repeats its values after 2π units. And each has a maximum value of 1 and a minimum value of -1 .

Remember that the sine function

- ✓ Crosses the origin
- ✓ Rises to a value of 1 at $x = \frac{\pi}{2}$
- ✓ Crosses the x -axis at all multiples of π

Remember that the cosine function

- ✓ Has a value of 1 at $x = 0$
- ✓ Drops to a value of 0 at $x = \frac{\pi}{2}$
- ✓ Crosses the x -axis at $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, $\frac{7\pi}{2}$, and so on

The graphs of other trig functions are also worth knowing. Figure 2-8 shows graphs of the trig functions $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$.

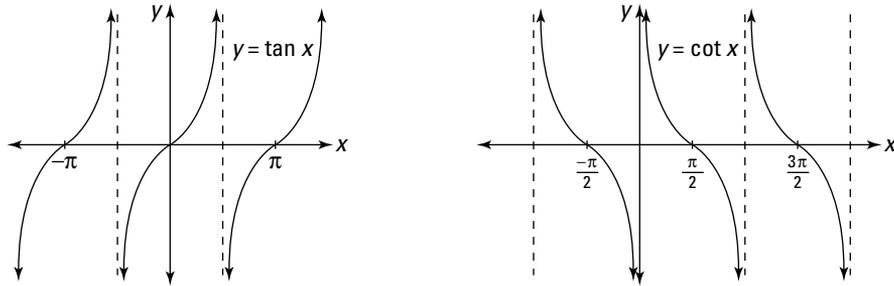
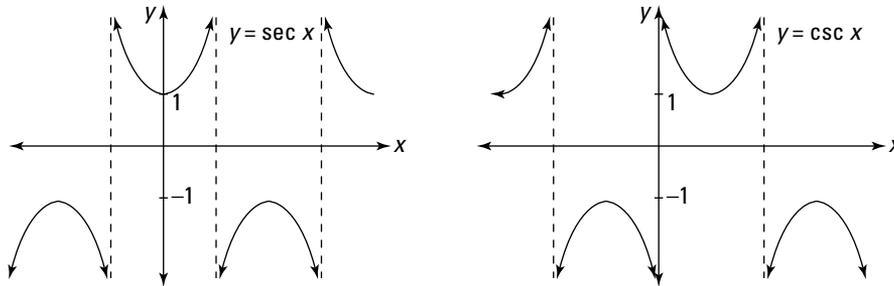


Figure 2-8:
Graphs of
the trig
functions
 $y = \tan x$,
 $y = \cot x$,
 $y = \sec x$,
and
 $y = \csc x$.



Asymptotes

An asymptote is any straight line on a graph that a curve approaches but doesn't touch. It's usually represented on a graph as a dashed line. For example, all four graphs in Figure 2-8 have vertical asymptotes.

Depending on the curve, an asymptote can run in any direction, including diagonally. When you're working with functions, however, horizontal and vertical asymptotes are more common.

Transforming continuous functions

When you know how to graph the most common functions, you can transform them by using a few simple tricks, as I show you in Table 2-4.

<i>Axis</i>	<i>Direction</i>	<i>Transformation</i>	<i>Example</i>
y-axis (vertical)	Shift Up	$y = f(x) + n$	$y = e^x + 1$
	Shift Down	$y = f(x) - n$	$y = x^2 - 2$
	Expand	$y = nf(x)$	$y = 5 \sec x$
	Contract	$y = \frac{f(x)}{n}$	$y = \frac{\sin x}{10}$
	Reflect	$y = -f(x)$	$y = -(\ln x)$
x-axis (horizontal)	Shift Right	$y = f(x - n)$	$y = e^{x-2}$
	Shift Left	$y = f(x + n)$	$y = (x + 4)^3$
	Expand	$y = f\left(\frac{x}{n}\right)$	$y = \sec \frac{x}{3}$
	Contract	$y = f(nx)$	$y = \sin(\pi x)$
	Reflect	$y = f(-x)$	$y = e^{-x}$



The vertical transformations are intuitive — that is, they take the function in the direction that you'd probably expect. For example, adding a constant shifts the function up and subtracting a constant shifts it down.

In contrast, the horizontal transformations are counterintuitive — that is, they take the function in the direction that you probably wouldn't expect. For example, adding a constant shifts the function left and subtracting a constant shifts it right.

Identifying some important trig identities

Memorizing trig identities is like packing for a camping trip.

When you're backpacking into the wilderness, there's a limit to what you can comfortably carry, so you should probably leave your pogo stick and your 30-pound dumbbells at home. At the same time, you don't want to find yourself miles from civilization without food, a tent, and a first-aid kit.

I know that committing trig identities to memory registers on the Fun Meter someplace between alphabetizing your spice rack and vacuuming the lint

How to avoid an identity crisis

Most students remember the first square identity without trouble:

$$\sin^2 x + \cos^2 x = 1$$

If you're worried that you might forget the other two square identities just when you need them most, don't despair. An easy way to remember them is to divide every term in the first square identity by $\sin^2 x$ and $\cos^2 x$:

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

Now, simplify these equations using the Basic Five trig identities:

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

filter on your dryer. But knowing a few important trig identities can be a lifesaver when you're lost out on the misty calculus trails, so I recommend that you take a few along with you. (It's nice when the metaphor really holds up, isn't it?)

For starters, here are the three *inverse identities*, which you probably know already:

$$\sin x = \frac{1}{\csc x}$$

$$\cos x = \frac{1}{\sec x}$$

$$\tan x = \frac{1}{\cot x}$$

You also need these two important identities:

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

I call these the Basic Five trig identities. By using them, you can express *any* trig expression in terms of sines and cosines. Less obviously, you can also express any trig expression in terms of tangents and secants (try it!). Both of these facts are useful in Chapter 7, when I discuss trig integration.

Equally indispensable are the three *square identities*. Most students remember the first and forget about the other two, but you need to know them all:

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

You also don't want to be seen in public without the two *half-angle identities*:

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

Finally, you can't live without the *double-angle identities for sines*:

$$\sin 2x = 2 \sin x \cos x$$

Beyond these, if you have a little spare time, you can include these *double-angle identities for cosines and tangents*:

$$\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Polar coordinates

Polar coordinates are an alternative to the Cartesian coordinate system. As with Cartesian coordinates, polar coordinates assign an ordered pair of values to every point on the plane. Unlike Cartesian coordinates, however, these values aren't (x, y) , but rather (r, θ) .

- ✓ The value r is the distance to the origin.
- ✓ The value θ is the angular distance from the polar axis, which corresponds to the positive x -axis in Cartesian coordinates. (Angular distance is always measured counterclockwise.)

Figure 2-9 shows how to plot points in polar coordinates. For example:

- ✓ To plot the point $(3, \frac{\pi}{4})$, travel 3 units from the origin on the polar axis, and then arc $\frac{\pi}{4}$ (equivalent to 45°) counterclockwise.
- ✓ To plot $(4, \frac{5\pi}{6})$, travel 4 units from the origin on the polar axis, and then arc $\frac{5\pi}{6}$ units (equivalent to 150°) counterclockwise.
- ✓ To plot the point $(2, \frac{3\pi}{2})$, travel 2 units from the origin on the polar axis, and then arc $\frac{3\pi}{2}$ units (equivalent to 270°) counterclockwise.

Polar coordinates allow you to plot certain shapes on the graph more simply than Cartesian coordinates. For example, here's the equation for a 3-unit circle centered at the origin in both Cartesian and polar coordinates:

$$y = \pm \sqrt{x^2 - 9} \quad r = 3$$

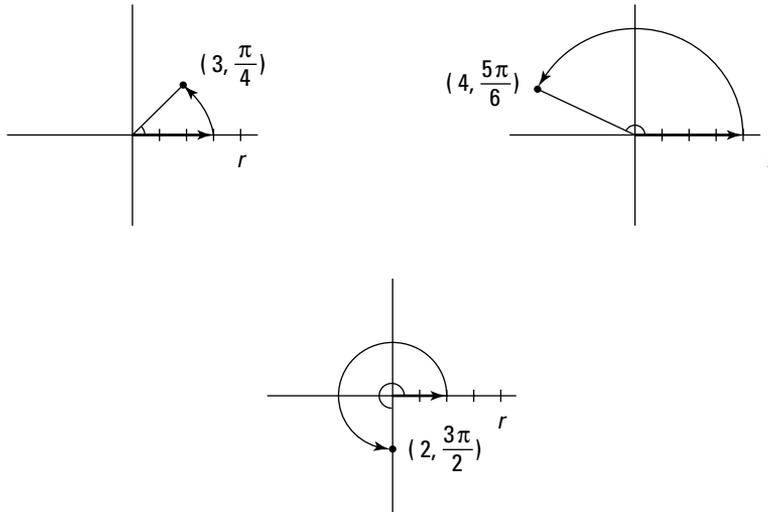


Figure 2-9:
Plotting
points in
polar
coordinates.

Some problems that would be difficult to solve expressed in terms of Cartesian variables (x and y) become much simpler when expressed in terms of polar variables (r and θ). To convert Cartesian variables to polar, use the following formulas:

$$x = r \cos \theta \quad y = r \sin \theta$$

To convert polar variables to Cartesian, use this formula:

$$r = \pm \sqrt{x^2 + y^2} \quad \theta = \arctan\left(\frac{y}{x}\right)$$

Polar coordinates are the basis of two alternative 3-D coordinate systems: cylindrical coordinates and spherical coordinates. See Chapter 14 for a look at these two systems.

Summing up sigma notation

Mathematicians just love sigma notation (Σ) for two reasons. First, it provides a convenient way to express a long or even infinite series. But even more important, it looks really cool and scary, which frightens nonmathematicians into revering mathematicians and paying them more money.

However, when you get right down to it, Σ is just fancy notation for adding, and even your little brother isn't afraid of adding, so why should you be?

For example, suppose that you want to add the even numbers from 2 to 10. Of course, you can write this expression and its solution this way:

$$2 + 4 + 6 + 8 + 10 = 30$$

Or you can write the same expression by using sigma notation:

$$\sum_{n=1}^5 2n$$

Here, n is the *variable of summation* — that is, the variable that you plug values into and then add them up. Below the Σ , you're given the starting value of n (1) and above it the ending value (5). So here's how to expand the notation:

$$\sum_{n=1}^5 2n = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) = 30$$

You can also use sigma notation to stand for the sum of an infinite number of values — that is, an *infinite series*. For example, here's how to add up all the positive square numbers:

$$\sum_{n=1}^{\infty} n^2$$

This compact expression can be expanded as follows:

$$= 1^2 + 2^2 + 3^2 + 4^2 + \dots = 1 + 4 + 9 + 16 + \dots$$

This sum is, of course, infinite. But not all infinite series behave in this way. In some cases, an infinite series equals a number. For example:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

This series expands and evaluates as follows:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$$

When a series evaluates to a number, the series is *convergent*. When a series isn't convergent, it's *divergent*. You find out all about divergent and convergent series in Chapter 12.

Recent Memories: A Review of Calculus I

Integration is the study of how to solve a single problem — the area problem. Similarly, differentiation, which is the focus of Calculus I, is the study of how to solve the *tangent problem*: how to find the slope of the tangent line at any point on a curve. In this section, I review the highlights of Calculus I. For a more thorough review, please see *Calculus For Dummies* by Mark Ryan (Wiley).

