

Chapter 8

Being Respectful of Rational Functions

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In This Chapter

- ▶ Investigating domains and related vertical asymptotes
 - ▶ Looking at limits and horizontal asymptotes
 - ▶ Removing discontinuities of rational functions
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The best way to investigate rational functions is to look at the intercepts, the asymptotes, any removable discontinuities, and the limits to tell where the function values have been, what they're doing for particular values of the domain, and what they'll be doing for large values of x . You also need all this information to discuss or graph a rational function.

Whether you're graphing rational functions by hand or with a graphing calculator, you need to be able to recognize the various characteristics (intercepts, asymptotes, and so on) of the rational function. And, if you don't know what these characteristics are and how to find them, your calculator is no better than a paperweight to you.

Examining Rational Functions

You see *rational functions* written, in general, in the form of a fraction:

$$y = \frac{f(x)}{g(x)}, \text{ where } f \text{ and } g \text{ are polynomials}$$

Rational functions (and more specifically their graphs) are distinctive because of what they do and don't have. The

graphs of rational functions *do* have *asymptotes* (dotted lines drawn in to help with the shape and direction of the curve), and the graphs often *don't* have all the real numbers in their domains. Polynomials and exponential functions (which I cover in Chapters 7 and 9, respectively) make use of all the real numbers — their domains aren't restricted.

Deliberating on domain

As I explain in Chapter 5, the *domain* of a function consists of all the real numbers that you can insert into the function equation. Values in the domain have to work in the equation and avoid producing imaginary or nonexistent answers.

The following list illustrates some examples of domains of rational functions:

✓ The domain of $y = \frac{x-1}{x-2}$ is all real numbers except 2.

✓ The domain of $y = \frac{x+1}{x(x+4)}$ is all real numbers except 0 and -4.

Investigating intercepts

Functions in algebra can have intercepts (where the graph of the function crosses or touches an axis). A rational function may have an x -intercept and/or a y -intercept, but it doesn't have to have either. You can determine whether a given rational function has intercepts by looking at its equation.

Introducing zero to find y -intercepts

The coordinates $(0, b)$ represent the y -intercept of a rational function. To find the value of b , you substitute a 0 for x and solve for y . If 0 is in the domain of a rational function, you can be sure that the function at least has a y -intercept.

Making X mark the spot

The coordinates $(a, 0)$ represent an x -intercept of a rational function. To find the value(s) of a , you let y equal 0 and solve for x . (Basically, you just set the numerator of the fraction equal to 0, after you completely reduce the fraction.)



Find the intercepts of the rational function $y = \frac{x^2 - 64}{x^2 + 3x - 4}$.

First, to find the y -intercept you replace each x with 0 to get $y = \frac{0 - 64}{0 + 0 - 4} = \frac{-64}{-4} = 16$. The y -intercept is $(0, 16)$. To find the x -intercepts, set the numerator equal to 0 and solve for x . You get that $x^2 - 64 = (x - 8)(x + 8) = 0$. The two solutions are $x = 8$ and $x = -8$, so the x -intercepts are at $(8, 0)$ and $(-8, 0)$. When setting the numerator equal to 0 to get the x -intercepts, you need to be sure that none of the factors in the numerator is also in the denominator.

Assigning Roles to Asymptotes

The graphs of rational functions take on some distinctive shapes because of asymptotes. An asymptote is a sort of ghost line. *Asymptotes* are drawn into the graph of a rational function to show the shape and direction of the function's graph. The asymptotes aren't really part of the graphs. You lightly sketch in the asymptotes when you're graphing to help you with the final product. The types of asymptotes that you usually find in a rational function include the following:

- ✓ Vertical asymptotes
- ✓ Horizontal asymptotes
- ✓ Oblique (slant) asymptotes

In this section, I explain how you crunch the numbers of rational equations to identify asymptotes and graph them.

