Chapter 4

Rolling Along with Rational and Radical Equations

In This Chapter

- Solving equations containing radicals and fractional exponents
- Working with negative exponents
- Recognizing quadratic-like equations and using unFOIL

Solving an algebraic equation requires some know-how. You need a game plan to solve equations with fractions, radicals, and negative or fractional exponents — one that involves careful planning and a final check of your answers. In this chapter, you find out how to tackle equations by changing them into new versions that are more familiar and easier to solve. You also see a recurring theme of *check your answers,* because changing equations into different forms can introduce mysterious strangers into the mix — in the form of false answers.

Rounding Up Rational Equations and Eliminating Fractions

An equation with one or more terms, at least one of which is rational, is called a *rational equation.* You probably hope that all your problems (and the people you associate with) are rational, but an equation that contains fractions isn't always easy to handle.

A general plan for solving a rational equation is to get rid of the fraction or fractions by changing the equation into an equivalent form with the same answer — a form that makes it easier to solve.

Two of the most common ways to get rid of the fractions are multiplying through by the least common denominator (LCD) and cross-multiplying proportions. I just happen to discuss both of these techniques in the sections that follow.

This mathematical sleight of hand — using alternate equations to solve more complicated problems — isn't without its potential problems. At times, the new equation produces an *extraneous solution* (also referred to as an *extraneous root*), a false solution that pops up because you messed around with the original format of the equation. To guard against including extraneous solutions in your answers, you need to check the solutions you come up with in the original equations.

Making your least common denominator work for you

You can solve many rational equations by simply getting rid of all the denominators (which gets rid of the fractions). To do so, you introduce the LCD into the problem. The LCD is the smallest number that all the denominators in the problem divide into evenly (such as 2, 3, and 4 all dividing the LCD 12 evenly).

To solve an equation using the LCD, you find the common denominator, write each fraction with that common denominator, and then multiply each side of the equation by that same denominator to get a nice fraction-less equation. The new equation is in an easier form to solve. I'll show you the step-by-step process with this example:

Solve for x in $\frac{3x+2}{2}-\frac{5}{2x-3}=\frac{x+3}{4}$.

1. **Find a common denominator.**

   The LCD is a multiple of each of the original denominators. To solve this equation, use $4(2x-3)$ as the LCD. All three denominators divide this product evenly.

2. **Write each fraction with the common denominator.**

   Multiply each fraction by the equivalent of 1. The numerator and denominator are the same, and the denominator is what is needed to change the original denominator into the LCD:

$$\frac{3x+2}{2}\cdot\frac{2(2x-3)}{2(2x-3)}-\frac{5}{2x-3}\cdot\frac{4}{4}=\frac{x+3}{4}\cdot\frac{2x-3}{2x-3}$$

   Completing the multiplication:

$$\frac{2(3x+2)(2x-3)}{4(2x-3)}-\frac{20}{4(2x-3)}=\frac{(x+3)(2x-3)}{4(2x-3)}$$

3. **Multiply each side of the equation by that same denominator.**

   Multiply each term in the equation by the LCD; then reduce each term and get rid of the denominators:

$$\cancel{4(2x-3)}\cdot\frac{2(3x+2)(2x-3)}{\cancel{4(2x-3)}}-\cancel{4(2x-3)}\cdot\frac{20}{\cancel{4(2x-3)}}=$$
$$\cancel{4(2x-3)}\cdot\frac{(x+3)(2x-3)}{\cancel{4(2x-3)}}$$
$$2(3x+2)(2x-3)-20=$$
$$(x+3)(2x-3)$$

4. **Solve the new equation.**

   To solve the new quadratic equation, you multiply out the terms, simplify, and set the equation equal to 0:

$$2(3x+2)(2x-3)-20=(x+3)(2x-3)$$
$$12x^2-10x-12-20=2x^2+3x-9$$
$$10x^2-13x-23=0$$

   Now you find out if the quadratic equation factors. If it doesn't factor, you can resort to the quadratic formula; fortunately, that isn't necessary here. After factoring, you set each factor equal to 0 and solve for x:

$$10x^2-13x-23=0$$
$$(10x-23)(x+1)=0$$
$$10x-23=0,\ x=\frac{23}{10}$$
$$x+1=0,\ x=-1$$

You find two solutions for the quadratic equation:

$x = \frac{23}{10}$ and $x = -1$.