Knowing your limits

An important thread that runs through Calculus I is the concept of a *limit*. Limits are also important in Calculus II. In this section, I give you a review of everything you need to remember but may have forgotten about limits.

Telling functions and limits apart

A function provides a link between two variables: the independent variable (usually x) and the dependent variable (usually y). A function tells you the value y when x takes on a specific value. For example, here's a function:

$$y = x^2$$

In this case, when x takes a value of 2, the value of y is 4.

In contrast, a limit tells you what happens to y as x approaches a certain number without actually reaching it. For example, suppose that you're working with the function $y = x^2$ and want to know the limit of this function as x approaches 2. The notation to express this idea is as follows:

$$\lim_{x \rightarrow 2} x^2$$

You can get a sense of what this limit equals by plugging successively closer approximations of 2 into the function (see Table 2-5).

Table 2-5		Approximating $\lim_{x \rightarrow 2} x^2$				
x	1.7	1.8	1.9	1.99	1.999	1.9999
y	2.89	3.24	3.61	3.9601	3.996001	3.99960001

This table provides strong evidence that the limit evaluates to 4. That is:

$$\lim_{x \rightarrow 2} x^2 = 4$$

Remember that this limit tells you nothing about what the function actually equals when $x = 2$. It tells you only that as x approaches 2, the value of the function gets closer and closer to 2. In this case, because the function and the limit are equal, the function is *continuous* at this point.



Evaluating limits

Evaluating a limit means either finding the value of the limit or showing that the limit doesn't exist.

You can evaluate many limits by replacing the limit variable with the number that it approaches. For example:

$$\lim_{x \rightarrow 4} \frac{x^2}{2x} = \frac{4^2}{2 \cdot 4} = \frac{16}{8} = 2$$

Sometimes this replacement shows you that a limit doesn't exist. For example:

$$\lim_{x \rightarrow \infty} x = \infty$$



When you find that a limit appears to equal either ∞ or $-\infty$, the limit *does not exist (DNE)*. DNE is a perfectly good way to complete the evaluation of a limit.

Some replacements lead to apparently untenable situations, such as division by zero. For example:

$$\lim_{x \rightarrow 0} \frac{e^x}{x} = \frac{e^0}{0} = \frac{1}{0}$$

This looks like a dead end, because division by zero is undefined. But, in fact, you can actually get an answer to this problem. Remember that this limit tells you nothing about what happens when x actually equals 0, only what happens as x *approaches* 0: The denominator shrinks toward 0, while the numerator never falls below 1, so the value fraction becomes indefinitely large. Therefore:

$$\lim_{x \rightarrow 0} \frac{e^x}{x} \text{ Does Not Exist (DNE)}$$

Here's another example:

$$\lim_{x \rightarrow \infty} \frac{1,000,000}{x} = \frac{1,000,000}{\infty}$$

This is another apparent dead end, because ∞ isn't really a number, so how can it be the denominator of a fraction? Again, the limit saves the day. It doesn't tell you what happens when x actually equals ∞ (if such a thing were possible), only what happens as x *approaches* ∞ . In this case, the denominator becomes indefinitely large while the numerator remains constant, so:

$$\lim_{x \rightarrow \infty} \frac{1,000,000}{x} = 0$$



Some limits are more difficult to evaluate because they're one of several *indeterminate forms*. The best way to solve them is to use L'Hospital's Rule, which I discuss in detail at the end of this chapter.

Hitting the slopes with derivatives

The *derivative* at a given point on a function is the slope of the tangent line to that function at that point. The derivative of a function provides a "slope map" of that function.

The best way to compare a function with its derivative is by lining them up vertically (see Figure 2-10 for an example).

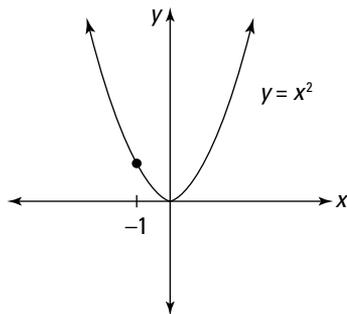
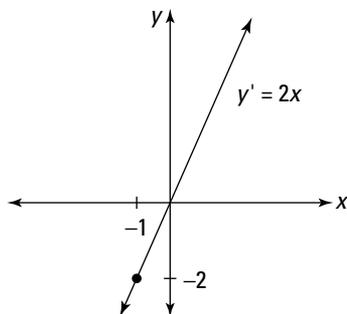


Figure 2-10:
Comparing
a graph of
the function
 $y = x^2$ with
its
derivative
function
 $y' = 2x$.



Looking at the top graph, you can see that when $x = 0$, the slope of the function $y = x^2$ is 0 — that is, no slope. The bottom graph verifies this because at $x = 0$, the derivative function $y = 2x$ is also 0.

You probably can't tell, however, what the slope of the top graph is at $x = -1$. To find out, look at the bottom graph and notice that at $x = -1$, the derivative function equals -2 , so -2 is also the slope of the top graph at this point. Similarly, the derivative function tells you the slope at every point on the original function.

Referring to the limit formula for derivatives

In Calculus I, you develop two formulas for the derivative of a function. These formulas are both based on limits, and they're both equally valid:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

You probably won't need to refer to these formulas much as you study Calculus II. Still, please keep in mind that the official definition of a function's derivative is always cast in terms of a limit.

For a more detailed look at how these formulas are developed, see *Calculus For Dummies* by Mark Ryan (Wiley).

Knowing two notations for derivatives

Students often find the notation for derivatives — especially Leibniz notation $\frac{d}{dx}$ — confusing. To make things simple, think of this notation as a *unary operator* that works in a similar way to a minus sign.

A minus sign attaches to the front of an expression, changing the value of that expression to its negative. Evaluating the effect of this sign on the expression is called *distribution*, which produces a new but equivalent expression. For example:

$$-(x^2 + 4x - 5) = -x^2 - 4x + 5$$

Similarly, the notation $\frac{d}{dx}$ attaches to the front of an expression, changing the value of that expression to its *derivative*. Evaluating the effect of this notation on the expression is called *differentiation*, which also produces a new but equivalent expression. For example:

$$\frac{d}{dx}(x^2 + 4x - 5) = 2x + 4$$

The basic notation remains the same even when an expression is recast as a function. For example, given the function $y = f(x) = x^2 + 4x - 5$, here's how you differentiate:

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = 2x + 4$$

The notation $\frac{dy}{dx}$, which means “the change in y as x changes,” was first used by Gottfried Leibniz, one of the two inventors of calculus (the other inventor was Isaac Newton). An advantage of Leibniz notation is that it explicitly tells you the variable over which you're differentiating — in this case, x . When this information is easily understood in context, a shorter notation is also available:

$$y' = f'(x) = 2x + 4$$

You should be comfortable with both of these forms of notation. I use them interchangeably throughout this book.

Understanding differentiation

Differentiation — the calculation of derivatives — is the central topic of Calculus I and makes an encore appearance in Calculus II.

In this section, I give you a refresher on some of the key topics of differentiation. In particular, the 17 need-to-know derivatives are here and, for your convenience, in the Cheat Sheet just inside the front cover of this book. And if you're shaky on the Chain Rule, I offer you a clear explanation that gets you up to speed.

Memorizing key derivatives

The derivative of any constant is always 0:

$$\frac{d}{dx} n = 0$$

The derivative of the variable by which you're differentiating (in most cases, x) is 1:

$$\frac{d}{dx} x = 1$$

Here are three more derivatives that are important to remember:

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} n^x = n^x \ln n$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

You need to know each of these derivatives as you move on in your study of calculus.

Derivatives of the trig functions

The derivatives of the six trig functions are as follows:

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

You need to know them all by heart.

Derivatives of the inverse trig functions

Two notations are commonly used for inverse trig functions. One is the addition of $^{-1}$ to the function: \sin^{-1} , \cos^{-1} , and so forth. The second is the addition of *arc* to the function: *arcsin*, *arccos*, and so forth. They both mean the same thing, but I prefer the *arc* notation, because it's less likely to be mistaken for an exponent.

I know that asking you to memorize these functions seems like a cruel joke. But you really need them when you get to trig substitution in Chapter 7, so at least have a looksie:

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \operatorname{arccot} x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{x \sqrt{x^2 - 1}}$$

$$\frac{d}{dx} \operatorname{arccsc} x = -\frac{1}{x \sqrt{x^2 - 1}}$$



Notice that derivatives of the three “co” functions are just negations of the three other functions, so your work is cut in half.

The Sum Rule

In textbooks, the Sum Rule is often phrased: The derivative of the sum of functions equals the sum of the derivatives of those functions:

$$\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

Simply put, the Sum Rule tells you that differentiating long expressions term by term is okay. For example, suppose that you want to evaluate the following:

$$\frac{d}{dx} (\sin x + x^4 - \ln x)$$

The expression that you’re differentiating has three terms, so by the Sum Rule, you can break this into three separate derivatives and solve them separately:

$$\begin{aligned} &= \frac{d}{dx} \sin x + \frac{d}{dx} x^4 - \frac{d}{dx} \ln x \\ &= \cos x + 4x^3 - \frac{1}{x} \end{aligned}$$

Note that the Sum Rule also applies to expressions of more than two terms. It also applies regardless of whether the term is positive or negative. Some books call this variation the Difference Rule, but you get the idea.

The Constant Multiple Rule

A typical textbook gives you this sort of definition for the Constant Multiple Rule: The derivative of a constant multiplied by a function equals the product of that constant and the derivative of that function:

$$\frac{d}{dx} n f(x) = n \frac{d}{dx} f(x)$$

In plain English, this rule tells you that moving a constant outside of a derivative before you differentiate is okay. For example:

$$\frac{d}{dx} 5 \tan x$$

To solve this, move the 5 outside the derivative, and then differentiate:

$$\begin{aligned} &= 5 \frac{d}{dx} \tan x \\ &= 5 \sec^2 x \end{aligned}$$

The Power Rule

The Power Rule tells you that to find the derivative of x raised to any power, bring down the exponent as the coefficient of x , and then subtract 1 from the exponent and use this as your *new* exponent. Here's the general form:

$$\frac{d}{dx} x^n = nx^{n-1}$$

Here are a few examples:

$$\frac{d}{dx} x^2 = 2x$$

$$\frac{d}{dx} x^3 = 3x^2$$

$$\frac{d}{dx} x^{10} = 10x^9$$

When the function that you're differentiating already has a coefficient, multiply the exponent by this coefficient. For example:

$$\frac{d}{dx} 2x^4 = 8x^3$$

$$\frac{d}{dx} 7x^6 = 42x^5$$

$$\frac{d}{dx} 4x^{100} = 400x^{99}$$

The Power Rule also extends to negative exponents, which allows you to differentiate many fractions. For example:

$$\begin{aligned} \frac{d}{dx} \frac{1}{x^5} &= \frac{d}{dx} x^{-5} \\ &= -5x^{-6} \\ &= -\frac{5}{x^6} \end{aligned}$$

It also extends to fractional exponents, which allows you to differentiate square roots and other roots:

$$\left(\frac{d}{dx}\right) x^{\frac{1}{3}} = \frac{1}{3} x^{-\frac{2}{3}}$$

The Product Rule

The derivative of the product of two functions $f(x)$ and $g(x)$ is equal to the derivative of $f(x)$ multiplied by $g(x)$ plus the derivative of $g(x)$ multiplied by $f(x)$. That is:

$$\begin{aligned}\frac{d}{dx} [f(x) \cdot g(x)] \\ = f'(x) \cdot g(x) + g'(x) \cdot f(x)\end{aligned}$$



Practice saying the Product Rule like this: “The derivative of the first function times the second *plus* the derivative of the second times the first.” This encapsulates the Product Rule and sets you up to remember the Quotient Rule (see the next section).