Validating vertical asymptotes

The equations of vertical asymptotes appear in the form $x = h$. This equation of a line has only the x variable — no y variable — and the number h . To find a vertical asymptote you establish, first, that in the rational function $y = \frac{f(x)}{g(x)}$, $f(x)$

and $g(x)$ have no common factors; then you determine when the denominator equals 0: $g(x) = 0$. The vertical asymptotes occur when the x -values make the denominator equal to 0.

EXAMPLE



Find the vertical asymptotes of the function $y = \frac{x}{x^2 - 4x + 3}$.

First note that there's no common factor in the numerator and denominator. Then set the denominator equal to 0. Factoring $x^2 - 4x + 3 = 0$, you get $(x - 1)(x - 3) = 0$. The solutions are $x = 1$ and $x = 3$, which are the equations of the vertical asymptotes.

Finding equations for horizontal asymptotes

The horizontal asymptote of a rational function has an equation that appears in the form $y = k$. This linear equation only has the variable y — no x — and the k is some number. A rational function has only one horizontal asymptote — if it has one at all (some rational functions have no horizontal asymptotes, others have one, and none of them has more than one). A rational function has a horizontal asymptote when the degree (highest power) of $f(x)$, the polynomial in the numerator, is less than or equal to the degree of $g(x)$, the polynomial in the denominator.

ALGEBRA RULES

$$\frac{1}{+1} / \frac{2}{2}$$

Here's a rule for determining the equation of a horizontal asymptote. The horizontal asymptote of

$$y = \frac{f(x)}{g(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_0} \text{ is:}$$

- ✓ $y = \frac{a_n}{b_m}$ when $n = m$, meaning that the highest degrees of the polynomials are the same. The fraction here is made up of the lead coefficients of the two polynomials.
- ✓ $y = 0$ when $n < m$, meaning that the degree in the numerator is less than the degree in the denominator.

EXAMPLE



Find the horizontal asymptote for $y = \frac{3x^4 - 2x^3 + 7}{x^4 - 3x^2 - 5}$.

The degree of the denominator is the same as the degree of the numerator. The horizontal asymptote is $y = 3$ (from $\frac{a_n}{b_m}$). The fraction formed by the lead coefficients is $y = \frac{3}{1} = 3$.

Taking vertical and horizontal asymptotes to graphs

When a rational function has one vertical asymptote and one horizontal asymptote, its graph usually looks like two flattened-out, C-shaped curves that appear diagonally opposite one another from the intersection of the asymptotes. Occasionally, the curves appear side by side, but that's the exception rather than the rule. Figure 8-1 shows you two examples of the more frequently found graphs in the one horizontal and one vertical classification.

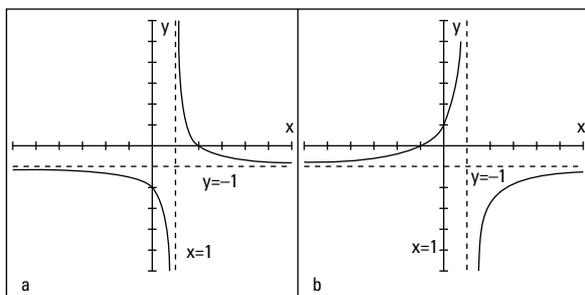


Figure 8-1: Rational functions approaching vertical and horizontal asymptotes.



Reconcile the rational equations $y = \frac{2-x}{x-1}$ and $y = \frac{x+1}{1-x}$ with the two graphs in Figure 8-1.

In both graphs, the vertical asymptotes are at $x = 1$, because the denominators are equal to 0 when $x = 1$. Also, in both graphs, the horizontal asymptotes are at $y = -1$.

In $y = \frac{2-x}{x-1}$, the highest power in both numerator and denominator is 1. You get $y = \frac{-1}{1}$. Letting $x = 0$, you get a y -intercept of $(0, -2)$. Letting $y = 0$, you get an x -intercept of $(2, 0)$. So this equation corresponds to Figure 8-1a.

The horizontal asymptote of function $y = \frac{x+1}{1-x}$ is $y = \frac{1}{-1}$. Letting $x = 0$, you get a y -intercept of $(0, 1)$. Letting $y = 0$, you get an x -intercept of $(-1, 0)$. So this equation corresponds to Figure 8-1b.