5. **Check your answers to avoid extraneous solutions.**

   You now have to check to be sure that both your solutions work in the *original* equation. ***Remember:*** One or both may be extraneous solutions.

   Checking the original equation to see if the two solutions work, you first look at $x = -1$. Replace each x with -1:

$$\frac{3(-1)+2}{2} - \frac{5}{2(-1)-3} = \frac{(-1)+3}{4}$$
$$\frac{-3+2}{2} - \frac{5}{-5} = \frac{2}{4}$$
$$-\frac{1}{2} + 1 = \frac{1}{2}$$
$$\frac{1}{2} = \frac{1}{2}$$

   Nice! The first solution works. The next check is to see if $x = \frac{23}{10}$ is a solution. And right now I'm going to take "author's privilege" and tell you that yes, the answer works. It takes more space than I have to show you all the steps, so I'm going to ask you to trust me and skip all the gory details. The two solutions of the rational equation are $x = -1$ and $x = \frac{23}{10}$.

Proposing proportions for solving rational equations

A *proportion* is an equation in which one fraction is set equal to another. Proportions have several very nice features that make them desirable to work with when you're solving rational equations because you can eliminate the fractions or change them so that they feature better denominators. Also, they factor in four different ways.

ALGEBRA RULES

When you have the proportion $\frac{a}{b} = \frac{c}{d}$, the following are also true:

- ad and bc, the cross-products, are equal, giving you $ad = bc$.

- $\frac{b}{a}$ and $\frac{d}{c}$, the reciprocals, are equal, giving $\frac{b}{a} = \frac{d}{c}$.
- You can divide out common factors both horizontally and vertically.

Solve for x in the proportion: $\frac{80x}{16} = \frac{30}{x-5}$.

First reduce across the numerators, and then reduce the left fraction:

$$\frac{{}^{8}\cancel{80}x}{16} = \frac{\cancel{30}^{3}}{x-5} \text{ becomes } \frac{8x}{16} = \frac{3}{x-5}$$

$$\frac{{}^{1}\cancel{8}x}{{}_{2}\cancel{16}} = \frac{3}{x-5} \text{ becomes } \frac{x}{2} = \frac{3}{x-5}$$

Now cross-multiply and solve the resulting quadratic equation:

$$\begin{aligned} x(x-5) &= 6 \\ x^2 - 5x &= 6 \\ x^2 - 5x - 6 &= 0 \\ (x-6)(x+1) &= 0 \\ x = 6 &\text{ or } x = -1 \end{aligned}$$

As usual, you need to check to be sure that you haven't introduced any extraneous roots. Both solutions work!

Reasoning with Radicals

A radical in an equation often indicates that you want to find a root — the square root of a number, its cube root, and so on. A radical (root) adds a whole new dimension to what could've been a perfectly nice equation to solve. In general, you deal with radicals in equations the same way you deal with fractions in equations — you get rid of them. But watch out: extraneous answers often crop up in your work, so you have to check your answers.

Squaring both sides of the equation

If you have an equation with a square root term in it, you square both sides of the equation to get rid of the radical.

EXAMPLE

Solve for x in $\sqrt{4x+21}-6=x$.

First, add 6 to both sides of the equation to get the radical by itself on the left. Then square both sides of the equation.

$$\begin{aligned}\sqrt{4x+21}&=x+6\\\left(\sqrt{4x+21}\right)^2&=(x+6)^2\\4x+21&=x^2+12x+36\end{aligned}$$

Now set the quadratic equation equal to 0 and solve it:

$$\begin{aligned}4x+21&=x^2+12x+36\\0&=x^2+8x+15\\0&=(x+3)(x+5)\\x&=-3\text{ or }x=-5\end{aligned}$$

The two solutions work for the quadratic equation that was created, but they don't necessarily work in the original equation. Check the work!

When $x=-3$, you get

$$\begin{aligned}\sqrt{4(-3)+21}-6&=\sqrt{-12+21}-6\\&=\sqrt{9}-6\\&=3-6\\&=-3\end{aligned}$$

The solution $x=-3$ works. Checking $x=-5$, you get

$$\begin{aligned}\sqrt{4(-5)+21}-6&=\sqrt{-20+21}-6\\&=\sqrt{1}-6\\&=-5\end{aligned}$$

This solution works, too.

Both solutions working out is more the exception rather than the rule. Most of the time, one solution or the other works, but not both. And, unfortunately, sometimes you go through all the calculations and find that neither solution works in the original equation. You get an answer, of course (that there is no answer), but it isn't very fulfilling.