For example, suppose that you want to differentiate $e^x \sin x$. Start by breaking the problem out as follows:

$$\frac{d}{dx} e^x \sin x = \left(\frac{d}{dx} e^x \right) \sin x + \left(\frac{d}{dx} \sin x \right) e^x$$

Now, you can evaluate both derivatives, which I underline, without much confusion:

$$= e^x \cdot \sin x + \cos x \cdot e^x$$

You can clean this up a bit as follows:

$$= e^x (\sin x + \cos x)$$

The Quotient Rule

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{g(x)^2}$$



Practice saying the Quotient Rule like this: “The derivative of the top function times the bottom *minus* the derivative of the bottom times the top, over the bottom squared.” This is similar enough to the Product Rule that you can remember it.

For example, suppose that you want to differentiate the following:

$$\frac{d}{dx} \left(\frac{x^4}{\tan x} \right)$$

As you do with the Product Rule example, start by breaking the problem out as follows:

$$= \frac{\left(\frac{d}{dx} x^4\right) \cdot \tan x - \left(\frac{d}{dx} \tan x\right) \cdot x^4}{\tan^2 x}$$

Now, evaluate the two derivatives:

$$= \frac{4x^3 \cdot \tan x - \sec^2 x \cdot x^4}{\tan^2 x}$$

This answer is fine, but you can clean it up by using some algebra plus the five basic trig identities from earlier in this chapter. (Don't worry too much about these steps unless your professor is particularly unforgiving.)

$$\begin{aligned} &= \frac{4x^3 \tan x}{\tan^2 x} - (x^4 \sec^2 x \cot^2 x) \\ &= \frac{4x^3}{\tan x} - x^4 \left(\frac{1}{\cos^2 x}\right) \left(\frac{\cos^2 x}{\sin^2 x}\right) \\ &= 4x^3 \cot x - x^4 \csc^2 x \\ &= x^3 (4 \cot x - x \csc^2 x) \end{aligned}$$

The Chain Rule

I'm aware that the Chain Rule is considered a major sticking point in Calculus I, so I take a little time to review it. (By the way, contrary to popular belief, the Chain Rule isn't "If you don't follow the *rules* in your Calculus class, the teacher gets to place you in *chains*." Such teaching methods are now considered questionable and have been out of use in the classroom since at least the 1970s.)

The Chain Rule allows you to differentiate nested functions — that is, functions within functions. It places no limit on how deeply nested these functions are. In this section, I show you an easy way to think about nested functions, and then I show you how to apply the Chain Rule simply.

Evaluating functions from the inside out

When you're evaluating a nested function, you begin with the *inner* function and move *outward*. For example:

$$f(x) = e^{2x}$$

In this case, $2x$ is the inner function. To see why, suppose that you want to evaluate $f(x)$ for a given value of x . To keep things simple, say that $x = 0$. After

plugging in 0 for x , your first step is to evaluate the inner function, which I underline:

$$\text{Step 1: } e^{\underline{2(0)}} = e^0$$

Your next step is to evaluate the outer function:

$$\text{Step 2: } \underline{e^0} = 1$$

The terms *inner function* and *outer function* are determined by the order in which the functions get evaluated. This is true no matter how deeply nested these functions are. For example:

$$g(x) = (\ln \sqrt{e^{3x-6}})^3$$

Suppose that you want to evaluate $g(x)$. To keep the numbers simple, this time let $x = 2$. After plugging in 2 for x , here's the order of evaluation from the inner function to the outer:

$$\text{Step 1: } (\ln \sqrt{e^{3(2)-6}})^3 = (\ln \sqrt{e^0})^3$$

$$\text{Step 2: } (\ln \sqrt{e^0})^3 = (\ln \sqrt{1})^3$$

$$\text{Step 3: } (\ln \sqrt{1})^3 = (\ln 1)^3$$

$$\text{Step 4: } (\ln 1)^3 = 0^3$$

$$\text{Step 5: } 0^3 = 0$$

The process of evaluation clearly lays out the five nested functions of $g(x)$ from inner to outer.

Differentiating functions from the outside in

In contrast to evaluation, differentiating a function by using the Chain Rule forces you to begin with the *outer* function and move *inward*.

Here's the basic Chain Rule the way that you find it in textbooks:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$



To differentiate nested functions by using the Chain Rule, write down the derivative of the outer function, copying everything inside it, and multiply this result by the derivative of the next function inward.

This explanation may seem a bit confusing, but it's a lot easier when you know how to find the outer function, which I explain in the previous section, "Evaluating functions from the inside out." A couple of examples should help.

For example, suppose that you want to differentiate the nested function $\sin 2x$. The outer function is the sine portion, so this is where you start:

$$\frac{d}{dx} \sin 2x = \cos 2x \cdot \frac{d}{dx} 2x$$

To finish, you still need to differentiate the underlined portion, $2x$:

$$= \cos 2x \cdot 2$$

Rearranging this solution to make it more presentable gives you your final answer:

$$= 2 \cos 2x$$

When you differentiate more than two nested functions, the Chain Rule really lives up to its name: As you break down the problem step by step, you string out a *chain* of multiplied expressions.

For example, suppose that you want to differentiate $\sin^3 e^x$. Remember from the earlier section, “Noting trig notation,” that the notation $\sin^3 e^x$ really means $(\sin e^x)^3$. This rearrangement makes clear that the outer function is the power of 3, so begin differentiating with this function:

$$\frac{d}{dx} (\sin e^x)^3 = 3(\sin e^x)^2 \cdot \frac{d}{dx} (\sin e^x)$$

Now, you have a smaller function to differentiate, which I underline:

$$= 3(\sin e^x)^2 \cdot \cos e^x \cdot \frac{d}{dx} e^x$$

Only one more derivative to go:

$$= 3(\sin e^x)^2 \cdot \cos e^x \cdot e^x$$

Again, rearranging your answer is customary:

$$= 3e^x \cos e^x \sin^2 e^x$$

Finding Limits by Using L'Hospital's Rule

L'Hospital's Rule is all about limits and derivatives, so it fits better with Calculus I than Calculus II. But some colleges save this topic for Calculus II.

So, even though I'm addressing this as a review topic, fear not: Here, I give you the full story of L'Hospital's Rule, starting with how to pronounce L'Hospital (low-pee-tahl).

L'Hospital's Rule provides a method for evaluating certain *indeterminate forms* of limits. First, I show you what an indeterminate form of a limit looks like, with a list of all common indeterminate forms. Next, I show you how to use L'Hospital's Rule to evaluate some of these forms. And finally, I show you how to work with the other indeterminate forms so that you can evaluate them.

Understanding determinate and indeterminate forms of limits

As you discover earlier in this chapter, in "Knowing your limits," you can evaluate many limits by simply replacing the limit variable with the number that it approaches. In some cases, this replacement results in a number, so this number is the value of the limit that you're seeking. In other cases, this replacement gives you an infinite value (either $+\infty$ or $-\infty$), so the limit does not exist (DNE).

Table 2-6 shows a list of some functions that often cause confusion.

Case	$f(x) =$	$g(x) =$	Function	Limit
#1	0	∞	$\frac{f(x)}{g(x)}$	0
#2	0	∞	$f(x)^{g(x)}$	0
#3	$c \neq 0$	0	$\frac{f(x)}{g(x)}$	DNE
#4	$\pm\infty$	0	$\frac{f(x)}{g(x)}$	DNE

To understand how to think about these four cases, remember that a limit describes the behavior of a function very close to, but not exactly at, a value of x .

In the first and second cases, $f(x)$ gets very close to 0 and $g(x)$ explodes to infinity, so both $\frac{f(x)}{g(x)}$ and $f(x)^{g(x)}$ approach 0. In the third case, $f(x)$ is a constant c other than 0 and $g(x)$ approaches 0, so the fraction $\frac{f(x)}{g(x)}$ explodes to infinity. And in the fourth case, $f(x)$ explodes to infinity and $g(x)$ approaches 0, so the fraction $\frac{f(x)}{g(x)}$ explodes to infinity.

In each of these cases, you have the answer you're looking for — that is, you know whether the limit exists and, if so, its value — so these are all called *determinate forms* of a limit.

In contrast, however, sometimes when you try to evaluate a limit by replacement, the result is an *indeterminate form* of a limit. Table 2-7 includes two common indeterminate forms.

Case	$f(x) =$	$g(x) =$	Function	Limit
#1	0	0	$\frac{f(x)}{g(x)}$	Indeterminate
#2	$\pm\infty$	$\pm\infty$	$\frac{f(x)}{g(x)}$	Indeterminate

In these cases, the limit becomes a race between the numerator and denominator of the fractional function. For example, think about the second example in the chart. If $f(x)$ crawls toward ∞ while $g(x)$ zooms there, the fraction becomes bottom heavy and the limit is 0.

But if $f(x)$ zooms to ∞ while $g(x)$ crawls there, the fraction becomes top heavy and the limit is ∞ — that is, DNE. And if both functions move toward 0 proportionally, this proportion becomes the value of the limit.

When attempting to evaluate a limit by replacement saddles you with either of these forms, you need to do more work. Applying L'Hospital's Rule is the most reliable way to get the answer that you're looking for.

Introducing L'Hospital's Rule



Suppose that you're attempting to evaluate the limit of a function of the form $\frac{f(x)}{g(x)}$. When replacing the limit variable with the number that it approaches results in either $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, L'Hospital's Rule tells you that the following equation holds true:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

Note that c can be any real number as well as ∞ or $-\infty$.

As an example, suppose that you want to evaluate the following limit:

$$\lim_{x \rightarrow 0} \frac{x^3}{\sin x}$$

Replacing x with 0 in the function leads to the following result:

$$\frac{0^3}{\sin 0} = \frac{0}{0}$$

This is one of the two indeterminate forms that L'Hospital's Rule applies to, so you can draw the following conclusion:

$$\lim_{x \rightarrow 0} \frac{x^3}{\sin x} = \lim_{x \rightarrow 0} \frac{(x^3)'}{(\sin x)'}$$

Next, evaluate the two derivatives:

$$= \lim_{x \rightarrow 0} \frac{3x^2}{\cos x}$$

Now, use this new function to try another replacement of x with 0 and see what happens:

$$\frac{3(0^2)}{\cos 0} = \frac{0}{1}$$

This time, the result is a determinate form, so you can evaluate the original limit as follows:

$$\lim_{x \rightarrow 0} \frac{x^3}{\sin x} = 0$$

In some cases, you may need to apply L'Hospital's Rule more than once to get an answer. For example:

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^5}$$

Replacement of x with ∞ results in the indeterminate form $\frac{\infty}{\infty}$, so you can use L'Hospital's Rule:

$$\lim_{x \rightarrow \infty} \frac{e^x}{e^5} = \lim_{x \rightarrow \infty} \frac{e^x}{5x^4}$$

In this case, the new function gives you the same indeterminate form, so use L'Hospital's Rule again:

$$= \lim_{x \rightarrow \infty} \frac{e^x}{20x^3}$$

The same problem arises, but again you can use L'Hospital's Rule. You can probably see where this example is going, so I fast forward to the end:

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{e^x}{60x^2} \\ &= \lim_{x \rightarrow \infty} \frac{e^x}{120x} \\ &= \lim_{x \rightarrow \infty} \frac{e^x}{120} \end{aligned}$$



When you apply L'Hospital's Rule repeatedly to a problem, make sure that every step along the way results in one of the two indeterminate forms that the rule applies to.