The graph of a rational function can cross a horizontal asymptote, but it never crosses a vertical asymptote. Horizontal asymptotes show what happens for very large or very small values of x .

Getting the scoop on oblique (slant) asymptotes

An *oblique* or *slant asymptote* takes the form $y = ax + b$. You may recognize this form as the slope-intercept form for the equation of a line. A rational function has a slant asymptote when the degree of the polynomial in the numerator is exactly one value greater than the degree in the denominator ($\frac{x^4}{x^3}$, for example).

You can find the equation of the slant asymptote by using long division. You divide the denominator of the rational function into the numerator and use the first two terms in the answer. Those two terms are the $ax + b$ part of the equation of the slant asymptote.



Find the slant asymptote of $y = \frac{x^4 - 3x^3 + 2x - 7}{x^3 + 3x - 1}$.

Doing the long division:

$$\begin{array}{r}
 x^3 + 3x - 1 \overline{) x^4 - 3x^3 } \\
 \underline{x^4 + 3x^2 - x } \\
 -3x^3 - 3x^2 + 3x - 7 \\
 \underline{-3x^3 - 9x + 3} \\
 -3x^2 + 12x - 10
 \end{array}$$

You can ignore the remainder at the bottom. The slant asymptote for this example is $y = x - 3$. (For more on long division of polynomials, see *Algebra Workbook For Dummies*, by yours truly and published by Wiley.)

An oblique (or slant) asymptote creates two new possibilities for the graph of a rational function. If a function has an oblique asymptote, its curve tends to be a very-flat C on opposite sides of the intersection of the slant asymptote

and a vertical asymptote (see Figure 8-2a), or the curve has U-shapes between the asymptotes (see Figure 8-2b).

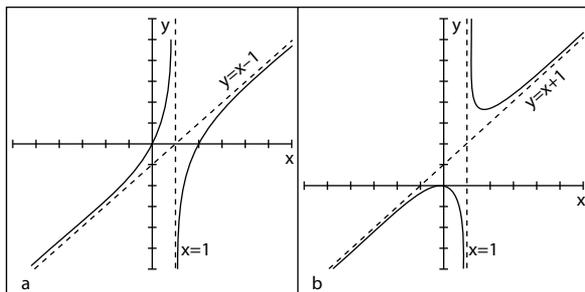


Figure 8-2: Rational graphs between vertical and oblique asymptotes.

Figure 8-2a has a vertical asymptote at $x = 1$ and a slant asymptote at $y = x - 1$; its intercepts are at $(0, 0)$ and $(2, 0)$. Figure 8-2b has a vertical asymptote at $x = 1$ and a slant asymptote at $y = x + 1$; its only intercept is at $(0, 0)$.

Discounting Removable Discontinuities

A *discontinuity* in the graph of a function is just what the word suggests: a break or pause in the action. Vertical asymptotes mark discontinuities. The domain does not include values at the vertical asymptotes. But rational functions can sometimes have *removable discontinuities*. The removable designation is, however, a bit misleading. The gap in the domain still exists at that “removable” spot, but the function values and graph of the curve tend to behave a little better at x -values where there’s a removable discontinuity. The function values stay close together — they don’t spread far apart — and the graphs just have tiny holes, not vertical asymptotes where the graphs rise to positive infinity or plunge to negative infinity.

Removable discontinuities are found when you’re factoring the original function statement — if it does factor. If the numerator and denominator don’t have a common factor, then there isn’t a removable discontinuity.

Finding removable discontinuities by factoring

Discontinuities are *removed* when they no longer have an effect on the rational function equation. You know this is the case when you find a factor that's common to both the numerator and the denominator. You accomplish the removal process by factoring the polynomials in the numerator and denominator of the rational function and then reducing the fraction.