Taking on two radicals

Some equations that contain radicals call for more than one application of squaring both sides. For example, you have to square both sides more than once when you can't isolate a radical term by itself on one side of the equation. And you usually need to square both sides more than once when you have three terms in the equation — two of them with radicals.

Solve $\sqrt{3x+19}-\sqrt{5x-1}=2$.

1. **Move the radicals so that only one appears on each side.**
2. **Square both sides of the equation.**

   After the first two steps, you have the following:

   $$\left(\sqrt{3x+19}\right)^2=\left(2+\sqrt{5x-1}\right)^2$$
   $$3x+19=4+4\sqrt{5x-1}+5x-1$$

3. **Move all the nonradical terms to the left and simplify.**

   This gives you the following:

   $$3x+19-4-5x+1=4\sqrt{5x-1}$$
   $$-2x+16=4\sqrt{5x-1}$$

4. **Make the job of squaring the binomial on the left easier by dividing each term by 2 — the common factor of all the terms on both sides. Then square both sides, simplify, set the quadratic equal to 0, and solve for *x*.**

$$\begin{aligned}
-x+8 &= 2\sqrt{5x-1}\\
(-x+8)^2 &= \left(2\sqrt{5x-1}\right)^2\\
x^2-16x+64 &= 4(5x-1)\\
x^2-16x+64 &= 20x-4\\
x^2-36x+68 &= 0\\
(x-2)(x-34) &= 0\\
x = 2 &\text{ or } x = 34
\end{aligned}$$

The two solutions you come up with are $x = 2$ and $x = 34$. Both have to be checked in the original equation. When $x = 2$,

$$\begin{aligned}
\sqrt{3(2)+19}-\sqrt{5(2)-1} &= \sqrt{25}-\sqrt{9}\\
&= 5-3\\
&= 2
\end{aligned}$$

When $x = 34$,

$$\begin{aligned}
\sqrt{3(34)+19}-\sqrt{5(34)-1} &= \sqrt{121}-\sqrt{169}\\
&= 11-13\\
&= -2
\end{aligned}$$

The solution $x = 2$ works. The other solution, $x = 34$, doesn't work in the equation. The number 34 is an extraneous solution.

Dealing with Negative Exponents

Equations with negative exponents offer some unique challenges. In general, negative exponents are easier to work with if they disappear. Yes, as wonderful as negative exponents are in the world of mathematics, solving equations that contain them is often easier if you can change the format to positive exponents and fractions and then deal with solving the fractional equations (as shown in the previous section). What I do in this section, though, is show you how to handle negative exponents without resorting to the fractions.

A common type of equation with negative exponents is one with a mixture of powers. I show you how to deal with these particular equations by factoring out a greatest common factor (GCF). Another common negative-exponent problem is one that's quadratic-like.

Factoring out a negative exponent as a greatest common factor

The next example shows you, step-by-step, how to deal with an equation with negative exponents that can be solved by factoring.

Solve $3x^{-3} - 5x^{-2} = 0$ for x.

1. **Factor out the GCF.**

   In this case, the GCF is x^{-3}:

   $x^{-3}(3 - 5x) = 0$

   Did you think the exponent of the GCF was –2? ***Remember:*** –3 is smaller than –2. When you factor out a GCF, you choose the smallest exponent out of all the choices and then divide each term by that common factor.

2. **Set each term in the factored form equal to 0 to solve for *x*.**

   You end up with:

   $$x^{-3}(3-5x)=0$$
   $$x^{-3}=0,\ \frac{1}{x^3}=0$$
   $$3-5x=0,\ x=\frac{3}{5}$$

   The first equation has no solution. The fraction with 1 in the numerator and x^3 in the denominator is never equal to 0. The only way a fraction is equal to 0 is if the numerator is 0 (and the denominator is some other number).

3. **Check your answers.**

   The only solution for this equation is $\frac{3}{5}$ — a perfectly dandy answer.

$$3\left(\frac{3}{5}\right)^{-3} - 5\left(\frac{3}{5}\right)^{-2} = 3\left(\frac{5}{3}\right)^{3} - 5\left(\frac{5}{3}\right)^{2}$$
$$= 3\left(\frac{125}{27}\right) - 5\left(\frac{25}{9}\right)$$
$$= \frac{125}{9} - \frac{125}{9}$$
$$= 0$$

Solving quadratic-like trinomials

Trinomials are expressions with three terms, with the highest term raised to the second degree, the expression is quadratic. You can simplify quadratic trinomials by factoring them into two binomial factors. (See Chapter 3 for details on factoring quadratic-like trinomials.)