At last! The process finally yields a function with a determinate form:

$$\frac{e^\infty}{120} = \frac{\infty}{120} = \infty$$

Therefore, the limit does not exist.

Alternative indeterminate forms

L'Hospital's Rule applies only to the two indeterminate forms $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$.

But limits can result in a variety of other indeterminate forms for which L'Hospital's Rule doesn't hold. Table 2-8 is a list of the indeterminate forms that you're most likely to see.

Table 2-8 Five Cases of Indeterminate Forms Where You Can't Apply L'Hospital's Rule Directly

Case	$f(x) =$	$g(x) =$	Function	Form
#1	0	∞	$f(x) \cdot g(x)$	Indeterminate
#2	∞	∞	$f(x) - g(x)$	Indeterminate
#3	0	0	$f(x)^{g(x)}$	Indeterminate
#4	∞	0	$f(x)^{g(x)}$	Indeterminate
#5	1	∞	$f(x)^{g(x)}$	Indeterminate



Because L'Hospital's Rule doesn't hold for these indeterminate forms, applying the rule directly gives you the wrong answer.

These indeterminate forms require special attention. In this section, I show you how to rewrite these functions so that you can then apply L'Hospital's Rule.

Case #1: $0 \cdot \infty$

When $f(x) = 0$ and $g(x) = \infty$, the limit of $f(x) \cdot g(x)$ is the indeterminate form $0 \cdot \infty$, which doesn't allow you to use L'Hospital's Rule. To evaluate this limit, rewrite this function as follows:

$$f(x) \cdot g(x) = \frac{f(x)}{\frac{1}{g(x)}}$$

The limit of this new function is the indeterminate form $\frac{0}{0}$, which allows you to use L'Hospital's Rule. For example, suppose that you want to evaluate the following limit:

$$\lim_{x \rightarrow 0^+} x \cot x$$

Replacing x with 0 gives you the indeterminate form $0 \cdot \infty$, so rewrite the limit as follows:

$$= \lim_{x \rightarrow 0^+} \frac{x}{\left(\frac{1}{\cot x}\right)}$$

This can be simplified a little by using the inverse trig identity for $\cot x$:

$$= \lim_{x \rightarrow 0^+} \frac{x}{\tan x}$$

Now, replacing x with 0 gives you the indeterminate for $\frac{0}{0}$, so you can apply L'Hospital's Rule.

$$\begin{aligned} &= \lim_{x \rightarrow 0^+} \frac{(x)'}{(\tan x)'} \\ &= \lim_{x \rightarrow 0^+} \frac{1}{\sec^2 x} \end{aligned}$$

At this point, you can evaluate the limit directly by replacing x with 0:

$$= \frac{1}{1} = 1$$

Therefore, the limit evaluates to 1.

Case #2: $\infty - \infty$

When $f(x) = \infty$ and $g(x) = \infty$, the limit of $f(x) - g(x)$ is the indeterminate form $\infty - \infty$, which doesn't allow you to use L'Hospital's Rule. To evaluate this limit, try to find a common denominator that turns the subtraction into a fraction. For example:

$$\lim_{x \rightarrow 0^+} \cot x - \csc x$$

In this case, replacing x with 0 gives you the indeterminate form $\infty - \infty$. A little tweaking with the Basic Five trig identities (see “Identifying some important trig identities” earlier in this chapter) does the trick:

$$\begin{aligned} &= \lim_{x \rightarrow 0^+} \frac{\cos x}{\sin x} - \frac{1}{\sin x} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos x - 1}{\sin x} \end{aligned}$$

Now, replacing x with 0 gives you the indeterminate form $\frac{0}{0}$, so you can use L'Hospital's Rule:

$$\begin{aligned} &= \lim_{x \rightarrow 0^+} \frac{(\cos x - 1)'}{(\sin x)'} \\ &= \lim_{x \rightarrow 0^+} \frac{-\sin x}{\cos x} \end{aligned}$$

At last, you can evaluate the limit by directly replacing x with 0.

$$= \frac{0}{1} = 0$$

Therefore, the limit evaluates to 0.

Cases #3, #4, and #5: 0^0 , ∞^0 , and 1^∞

In the following three cases, the limit of $f(x)^{g(x)}$ is an indeterminate form that doesn't allow you to use L'Hospital's Rule:

- ✓ When $f(x) = 0$ and $g(x) = 0$
- ✓ When $f(x) = \infty$ and $g(x) = 0$
- ✓ When $f(x) = 1$ and $g(x) = \infty$



This indeterminate form 1^∞ is easy to forget because it seems weird. After all, $1^x = 1$ for every real number, so why should 1^∞ be any different? In this case, infinity plays one of its many tricks on mathematics. You can find out more about some of these tricks in Chapter 16.

For example, suppose that you want to evaluate the following limit:

$$\lim_{x \rightarrow 0} x^x$$

As it stands, this limit is of the indeterminate form 0^0 .

Fortunately, I can show you a trick to handle these three cases. As with so many things mathematical, mere mortals such as you and me probably wouldn't discover this trick, short of being washed up on a desert island with nothing to do

but solve math problems and eat coconuts. However, somebody did the hard work already. Remembering this following recipe is a small price to pay:

1. Set the limit equal to y .

$$y = \lim_{x \rightarrow 0} x^x$$

2. Take the natural log of both sides, and then do some *log rolling*:

$$\ln y = \ln \lim_{x \rightarrow 0} x^x$$

Here are the two log rolling steps:

- First, *roll* the log inside the limit:

$$= \lim_{x \rightarrow 0} \ln x^x$$

This step is valid because the limit of a log equals the log of a limit (I know, those words veritably *roll* off the tongue).

- Next, *roll* the exponent over the log:

$$= \lim_{x \rightarrow 0} x \ln x$$

This step is also valid, as I show you earlier in this chapter when I discuss the log function in “Graphing common functions.”

3. Evaluate this limit as I show you in “Case #1: $0 \cdot \infty$.”

Begin by changing the limit to a determinate form:

$$= \lim_{x \rightarrow 0} \frac{\ln x}{\frac{1}{x}}$$

At last, you can apply L'Hospital's Rule:

$$= \lim_{x \rightarrow 0} \frac{(\ln x)^1}{\left(\frac{1}{x}\right)^1}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

Now, evaluating the limit isn't too bad:

$$= \lim_{x \rightarrow 0} -\frac{x^2}{x}$$

$$= \lim_{x \rightarrow 0} -x = 0$$

Wait! Remember that way back in Step 2 you set this limit equal to $\ln y$. So you have one more step!

4. Solve for y .

$$\ln y = 0$$

$$y = 1$$

Yes, this is your final answer, so $\lim_{x \rightarrow 0} x^x = 1$.

This recipe works with all three indeterminate forms that I talk about at the beginning of this section. Just make sure that you keep tweaking the limit until you have one of the two forms that are compatible with L'Hospital's Rule.

Chapter 3

From Definite to Indefinite: The Indefinite Integral

In This Chapter

- ▶ Approximating area in five different ways
 - ▶ Calculating sums and definite integrals
 - ▶ Looking at the Fundamental Theorem of Calculus (FTC)
 - ▶ Seeing how the indefinite integral is the inverse of the derivative
 - ▶ Clarifying the differences between definite and indefinite integrals
-

The first step to solving an area problem — that is, finding the area of a complex or unusual shape on the graph — is expressing it as a definite integral. In turn, you can evaluate a definite integral by using a formula based on the limit of a Riemann sum (as I show you in Chapter 1).

In this chapter, you get down to business calculating definite integrals. First, I show you a variety of different ways to estimate area. All these methods lead to a better understanding of the Riemann sum formula for the definite integral. Next, you use this formula to find exact areas. This rather hairy method of calculating definite integrals prompts a search for a better way.

This better way is the indefinite integral. I show you how the indefinite integral provides a much simpler way to calculate area. Furthermore, you find a surprising link between differentiation (which is the focus of Calculus I) and integration. This link, called the Fundamental Theorem of Calculus, shows that the indefinite integral is really an *anti-derivative* (the inverse of the derivative).

To finish up, I show you how using an indefinite integral to evaluate a definite integral results in signed area. I also clarify the differences between definite and indefinite integrals so that you never get them confused. By the end of this chapter, you're ready for Part II, which focuses on an abundance of methods for calculating the indefinite integral.

Approximate Integration

Finding the *exact* area under a curve — that is, solving an area problem (see Chapter 1) — is one of the main reasons that integration was invented. But you can approximate area by using a variety of methods. Approximating area is a good first step toward understanding how integration works.

In this section, I show you five different methods for approximating the solution to an area problem. Generally speaking, I introduce these methods in the order of increasing difficulty and effectiveness. The first three involve manipulating rectangles.

- ✔ The first two methods — left and right rectangles — are the easiest to use, but they usually give you the greatest margin of error.
- ✔ The Midpoint Rule (slicing rectangles) is a little more difficult, but it usually gives you a slightly better estimate.
- ✔ The Trapezoid Rule requires more computation, but it gives an even better estimate.
- ✔ Simpson's Rule is the most difficult to grasp, but it gives the best approximation and, in some cases, provides you with an exact measurement of area.

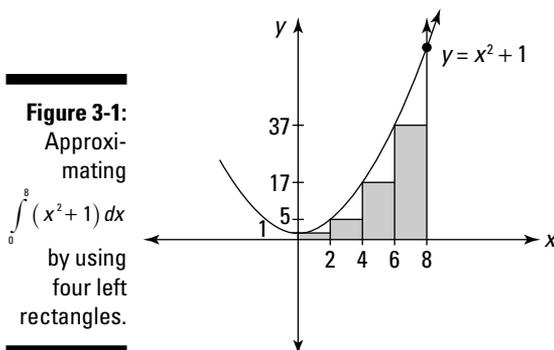
Three ways to approximate area with rectangles

Slicing an irregular shape into rectangles is the most common approach to approximating its area (see Chapter 1 for more details on this approach). In this section, I show you three different techniques for approximating area with rectangles.

Using left rectangles

You can use left rectangles to approximate the solution to an area problem (see Chapter 1). For example, suppose that you want to approximate the shaded area in Figure 3-1 by using four left rectangles.

To draw these four rectangles, start by dropping a vertical line from the function to the x -axis at the *left-hand* limit of integration — that is, $x = 0$. Then drop three more vertical lines from the function to the x -axis at $x = 2$, 4 , and 6 . Next, at the four points where these lines cross the function, draw horizontal lines *from left to right* to make the top edges of the four rectangles. The left and top edges define the size and shape of each left rectangle.



To measure the areas of these four rectangles, you need the width and height of each. The width of each rectangle is obviously 2. The height and area of each is determined by the value of the function at its *left* edge, as shown in Table 3-1.