To remove the discontinuity in the rational function

$y = \frac{x^2 - 4}{x^2 - 5x - 14}$, for example, you first factor the fraction and reduce:

$$y = \frac{(x-2)(x+2)}{(x-7)(x+2)} = \frac{(x-2)\cancel{(x+2)}}{(x-7)\cancel{(x+2)}}$$

The *removable discontinuity* occurs when $x = -2$. Now you have a new function statement:

$$y = \frac{x-2}{x-7}$$

By getting rid of the removable discontinuity, you simplify the equation that you're graphing. Now you need only work with the new equation which shows you a function with vertical asymptote of $x = 7$, horizontal asymptote of $y = 1$, y -intercept of $(0, \frac{2}{7})$, and x -intercept of $(2, 0)$. You also will have a "hole" in the graph when $x = -2$. Substituting that x -value into the new equation, you get $y = \frac{4}{9}$. So the small hole in the graph, marking the discontinuity, is at $(-2, \frac{4}{9})$.

Evaluating the removals

You need to take care when *removing* discontinuities. Numbers excluded from the domain stay excluded even after you remove the discontinuity. The function still isn't defined for any values you find before the procedure. It's just that the function behaves differently at the different types of discontinuities. When the graph of a function has a hole, the curve approaches the value,

skips it, and goes on. It behaves in a reasonable fashion: The function values skip over the discontinuity, but the x -values can get really close to it. When a vertical asymptote appears, though, the discontinuity doesn't go away. The function values go haywire — they're unrestrained as the x -values get close.

Figure 8-3 shows a rational function with a vertical asymptote at $x = -2$ and a removable discontinuity at $x = 3$. The horizontal asymptote is the x -axis (written $y = 0$). Unfortunately, graphing calculators don't show the little hollow circles indicating removable discontinuities. Oh, sure, they leave a gap there, but the gap is only 1 pixel wide, so you can't see it with the naked eye. You just have to know that the discontinuity is there. We're still better than the calculators!

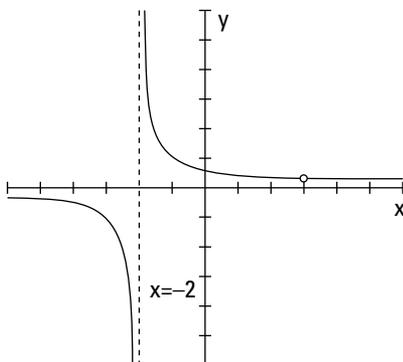


Figure 8-3: A removable discontinuity at the coordinates (3, 0.2).

Looking at Limits of Rational Functions

The *limit* of a rational function is a description of its behavior — telling you what the function equation is doing as you get nearer and nearer to some input value. If a function has a limit as you approach a number, then, as you get closer from the left or from the right, you home in on a particular function value. The function doesn't have to be defined at the number you're approaching (sometimes they are and sometimes not) — there could be a discontinuity at the point you're investigating. But, if a limit exists, the values of the function have a recognizable pattern.



Here is the special notation for limits:

$$\lim_{x \rightarrow a} f(x) = L$$

You read the notation as, “The limit of the function, $f(x)$, as x approaches the number a , is equal to L .” The number a doesn’t have to be in the domain of the function. You can talk about a limit of a function regardless of whether a is in the domain. And you can approach a ; you don’t actually reach it.

Determining limits at function discontinuities

The beauty of a limit is that it can also work when a rational function isn’t defined at a particular number. The function $y = \frac{x-2}{x^2-2x}$, for example, is discontinuous at $x = 0$ and at $x = 2$. You find these numbers by factoring the denominator, setting it equal to 0 — $x(x-2) = 0$ — and solving for x . This function has no limit when x approaches 0, but it has a limit when x approaches 2. Sometimes it’s helpful to actually see the numbers — see what you get from evaluating a function at different values — so I’ve included Table 8-1. It shows what happens as x approaches 0 from the left and right, and it illustrates that the function has no limit at that value.