EXAMPLE

Solve the trinomial equation $3x^{-2} + 5x^{-1} - 2 = 0$.

You find the quadratic-like pattern: $ax^{-2n} + bx^{-n} + c$. Factoring and setting the two factors equal to 0:

$$\left(3x^{-1} - 1\right)\left(x^{-1} + 2\right) = 0$$
$$3x^{-1} - 1 = 0,\ \frac{3}{x} = 1,\ x = 3$$
$$x^{-1} + 2 = 0,\ \frac{1}{x} = -2,\ x = -\frac{1}{2}$$

You produce two solutions, and both work when substituted into the original equation.

WARNING!

Be careful when solving an equation containing negative exponents — when the equation involves taking an even root (square root, fourth root, and so on). Watch out for zeros in the denominator, because those numbers don't exist, and be wary of imaginary numbers — they exist somewhere, in some mathematician's imagination. Factoring into binomials is a nifty way of solving equations with negative exponents — just be sure to proceed cautiously.

Fiddling with Fractional Exponents

You use fractional exponents ($x^{\frac{1}{2}}$, for example) to replace radicals and powers under radicals. Writing terms with fractional exponents allows you to perform operations on terms more easily when they have the same base or variable.

Solving equations by factoring fractional exponents

You can easily factor expressions that contain variables with fractional exponents if you know the rule for dividing numbers with the same base. To factor the expression $2x^{\frac{1}{2}} - 3x^{\frac{1}{3}}$, for example, you note that the smaller of the two exponents is the fraction $\frac{1}{3}$. Factor out *x* raised to that lower power, changing to a common denominator where necessary:

$$2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} = x^{\frac{1}{3}}\left(2x^{\frac{1}{6}} - 3\right)$$

A good way to check your factoring work is to mentally distribute the first term through the terms in parentheses to be sure that the product is what you started with.

Promoting techniques for working with fractional exponents

Fractional exponents represent radicals and powers. Some equations with fractional exponents are solved by raising each side to an appropriate power to get rid of the fraction in the exponent. Other equations require various methods for solving equations, such as factoring.

Factoring out the greatest common factor

You don't always have the luxury of being able to raise each side of an equation to a power to get rid of the fractional exponents. Your next best plan of attack involves factoring out the variable with the smaller exponent and setting the two factors equal to 0.

Solve $x^{\frac{5}{6}} - 3x^{\frac{1}{2}} = 0$.

First factor out an x with the exponent of $\frac{1}{2}$. Then set the two factors equal to 0 to solve for x.

$$x^{\frac{5}{6}} - 3x^{\frac{1}{2}} = 0$$
$$x^{\frac{1}{2}}\left(x^{\frac{1}{3}} - 3\right) = 0$$
$$x^{\frac{1}{2}} = 0,\ x = 0$$
$$x^{\frac{1}{3}} - 3 = 0,\ x^{\frac{1}{3}} = 3,\ \left(x^{\frac{1}{3}}\right)^3 = (3)^3,\ x = 27$$

You come up with two perfectly civilized answers: $x = 0$ and $x = 27$.

Factoring quadratic-like fractional terms

Often, you can factor trinomials with fractional exponents into the product of two binomials. This is another version of the quadratic-like trinomials. After the factoring, you set the two binomials equal to 0 to determine if you can find any solutions.

Solve $x^{\frac{1}{2}} - 6x^{\frac{1}{4}} + 5 = 0$.

First, factor the left side into the product of two binomials. The exponent of the first term is twice that of the second, which should indicate to you that the trinomial has factoring potential. After you factor, you set the expression equal to 0 and solve for x:

$$\left(x^{\frac{1}{4}} - 1\right)\left(x^{\frac{1}{4}} - 5\right) = 0$$
$$x^{\frac{1}{4}} - 1 = 0,\ x^{\frac{1}{4}} = 1,\ \left(x^{\frac{1}{4}}\right)^4 = (1)^4,\ x = 1$$
$$x^{\frac{1}{4}} - 5 = 0,\ x^{\frac{1}{4}} = 5,\ \left(x^{\frac{1}{4}}\right)^4 = (5)^4,\ x = 625$$

Check your answers in the original equation; you find that both $x = 1$ and $x = 625$ work.