<i>Rectangle</i>	<i>Width</i>	<i>Height</i>	<i>Area</i>
#1	2	$0^2 + 1 = 1$	2
#2	2	$2^2 + 1 = 5$	10
#3	2	$4^2 + 1 = 17$	34
#4	2	$6^2 + 1 = 37$	74

To approximate the shaded area, add up the areas of these four rectangles:

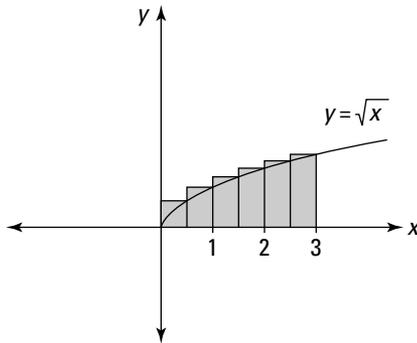
$$\int_0^8 (x^2 + 1) dx \approx 2 + 10 + 34 + 74 = 120$$

Using right rectangles

Using right rectangles to approximate the solution to an area problem is virtually the same as using left rectangles. For example, suppose that you want to use six right rectangles to approximate the shaded area in Figure 3-2.

To draw these rectangles, start by dropping a vertical line from the function to the *x*-axis at the *right-hand* limit of integration — that is, $x = 3$. Next, drop five more vertical lines from the function to the *x*-axis at $x = 0.5, 1, 1.5, 2,$ and 2.5 . Then, at the six points where these lines cross the function, draw horizontal lines *from right to left* to make the top edges of the six rectangles. The right and top edges define the size and shape of each left rectangle.

Figure 3-2:
Approximating
 $\int_0^3 \sqrt{x} dx$
by using
six right
rectangles.



To measure the areas of these six rectangles, you need the width and height of each. Each rectangle's width is 0.5. Its height and area are determined by the value of the function at its *right* edge, as shown in Table 3-2.

<i>Rectangle</i>	<i>Width</i>	<i>Height</i>	<i>Area</i>
#1	0.5	$\sqrt{0.5} \approx 0.707$	0.354
#2	0.5	$\sqrt{1} = 1$	0.5
#3	0.5	$\sqrt{1.5} \approx 1.225$	0.613
#4	0.5	$\sqrt{2} \approx 1.414$	0.707
#5	0.5	$\sqrt{2.5} \approx 1.581$	0.791
#6	0.5	$\sqrt{3} \approx 1.732$	0.866

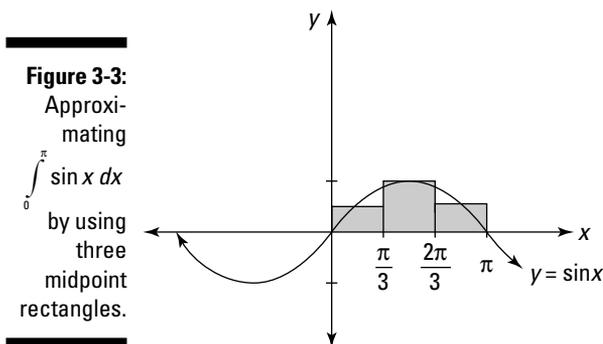
To approximate the shaded area, add up the areas of these six rectangles:

$$\int_0^3 \sqrt{x} dx \approx 0.354 + 0.5 + 0.613 + 0.707 + 0.791 + 0.866 = 3.831$$

Finding a middle ground: The Midpoint Rule

Both left and right rectangles give you a decent approximation of area. So, it stands to reason that slicing an area vertically and measuring the height of each rectangle from the *midpoint* of each slice might give you a slightly better approximation of area.

For example, suppose that you want to use midpoint rectangles to approximate the shaded area in Figure 3-3.



To draw these three rectangles, start by drawing vertical lines that intersect both the function and the x -axis at $x = 0$, $\frac{\pi}{3}$, $\frac{2\pi}{3}$, and π . Next, find where the midpoints of these three regions — that is, $\frac{\pi}{6}$, $\frac{\pi}{2}$, and $\frac{5\pi}{6}$ — intersect the function. Now, draw horizontal lines through these three points to make the tops of the three rectangles.

To measure these three rectangles, you need the width and height of each to compute the area. The width of each rectangle is $\frac{\pi}{3}$, and the height is given in Table 3-3.

<i>Rectangle</i>	<i>Width</i>	<i>Height</i>	<i>Area</i>
#1	$\frac{\pi}{3}$	$\sin \frac{\pi}{6} = \frac{1}{2}$	$\frac{\pi}{6}$
#2	$\frac{\pi}{3}$	$\sin \frac{\pi}{2} = 1$	$\frac{\pi}{3}$
#3	$\frac{\pi}{3}$	$\sin \frac{\pi}{6} = \frac{1}{2}$	$\frac{\pi}{6}$

To approximate the shaded area, add up the areas of these three rectangles:

$$\int_0^{\pi} \sin x \, dx \approx \frac{\pi}{6} + \frac{\pi}{3} + \frac{\pi}{6} = \frac{2\pi}{3} \approx 2.0944$$

The slack factor

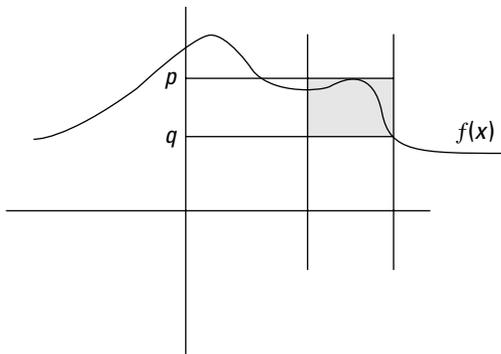
The formula for the definite integral is based on Riemann sums (see Chapter 1). This formula allows you to add up the area of infinitely many infinitely thin rectangular slices to find the exact solution to an area problem.

And here's the strange part: Within certain parameters, the Riemann sum formula doesn't care *how* you do the slicing. All three slicing methods that I discuss in the previous section work equally well. That is, although each method yields a different *approximate* area for a given *finite* number of slices, all these differences are smoothed over when the limit is applied. In other words, all three methods work to provide you the *exact* area for *infinitely many* slices.

I call this feature of measuring rectangles the slack factor. Understanding the slack factor helps you understand why using rectangles drawn at the left endpoint, right endpoint, or midpoint all lead to the same *exact* value of an area: As you measure progressively thinner slices, the slack factor never increases and tends to decrease. As the number of slices approaches ∞ , the width of each slice approaches 0, so the slack factor also approaches 0.

Figure 3-4 shows the range of this slack in choosing a rectangle. In this example, to find the area under $f(x)$, you need to measure a rectangle inside the given slice. The height of this rectangle must be inclusively between p and q , the local maximum and minimum of $f(x)$. Within these parameters, however, you can measure *any* rectangle.

Figure 3-4:
For each slice you're measuring, you can use *any* rectangle that passes through the function at one point or more.



Two more ways to approximate area

Although slicing a region into rectangles is the simplest way to approximate its area, rectangles aren't the only shape that you can use. For finding many areas, other shapes can yield a better approximation in fewer slices.

In this section, I show you two common alternatives to rectangular slicing: the Trapezoid Rule (which, not surprisingly, uses trapezoids) and Simpson's Rule (which uses rectangles topped with parabolas).

Feeling trapped? The Trapezoid Rule

In case you feel restricted — dare I say *boxed in*? — by estimating areas with only rectangles, you can get an even closer approximation by drawing trapezoids instead of rectangles.

For example, suppose that you want to use six trapezoids to estimate this area:

$$\int_{-3}^3 9 - x^2 dx$$

You can probably tell just by looking at the graph in Figure 3-5 that using trapezoids gives you a closer approximation than rectangles. In fact, the area of a trapezoid drawn on any slice of a function will be the average of the areas of the left and right rectangles drawn on that slice.

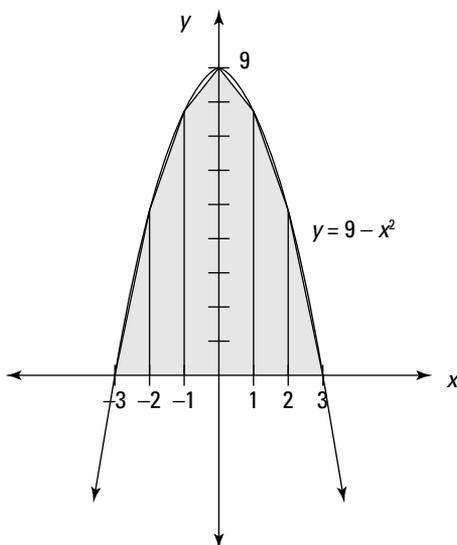


Figure 3-5:
Approximating
 $\int_{-3}^3 9 - x^2 dx$
by using six
trapezoids.



To draw these six trapezoids, first plot points along the function at $x = -3, -2, -1, 0, 1, 2,$ and 3 . Next, connect adjacent points to make the top edges of the trapezoids. Finally, draw vertical lines through these points.

Two of the six “trapezoids” are actually triangles. This fact doesn’t affect the calculation; just think of each triangle as a trapezoid with one height equal to zero.

To find the area of these six trapezoids, use the formula for the area of a trapezoid that you know from geometry: $\frac{w(b_1 + b_2)}{2}$. In this case, however, the two bases — that is, the parallel sides of the trapezoid — are the heights on the left and right sides. As always, the width is easy to calculate — in this case, it’s 1. Table 3-4 shows the rest of the information for calculating the area of each trapezoid.

<i>Trapezoid</i>	<i>Width</i>	<i>Left Height</i>	<i>Right Height</i>	<i>Area</i>
#1	1	$9 - (-3)^2 = 0$	$9 - (-2)^2 = 5$	$\frac{1(0 + 5)}{2} = 2.5$
#2	1	$9 - (-2)^2 = 5$	$9 - (-1)^2 = 8$	$\frac{1(5 + 8)}{2} = 6.5$
#3	1	$9 - (-1)^2 = 8$	$9 - (0)^2 = 9$	$\frac{1(8 + 9)}{2} = 8.5$
#4	1	$9 - (0)^2 = 9$	$9 - (1)^2 = 8$	$\frac{1(9 + 8)}{2} = 8.5$
#5	1	$9 - (1)^2 = 8$	$9 - (2)^2 = 5$	$\frac{1(8 + 5)}{2} = 6.5$
#6	1	$9 - (2)^2 = 5$	$9 - (3)^2 = 0$	$\frac{1(5 + 0)}{2} = 2.5$

To approximate the shaded area, find the sum of the six areas of the trapezoids:

$$\int_{-3}^3 9 - x^2 dx \approx 2.5 + 6.5 + 8.5 + 8.5 + 6.5 + 2.5 = 35$$

Don't have a cow! Simpson's Rule

You may recall from geometry that you can draw exactly one circle through any three nonlinear points. You may not recall, however, that the same is true of parabolas: Just three nonlinear points determine a parabola.

Simpson’s Rule relies on this geometric theorem. When using Simpson’s Rule, you use left and right endpoints as well as midpoints as these three points for each slice.

1. Begin slicing the area that you want to approximate into strips that intersect the function.
2. Mark the left endpoint, midpoint, and right endpoint of each strip.
3. Top each strip with the section of the parabola that passes through these three points.
4. Add up the areas of these parabola-topped strips.