Table 8-1 **Approaching $x = 0$ from Both Sides in $y = \frac{x-2}{x^2-2x}$**

<i>x</i> <i>Approaching</i> <i>0 from the</i> <i>Left</i>	<i>Corresponding</i> <i>Behavior of</i> $\lim_{x \rightarrow 0} \frac{x-2}{x^2-2x} \rightarrow -\infty$	<i>x</i> <i>Approaching</i> <i>0 from the</i> <i>Right</i>	<i>Corresponding</i> <i>Behavior of</i> $\lim_{x \rightarrow 0} \frac{x-2}{x^2-2x} \rightarrow \infty$
-1.0	-1	1.0	1
-0.5	-2	0.5	2
-0.1	-10	0.1	10
-0.001	-1,000	0.001	1,000
-0.00001	-100,000	0.00001	100,000

Table 8-1 shows you that $\lim_{x \rightarrow 0} \frac{x-2}{x^2-2x}$ doesn't exist. As x approaches from below the value of 0, the values of the function drop down lower and lower toward negative infinity. Coming from above the value of 0, the values of the function raise higher and higher toward positive infinity. The sides will never come to an agreement; no limit exists.

Table 8-2 shows you how a function can have a limit even when the function isn't defined at a particular number. Sticking with the previous example function, you find a limit as x approaches 2.

<i>x</i> <i>Approaching 2 from the Left</i>	<i>Corresponding Behavior in $\lim_{x \rightarrow 0} \frac{x-2}{x^2-2x} \rightarrow 0.5$</i>	<i>x</i> <i>Approaching 2 from the Right</i>	<i>Corresponding Behavior in $\lim_{x \rightarrow 0} \frac{x-2}{x^2-2x} \rightarrow 0.5$</i>
1.0	1.0	3.0	0.3333...
1.5	0.6666...	2.5	0.4
1.9	0.526316...	2.1	0.476190...
1.99	0.502513...	2.001	0.499750...
1.999	0.500250...	2.00001	0.4999975...

Table 8-2 shows $\lim_{x \rightarrow 2} \frac{x-2}{x^2-2x} = 0.5$.

The numbers get closer and closer to 0.5 as x gets closer and closer to 2 from both directions. You find a limit at $x = 2$, even though the function isn't defined there.

Determining a limit algebraically

If you've examined the two tables from the previous section, you may think that the process of finding limits is exhausting. Allow me to tell you that algebra offers a much easier way to find limits — if they exist.

Functions with removable discontinuities have limits at the values where the discontinuities exist. In the "Finding

removable discontinuities by factoring” section, earlier in this chapter, I show you the process needed to remove a discontinuity. I apply the same technique here.

To solve for the limit when $x = 2$ in the rational function

$y = \frac{x-2}{x^2-2x}$, you first factor and then reduce the fraction:

$$y = \frac{x-2}{x^2-2x} = \frac{\cancel{x-2}}{x(\cancel{x-2})} = \frac{1}{x}$$

Now you replace the x with 2 and get $y = 0.5$, the limit when

$x = 2$. I write that as $\lim_{x \rightarrow 2} \frac{x-2}{x^2-2x} = 0.5$. In general, if a ratio-

nal function factors, then you’ll find a limit at the number excluded from the domain if the factoring makes that exclusion seem to disappear.

Determining whether the function has a limit

Some rational functions have limits at discontinuities and some don’t. You can determine whether to look for a removable discontinuity in a particular function by first trying the x -value in the function. Replace all the x ’s in the function with the number in the limit (what x is approaching). The result of that substitution may tell you if you have a limit or not. You use the following rules of thumb:

✔ If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\text{some nonzero number}}{0}$, the function has no limit at a .

✔ If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$, the function may have a limit at a . You reduce the fraction and evaluate the newly formed function equation at a .

Finding infinity

When a rational function doesn’t have a limit at a particular value, the function values and graph have to go somewhere. Even though the function has no limit at some value, you can still say something about the behavior of the function. The behavior is described with *one-sided limits*.



A one-sided limit tells you what a function is doing as the x -value of the function approaches some number from one side or the other. One-sided limits are more restrictive; they only work from the left or from the right.