At first glance, Simpson's Rule seems a bit circular: You're trying to approximate the area under a curve, but this method forces you to measure the area inside a region that includes a curve. Fortunately, Thomas Simpson, who invented this rule, is way ahead on this one. His method allows you to measure these strangely shaped regions without too much difficulty.

Without further ado, here's Simpson's Rule:

Given that n is an even number,

$$\int f(x) dx \approx f(x) dx$$

$$\approx \frac{b-a}{3n} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 4f(x_{n-3}) + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

What does it all mean? As with every approximation method you've encountered, the key to Simpson's Rule is measuring the width and height of each of these regions (with some adjustments):

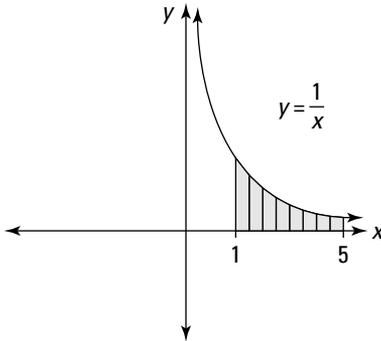
- ✓ The width is represented by $\frac{b-a}{n}$ — but Simpson's Rule adjusts this value to $\frac{b-a}{3n}$.
- ✓ The heights are represented by $f(x)$ taken at various values of x — but Simpson's Rule multiplies some of these by a coefficient of either 4 or 2. (By the way, these choices of coefficients are based on the known result of the area under a parabola — not just picked out of the air!)

The best way to show you how this rule works is with an example. Suppose that you want to use Simpson's Rule to approximate the following:

$$\int_1^5 \frac{1}{x} dx$$

First, divide the area that you want to approximate into an *even* number of regions — say, eight — by drawing nine vertical lines from $x = 1$ to $x = 5$. Now top these regions off with parabolas as I show you in Figure 3-6.

Figure 3-6:
Approximating
 $\int_1^5 \frac{1}{x} dx$
by using
Simpson's
Rule.



The width of each region is 0.5, so adjust this by dividing by 3:

$$\frac{b-a}{3n} = \frac{0.5}{3} \approx 0.167$$

Moving on to the heights, find $f(x)$ when $x = 1, 1.5, 2, \dots, 4.5,$ and 5 (see the second column of Table 3-5). Adjust all these values except the first and the last by multiplying by 4 or 2, alternately.

n	$f(x_n)$	<i>Coefficient</i>	<i>Total</i>
0	$f(1) = 1$	1	$f(1) = 1$
1	$f(1.5) = 0.667$	4	$4f(1.5) = 2.668$
2	$f(2) = 0.5$	2	$2f(2) = 1$
3	$f(2.5) = 0.4$	4	$4f(2.5) = 1.6$
4	$f(3) = 0.333$	2	$2f(3) = 0.666$
5	$f(3.5) = 0.290$	4	$4f(3.5) = 1.160$
6	$f(4) = 0.25$	2	$2f(4) = 0.5$
7	$f(4.5) = 0.222$	4	$4f(4.5) = 0.888$
8	$f(5) = 0.2$	1	$f(5) = 0.2$

Now, apply Simpson's Rule as follows:

$$\begin{aligned} \int_1^5 \frac{1}{x} dx \\ \approx 0.167 (1 + 2.668 + 1 + 1.6 + 0.666 + 1.16 + 0.5 + 0.888 + 0.2) \\ = 0.167 (9.682) \approx 1.617 \end{aligned}$$

So Simpson's Rule approximates the area of the shaded region in Figure 3-6 as 1.617. (By the way, the actual area to three decimal places is about 1.609 — so Simpson's Rule provides a pretty good estimate.)



In fact, Simpson's Rule often provides an even better estimate than this example leads you to believe, because a lot of inaccuracy arises from rounding off decimals. In this case, when you perform the calculations with enough precision, Simpson's Rule provides the correct area to three decimal places!

Knowing Sum-Thing about Summation Formulas

In Chapter 1, I introduce you to the Riemann sum formula for the definite integral. This formula includes a summation using sigma notation (Σ). (Please flip to Chapter 2 if you need a refresher on this topic.)

In practice, evaluating a summation can be a little tricky. Fortunately, three important summation formulas exist to help you. In this section, I introduce you to these formulas and show you how to use them. In the next section, I show you how and when to apply them when you're using the Riemann sum formula to solve an area problem.

The summation formula for counting numbers

The summation formula for counting numbers gives you an easy way to find the sum $1 + 2 + 3 + \dots + n$ for any value of n :

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

To see how this formula works, suppose that $n = 9$:

$$\sum_{i=1}^9 i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

The summation formula for counting numbers also produces this result:

$$\frac{n(n+1)}{2} = \frac{9(10)}{2} = 45$$



According to a popular story, mathematician Karl Friedrich Gauss discovered this formula as a schoolboy, when his teacher gave the class the boring task of adding up all the counting numbers from 1 to 100 so that he (the teacher) could nap at his desk. Within minutes, Gauss arrived at the correct answer, 5,050, disturbing his teacher's snooze time and making mathematical history.

The summation formula for square numbers

The summation formula for square numbers gives you a quick way to add up $1 + 4 + 9 + \dots + n^2$ for any value of n :

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

For example, suppose that $n = 7$:

$$\sum_{i=1}^7 i^2 = 1 + 4 + 9 + 16 + 25 + 36 + 49 = 140$$

The summation formula for square numbers gives you the same answer:

$$\frac{n(n+1)(2n+1)}{6} = \frac{7(8)(15)}{6} = 140$$

The summation formula for cubic numbers

The summation formula for cubic numbers gives you a quick way to add up $1 + 8 + 27 + \dots + n^3$ for any value of n :

$$\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2$$

For example, suppose that $n = 5$:

$$\sum_{i=1}^5 i^3 = 1 + 8 + 27 + 64 + 125 = 225$$

The summation formula for cubic numbers produces the same result:

$$\left[\frac{n(n+1)}{2} \right]^2 = \left[\frac{5(6)}{2} \right]^2 = 15^2 = 225$$

As Bad as It Gets: Calculating Definite Integrals by Using the Riemann Sum Formula

In Chapter 1, I introduce you to this hairy equation for calculating the definite integral:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[f(x_i^*) \left(\frac{b-a}{n} \right) \right]$$

You may be wondering how practical this little gem is for calculating area. That's a valid concern. The bad news is that this formula is, indeed, hairy and you'll need to understand how to use it to pass your first Calculus II exam.

But I have good news, too. In the beginning of Calculus I, you work with an equally hairy equation for calculating derivatives (see Chapter 2 for a refresher). Fortunately, later on, you find a bunch of easier ways to calculate derivatives.

This good news applies to integration, too. Later in this chapter, I show you how to make your life easier. In this section, however, I focus on how to use the Riemann sum formula to calculate the definite integral.

Before I get started, take another look at the Riemann sum formula and notice that the right side of this equation breaks down into four separate “chunks”:

- ✓ The limit: $\lim_{n \rightarrow \infty}$
- ✓ The sum: $\sum_{i=1}^n$
- ✓ The function: $f(x_i^*)$
- ✓ The limits of integration: $\frac{b-a}{n}$

To solve an integral by using this formula, work backwards, step by step, as follows:

1. **Plug the limits of integration into the formula.**
2. **Rewrite the function $f(x_i^*)$ as a summation in terms of i and n .**
3. **Calculate the sum.**
4. **Evaluate the limit.**

Plugging in the limits of integration

In this section, I show you how to calculate the following integral:

$$\int_0^4 x^2 dx$$

This step is a no-brainer: You just plug the limits of integration — that is, the values of a and b — into the formula:

$$\int_0^4 x^2 dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[f(x_i^*) \left(\frac{4-0}{n} \right) \right]$$

Before moving on, I know that you just can't go on living until you simplify $4 - 0$:

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[f(x_i^*) \frac{4}{n} \right]$$

That's it!

Expressing the function as a sum in terms of i and n

This is the tricky step. It's more of an art than a science, so if you're an art major who just happens to be taking a Calculus II course, this just might be your lucky day (or maybe not).

To start out, think about how you would estimate $\int_0^4 x^2 dx$ by using right rectangles, as I explain earlier in this chapter. Table 3-6 shows you how to do this, using one, two, four, and eight rectangles.

n	Height	Width	Expression
1	4^2	4	$\sum_{i=1}^1 (4i)^2 (4)$
2	$2^2 + 4^2$	2	$\sum_{i=1}^2 (2i)^2 (2)$
4	$1^2 + 2^2 + 3^2 + 4^2$	1	$\sum_{i=1}^4 i^2 (1)$
8	$0.5^2 + 1^2 + 1.5^2 + 2^2 + 2.5^2 + 3^2 + 3.5^2 + 4^2$	0.5	$\sum_{i=1}^8 (0.5i)^2 (0.5)$

Your goal now is to find a *general* expression of the form $\sum_{i=1}^n$ that works for every value of n . In the last section, you find that $\frac{4}{n}$ produces the correct width. So, here's the general expression that you're looking for:

$$\sum_{i=1}^n \left(\frac{4i}{n} \right)^2 \left(\frac{4}{n} \right)$$

Make sure that you understand why this expression works for all values of n before moving on. The first fraction represents the height of the rectangles and the second fraction represents the width, expressed as $\frac{b-a}{n}$.

You can simplify this expression as follows:

$$= \sum_{i=1}^n \frac{64i^2}{n^3}$$

Don't forget before moving on that the entire expression is a limit as n approaches infinity:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{64i^2}{n^3}$$



At this point in the problem, you have an expression that's based on two variables: i and n . Remember that the two variables i and n are *in* the sum, and the variable x should already have exited.

Calculating the sum

Now you need a few tricks for calculating the summation portion of this expression:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{64i^2}{n^3}$$

You can ignore the limit in this section — it's just coming along for the ride. You can move a constant outside of a summation without changing the value of that expression:

$$= \lim_{n \rightarrow \infty} 64 \sum_{i=1}^n \frac{i^2}{n^3}$$

At this point, only the variables i and n are left inside the summation.



Remember that i stands for *icky* and n stands for *nice*. The variable n is nice because you can move it outside the summation just as if it were a constant:

$$= \lim_{n \rightarrow \infty} \frac{64}{n^3} \sum_{i=1}^n i^2$$

Solving the problem with a summation formula

To handle the icky variable, i , you need a little help. Earlier in the chapter, in “Knowing Sum-Thing about Summation Formulas,” I give you some important formulas for handling this summation and others like it.

Getting back to the example, here's where you left off:

$$\lim_{n \rightarrow \infty} \frac{64}{n^3} \sum_{i=1}^n i^2$$

To evaluate the sum $\sum_{i=1}^n i^2$, use the summation formula for square numbers:

$$\lim_{n \rightarrow \infty} \frac{64}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

A bit of algebra — which I omit because I know you can do it! — makes the problem look like this:

$$\lim_{n \rightarrow \infty} \frac{64}{3} + \frac{1}{n} + \frac{1}{3n^3}$$

You're now set up for the final — and easiest — step.