Here is the notation for indicating one-sided limits from the left or right:

✓ The limit as x approaches the value a from the left is $\lim_{x \rightarrow a^-} f(x)$.

✓ The limit as x approaches the value a from the right is $\lim_{x \rightarrow a^+} f(x)$.



Do you see the little positive or negative sign after the a ? You can think of *from the left* as coming from the same direction as all the negative numbers on the number line and *from the right* as coming from the same direction as all the positive numbers.

Table 8-3 shows some values of the function $y = \frac{1}{x-3}$, which has a vertical asymptote at $x = 3$.

Table 8-3 Approaching $x = 3$ from Both Sides in $y = \frac{1}{x-3}$

<i>Approaching 3 from the Left</i>	<i>Corresponding Behavior in</i> $y = \frac{1}{x-3}$	<i>x Approaching 3 from the Right</i>	<i>Corresponding Behavior in</i> $y = \frac{1}{x-3}$
2.0	-1	4.0	1
2.5	-2	3.5	2
2.9	-10	3.1	10
2.999	-1,000	3.001	1,000
2.99999	-100,000	3.00001	100,000

You express the one-sided limits for the function from Table 8-3 as follows:

$$\lim_{x \rightarrow 3^-} \frac{1}{x-3} = -\infty, \quad \lim_{x \rightarrow 3^+} \frac{1}{x-3} = +\infty$$

The function goes down to negative infinity as it approaches 3 from below the value and up to positive infinity as it approaches 3 from above the value. “And nary the twain shall meet.”

Looking at infinity

The previous section describes how function values can go to positive or negative infinity as x approaches some specific number. This section also talks about infinity, but it focuses on what rational functions do as their x -values become very large or very small (approaching infinity themselves).

A function such as the parabola $y = x^2 + 1$ opens upward. If you let x be some really big number, y gets very big, too. Also, when x is very small (a “big” negative number), you square the value, making it positive, so y is very big for the small x . In function notation, you describe what’s happening to this function as the x -values approach infinity with $\lim_{x \rightarrow \infty} (x^2 + 1) = +\infty$.

In the case of rational functions, the limits at infinity — as x gets very large or very small — may be specific, finite, describable numbers. In fact, when a rational function has a horizontal asymptote, its limit at infinity is the same value as the number in the equation of the asymptote.

If you’re looking for the horizontal asymptote of the function $y = \frac{4x^2 + 3}{2x^2 - 3x - 7}$, for example, you can use the rules in the section “Taking vertical and horizontal asymptotes to graphs” to determine that the horizontal asymptote of the function is $y = 2$. Using limit notation, you can write the solution as $\lim_{x \rightarrow \infty} \frac{4x^2 + 3}{2x^2 - 3x - 7} = 2$.



The proper algebraic method for evaluating limits at infinity is to divide every term in the rational function by the highest power of x in the fraction and then look at each term. Here’s an important property to use: As x approaches infinity, any term with $\frac{1}{x}$ or $\frac{1}{x^2}$ or $\frac{1}{x^3}$, and so on in it approaches 0 — in other words, gets very small — so you can replace those terms with 0 and simplify.

Here's how the property works when evaluating the limit of the previous example function, $y = \frac{4x^2 + 3}{2x^2 - 3x - 7}$. The highest power of the variable in the fraction is x^2 , so every term is divided by x^2 :

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{4x^2 + 3}{2x^2 - 3x - 7} &= \lim_{x \rightarrow \infty} \frac{\frac{4x^2}{x^2} + \frac{3}{x^2}}{\frac{2x^2}{x^2} - \frac{3x}{x^2} - \frac{7}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{4 + \frac{3}{x^2}}{2 - \frac{3}{x} - \frac{7}{x^2}} \\ &= \frac{4 + 0}{2 - 0 - 0} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

The limit as x approaches infinity is 2. As predicted, the number 2 is the number in the equation of the horizontal asymptote. The quick method for determining horizontal asymptotes is an easier way to find limits at infinity; this algebraic procedure is the correct *mathematical* way of doing it — and it shows why the other rule (the quick method) works.