Evaluating the limit

At this point, the limit that you've probably been dreading all this time turns out to be the simplest part of the problem. As n approaches infinity, the two terms with n in the denominator approach 0, so they drop out entirely:

$$\lim_{n \rightarrow \infty} \frac{64}{3} + \frac{1}{n} + \frac{1}{3n^3} = \frac{64}{3}$$

Yes, this is your final answer! Please note that because you used the Riemann sum formula, this is *not an approximation*, but the *exact* area under the curve $y = x^2$ from 0 to 4.

Light at the End of the Tunnel: The Fundamental Theorem of Calculus

Finding the area under a curve — that is, solving an area problem — can be formalized by using the definite integral (as you discover in Chapter 1). And the definite integral, in turn, is defined in terms of the Riemann sum formula. But, as you find out earlier in this chapter, the Riemann sum formula usually results in lengthy and difficult calculations.

There must be a better way! And, indeed, there is.

The Fundamental Theorem of Calculus (FTC) provides the link between derivatives and integrals. At first glance, these two ideas seem entirely unconnected, so the FTC seems like a bit of mathematical black magic. On closer examination, however, the connection between a function's derivative (its slope) and its integral (the area underneath it) becomes clearer.

In this section, I show you the connection between slope and area. After you see this, the FTC will make more intuitive sense. At that point, I introduce the exact theorem and show you how to use it to evaluate integrals as *anti-derivatives* — that is, by understanding integration as the inverse of differentiation.

Without further ado, here's the Fundamental Theorem of Calculus (FTC) in its most useful form:

$$\int_a^b f'(x) = f(b) - f(a)$$



The mainspring of this equality is the connection between f and its derivative function f' . To solve an integral, you need to be able to *undo* differentiation and find the original function f .

Many math books use the following notation for the FTC:

$$\int_a^b f(x) dx = F(b) - F(a) \text{ where } F'(x) = f(x)$$

Both notations are equally valid, but I find this version a bit less intuitive than the version that I give you.

The FTC makes evaluating integrals a whole lot easier. For example, suppose that you want to evaluate the following:

$$\int_0^{\pi} \sin x dx$$

This is the function that you see in Figure 3-3. The FTC allows you to solve this problem by thinking about it in a new way. First notice that the following statement is true:

$$f(x) = -\cos x \rightarrow f'(x) = \sin x$$

So the FTC allows you to draw this conclusion:

$$\int_0^{\pi} \sin x dx = (-\cos \pi) - (-\cos 0)$$

Now you can solve this problem by using simple trig:

$$= 1 + 1 = 2$$

So the *exact* (not approximate) shaded area in Figure 3-3 is 2 — all without drawing rectangles! The approximation using the Midpoint Rule (see “Finding a middle ground: The Midpoint Rule” earlier in this chapter) is 2.0944.

As another example, here’s the integral that, earlier in the chapter, you solved by using the Riemann sum formula:

$$\int_0^4 x^2 dx$$

Begin by noticing that the following statement is true:

$$f(x) = \frac{1}{3} x^3 \rightarrow f'(x) = x^2$$

Now use the FTC to write this equation:

$$\int_0^4 x^2 dx = \left(\frac{1}{3} 4^3\right) - \left(\frac{1}{3} 0^3\right)$$

At this point, the solution becomes a matter of arithmetic:

$$\frac{64}{3} - 0 = \frac{64}{3}$$

In just three simple steps, the definite integral is solved without resorting to the hairy Riemann sum formula!

Understanding the Fundamental Theorem of Calculus

In the previous section, I show you just how useful the Fundamental Theorem of Calculus (FTC) can be for finding the exact value of a definite integral without using the Riemann sum formula. But *why* does the theorem work?

The FTC implies a connection between derivatives and integrals that isn't intuitively obvious. In fact, the theorem implies that derivatives and integrals are inverse operations. It's easy to see why other pairs of operations — such as addition and subtraction — are inverses. But how do you see this same connection between derivatives and integrals?

In this section, I give you a few ways to better understand this connection.

Solving a 200-year-old problem

The connection between derivatives and integrals as inverse operations was first noticed by Isaac Barrow (the teacher of Isaac Newton) in the 17th century. Newton and Gottfried Leibniz (the two key inventors of calculus) both made use of it as a conjecture — that is, as a mathematical statement that's suspected to be true but hasn't been proven yet.

But the FTC wasn't officially proven in all its glory until your old friend Bernhard Riemann demonstrated it in the 19th century. During this 200-year lag, a lot of math — most notably, real analysis — had to be invented before Riemann could prove that derivatives and integrals are inverses.

What's slope got to do with it?

The idea that derivatives and integrals are connected — that is, the slope of a curve and the area under it are linked mathematically — seems odd until you spend some time thinking about it.

If you have a head for business, here's a practical way to understand the connection. Imagine that you own your own company. Envision a graph with a line as your net income (money coming in) and the area under the graph as your net savings (money in the bank). To keep this simple, imagine for the moment that this is a happy world where you have *no* expenses draining your savings account.

When the line on the graph is horizontal, your net income stays the same, so money comes in at a steady rate — that is, your paycheck every week or month is the same. So, your bank account (the area under the line) grows at a steady rate as time passes — that is, as your x -value moves to the right.

But suppose that business starts booming. As the line on the graph starts to rise, your paychecks rise proportionally. So, your bank account begins growing at a faster rate.

Now suppose that business slows down. As the line on the graph starts to fall, your paychecks fall proportionally. So, your bank account still grows, but its rate of growth slows down. But beware: If business goes so sour that it can no longer support itself, you may find that you're dipping into savings to support the business, so for the first time your savings goes down.

In this analogy, every *paycheck* is like the area inside a one-unit-wide slice of the graph. And the *bank account* on any particular day is like the total area between the y -axis and that day as shown on the graph.

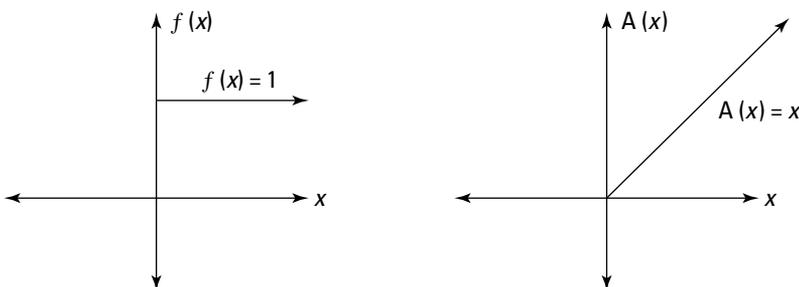
So, when you give it some thought, it would be hard to imagine how slope and area could *not* be connected. The Fundamental Theorem of Calculus is just the exact mathematical representation of this connection.

Introducing the area function

This connection between income (the size of your paycheck) and savings (the amount in your bank account) is a perfect analogy for two important, connected ideas. The income graph represents a function $f(x)$ and the savings graph represents that function's area function $A(x)$.

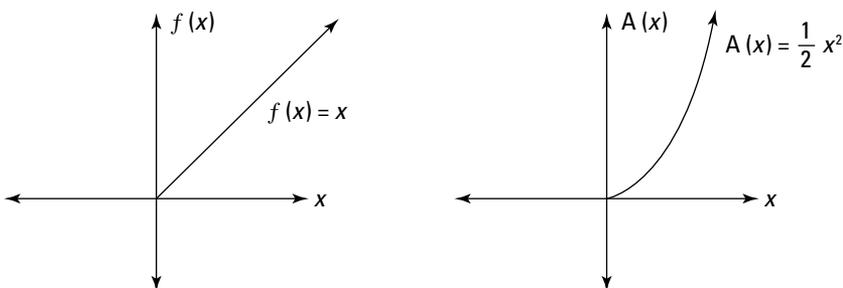
Figure 3-7 illustrates this connection between $f(x)$ and $A(x)$. This figure represents the *steady income* situation that I describe in the previous section. I choose $f(x) = 1$ to represent income. The resulting savings graph is $A(x) = x$, which rises steadily.

Figure 3-7:
The function
 $f(x) = 1$
produces an
area
function
 $A(x) = x$.



In comparison, look at Figure 3-8, which represents *rising income*. This time, I choose $f(x) = x$ to represent income. This function produces the area function $A(x) = \frac{1}{2}x^2$, which rises at an increasing rate.

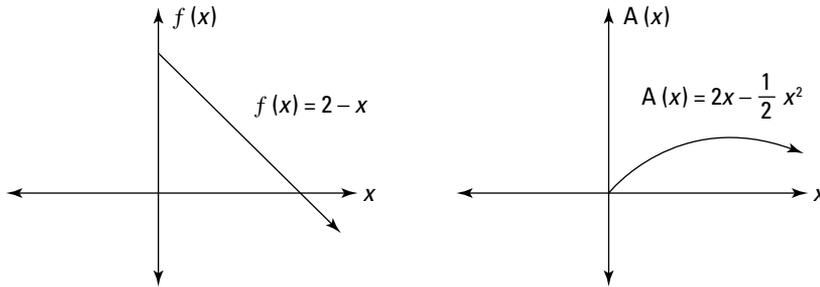
Figure 3-8:
The function
 $f(x) = x$
produces an
area
function
 $A(x) = \frac{1}{2}x^2$.



Finally, take a peek at Figure 3-9, which represents *falling income*. In this case, I use $f(x) = 2 - x$ to represent income. This function results in the area function $A(x) = 2x - \frac{1}{2}x^2$, which rises at a decreasing rate until the original function drops below 0, and then starts falling.

Take a moment to think about these three examples. Make sure that you see how, in a very practical sense, slope and area are connected: In other words, the slope of a function is the qualitative factor that governs what the related area function looks like.

Figure 3-9:
The function
 $f(x) = 2 - x$
produces an
area
function
 $A(x) = 2x - \frac{1}{2}x^2$.



Connecting slope and area mathematically

In the previous section, I discuss three functions $f(x)$ and their related area functions $A(x)$. Table 3-7 summarizes this information.

Table 3-7 A Closer Look at Functions and Their Area Functions

Description of Function	Equation of Function	Description of Area Function	Equation of Area Function	Derivative of Area Function
Constant	$f(x) = 1$	Rising steadily	$A(x) = x$	$A'(x) = 1$
Rising	$f(x) = x$	Rising at increasing rate	$A(x) = \frac{1}{2}x^2$	$A'(x) = x$
Falling	$f(x) = 2 - x$	Rising at decreasing rate, and then falling when $f(x) < 0$	$A(x) = 2x - \frac{1}{2}x^2$	$A'(x) = 2 - x$

At this point, the big connection is only a heartbeat away. Notice that each function is the *derivative* of its area function:

$$A'(x) = f(x)$$

Is this mere coincidence? Not at all. Table 3-7 just adds mathematical precision to the intuitive idea that slope of a function (that is, its derivative) is related to the area underneath it.

Because area is mathematically described by the definite integral, as I discuss in Chapter 1, this connection between differentiation and integration makes a whole lot of sense. That's why finding the area under a function — that is, *integration* — is essentially *undoing* a derivative — that is, *anti-differentiation*.

Seeing a dark side of the FTC

Earlier in this chapter, I give you this piece of the Fundamental Theorem of Calculus:

$$\int_a^b f'(x) = f(b) - f(a)$$

Now that you understand the connection between a function $f(x)$ and its area function $A(x)$, here's another piece of the FTC:

$$A_i(x) = \int_s^x f(t) dt$$

This piece of the theorem is generally regarded as less useful than the first piece, and it's also harder to grasp because of all the extra variables. I won't belabor it too much, but here are a few points that may help you understand it better:

- ✓ The variable s — the lower limit of integration — is an arbitrary starting point where the area function equals zero. In my examples in the previous section, I start the area function at the origin, so $s = 0$. This point represents the day when you opened your bank account, before you deposited any money.
- ✓ The variable x — the upper limit of integration — represents any time after you opened your bank account. It's also the independent variable of the area function.
- ✓ The variable t is the variable of the function. If you were to draw a graph, t would be the independent variable and $f(t)$ the dependent variable.

In short, don't worry too much about this version of the FTC. The most important thing is that you remember the first version and know how to use it. The other important thing is that you understand how slope and area — that is, derivatives and integrals — are intimately related.

Your New Best Friend: The Indefinite Integral

The Fundamental Theorem of Calculus gives you insight into the connection between a function's slope and the area underneath it — that is, between differentiation and integration.

On a practical level, the FTC gives you an easier way to integrate, without resorting to the Riemann sum formula. This easier way is called *anti-differentiation* — in other words, undoing differentiation. Anti-differentiation is the method that you'll use to integrate throughout the remainder of Calculus II. It leads quickly to a new key concept: the *indefinite integral*.

In this section, I show you step by step how to use the indefinite integral to solve definite integrals, and I introduce the important concept of signed area. To finish the chapter, I make sure that you understand the important distinctions between definite and indefinite integrals.

Introducing anti-differentiation

Integration without resorting to the Riemann sum formula depends upon undoing differentiation (anti-differentiation). Earlier in this chapter, in “Light at the End of the Tunnel: The Fundamental Theorem of Calculus,” I calculate a few areas informally by reversing a few differentiation formulas that you know from Calculus I. But anti-differentiation is so important that it deserves its own notation: the indefinite integral.

An *indefinite integral* is simply the notation representing the inverse of the derivative function:

$$\frac{d}{dx} \int f(x) dx = f(x)$$



Be careful not to confuse the indefinite integral with the definite integral. For the moment, notice that the indefinite integral has *no limits of integration*. Later in this chapter, in “Distinguishing definite and indefinite integrals,” I outline the differences between these two types of integrals.

Here are a few examples that informally connect derivatives that you know with indefinite integrals that you want to be able to solve:

$$\frac{d}{dx} \sin x = \cos x \rightarrow \int \cos x dx = \sin x$$

$$\frac{d}{dx} e^x = e^x \rightarrow \int e^x dx = e^x$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x} \rightarrow \int \frac{1}{x} dx = \ln x$$

There's a small but important catch in this informal analysis. Notice that the following three statements are all true:

$$\frac{d}{dx} \sin x + 1 = \cos x$$

$$\frac{d}{dx} \sin x - 100 = \cos x$$

$$\frac{d}{dx} \sin x + 1,000,000 = \cos x$$

Because any constant differentiates to 0, you need to account for the possible presence of a constant when integrating. So, here are the more precise formulations of the indefinite integrals I just introduced:

$$\int \cos x \, dx = \sin x + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \frac{1}{x} \, dx = \ln x + C$$



The formal solution of every indefinite integral is an anti-derivative up to the addition of a constant C , which is called the *constant of integration*. So, just mechanically attach a $+ C$ whenever you evaluate an indefinite integral.

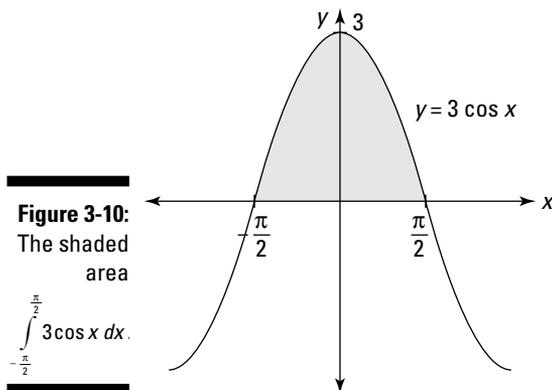
Solving area problems without the Riemann sum formula

After you know how to solve an indefinite integral by using anti-differentiation (as I show you in the previous section), you have at your disposal a very useful method for solving area problems. This announcement should come as a great relief, especially after reading the earlier section “As Bad as It Gets: Calculating Definite Integrals by Using the Riemann Sum Formula.”

Here’s how you solve an area problem by using indefinite integrals — that is, *without* resorting to the Riemann sum formula:

- 1. Formulate the area problem as a definite integral (as I show you in Chapter 1).**
- 2. Solve the definite integral as an indefinite integral evaluated between the given limits of integration.**
- 3. Plug the limits of integration into this expression and simplify to find the area.**

This method is, in fact, the one that you use for solving area problems for the rest of Calculus II. For example, suppose that you want to find the shaded area in Figure 3-10.



Here's how you do it:

1. Formulate the area problem as a definite integral:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3 \cos x \, dx$$

2. Solve this definite integral as an indefinite integral:

$$= 3 \sin x \Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}}$$

I replace the integral with the expression $3 \sin x$, because $\frac{d}{dx} 3 \sin x = 3 \cos x$. I also introduce the notation $\Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}}$. You can read it as *evaluated from x equals $-\frac{\pi}{2}$ to x equals $\frac{\pi}{2}$* . This notation is commonly used so that you can show your teacher that you know how to integrate and postpone worrying about the limits of integration until the next step.

3. Plug these limits of integration into the expression and simplify:

$$= 3 \sin \frac{\pi}{2} - 3 \sin -\frac{\pi}{2}$$

As you can see, this step comes straight from the FTC, subtracting $f(b) - f(a)$. Now, I just simplify this expression to find the area:

$$= 3 - (-3) = 6$$

So, the area of the shaded region in Figure 3-10 equals 6.

No C , no problem!

You may wonder why the constant of integration C —which is so important when you're evaluating an indefinite integral—gets dropped when you're evaluating a definite integral. This one is easy to explain.

Remember that every definite integral is expressed as the difference between a function evaluated at one point and *the same function* evaluated at another point. If this function includes a constant C , one C cancels out the other.

For example, take the definite integral

$\int_0^{\frac{\pi}{6}} \cos x \, dx$. Technically speaking, this integral is evaluated as follows:

$$\begin{aligned} & \sin x + C \Big|_{x=0}^{x=\frac{\pi}{6}} \\ &= \left(\sin \frac{\pi}{6} + C \right) - \left(\sin 0 + C \right) \\ &= \frac{1}{2} + C - 0 - C = \frac{1}{2} \end{aligned}$$

As you can clearly “ C ,” the two C s cancel each other out, so there's no harm in dropping them at the beginning of the evaluation rather than at the end.

Understanding signed area

In the real world, the smallest possible area is 0, so area is always a nonnegative number. On the graph, however, area can be either positive or negative.

This idea of negative area relates back to a discussion earlier in this chapter, in “Introducing the area function,” where I talk about what happens when a function dips below the x -axis.

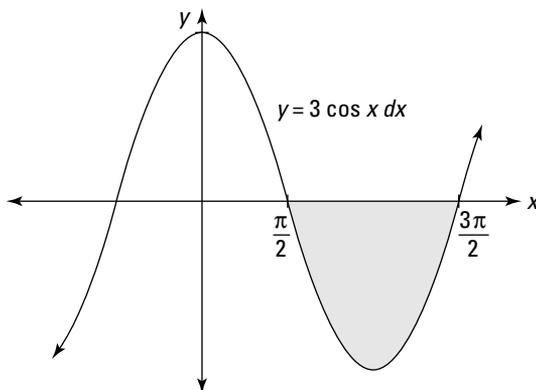
To use the analogy of income and savings, this is the moment when your income dries up and money starts flowing out. In other words, you're spending your savings, so your savings account balance starts to fall.

So, area above the x -axis is positive, but area below the x -axis is measured as negative area.

The definite integral takes this important distinction into account. It provides not just the area but the *signed area* of a region on the graph. For example, suppose that you want to measure the shaded area in Figure 3-11.

Figure 3-11:
Measuring
signed area
on the graph

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3 \cos x \, dx.$$



Here's how you do it using the steps that I outline in the previous section:

1. Formulate the area problem as a definite integral:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3 \cos x \, dx$$

2. Solve this definite integral as an indefinite integral:

$$= 3 \sin x \Big|_{x=\frac{\pi}{2}}^{x=\frac{3\pi}{2}}$$

3. Plug these limits of integration into the expression and simplify:

$$\begin{aligned} &= 3 \sin \frac{3\pi}{2} - 3 \sin \frac{\pi}{2} \\ &= -3 - 3 = -6 \end{aligned}$$

So, the signed area of the shaded region in Figure 3-11 equals -6 . As you can see, the computational method for evaluating the definite integral gives the signed area automatically.

As another example, suppose that you want to find the total area of the two shaded regions in Figure 3-10 and Figure 3-11. Here's how you do it using the steps that I outline in the previous section:

1. Formulate the area problem as a definite integral:

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \, dx$$

2. Solve this definite integral as an indefinite integral:

$$= 3 \sin x \Big|_{x=-\frac{\pi}{2}}^{x=\frac{3\pi}{2}}$$

3. Plug these limits of integration into the expression and simplify:

$$\begin{aligned} &= -3 \sin \frac{3\pi}{2} - 3 \sin \frac{\pi}{2} \\ &= 3 - 3 = 0 \end{aligned}$$

This time, the signed area of the shaded region is 0. This answer makes sense, because the unsigned area above the x -axis equals the unsigned area below it, so these two areas cancel each other out.

Distinguishing definite and indefinite integrals

Don't confuse the definite and indefinite integrals. Here are the key differences between them:

A definite integral

- ✓ Includes limits of integration (a and b)
- ✓ Represents the exact area of a specific set of points on a graph
- ✓ Evaluates to a number

An indefinite integral

- ✓ Doesn't include limits of integration
- ✓ Can be used to evaluate an infinite number of related definite integrals
- ✓ Evaluates to a function

For example, here's a *definite* integral:

$$\int_0^{\frac{\pi}{4}} \sec^2 x \, dx$$

As you can see, it includes limits of integration (0 and $\frac{\pi}{4}$), so you can draw a graph of the area that it represents. You can then use a variety of methods to evaluate this integral as a *number*. This number equals the signed area between the function and the x -axis inside the limits of integration, as I discuss earlier in "Understanding signed area."

In contrast, here's an *indefinite* integral:

$$\int \sec^2 x \, dx$$

This time, the integral doesn't include limits of integration, so it doesn't represent a specific area. Thus, it doesn't evaluate to a number, but to a function:

$$= \tan x + C$$

You can use this function to evaluate any related definite integral. For example, here's how to use it to evaluate the definite integral I just gave you:

$$\begin{aligned} & \int_0^{\frac{\pi}{4}} \sec^2 x \, dx \\ &= \tan x \Big|_{x=0}^{x=\frac{\pi}{4}} \\ &= 2 \tan \frac{\pi}{4} - \tan 0 \\ &= 1 - 0 = 1 \end{aligned}$$

So, the area of the shaded region in the graph is 1.

As you can see, the indefinite integral encapsulates an infinite number of related definite integrals. It also provides a practical means for evaluating definite integrals. Small wonder that much of Calculus II focuses on evaluating indefinite integrals. In Part II, I give you an ordered approach to evaluating indefinite integrals.