Chapter 10

Getting Creative with Conics

In This Chapter

- Determining the centers of circles, ellipses, and hyperbolas
- Graphing parabolas using vertex and direction
- Using equations to sketch graphs of all conics

Conic is the name given to a special group of curves. The four conic sections are a parabola, circle, ellipse, and hyperbola.

Each conic section has a specific form or type of equation, and I cover each in this chapter. You can glean a good deal of valuable information from a conic section's equation, such as where it's centered in a graph, how wide it opens, and its general shape. I also discuss the techniques that work best for you when you're called on to graph conics.

The graphs of circles and ellipses are closed curves. Parabolas and hyperbolas open upward, downward, left, or right — depending on the type you're graphing. Just to acquaint you with what conic sections look like, I show you some graphs in Figure 10-1. Then, in subsequent sections, I give you all the details in terms of the characteristics and important features of the individual conics.

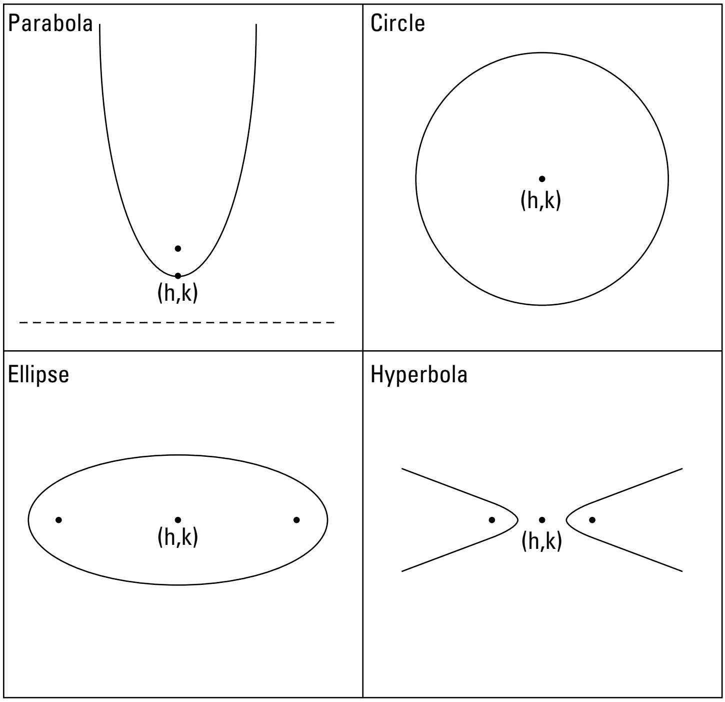

Figure 10-1: The four conic sections.

Posing with Parabolas

A *parabola,* a U-shaped conic that I first introduce in Chapter 6 (the parabola is the only conic section that fits the definition of a polynomial), is defined as all the points that fall the same distance from some fixed point, called its *focus,* and a fixed line, called its *directrix.* The focus is denoted by F, and the directrix by $y = d$ (assuming the parabola opens up or down).

A parabola has a couple other defining features. The *axis of symmetry* of a parabola is a line that runs through the focus and is perpendicular to the directrix. The axis of symmetry does just what its name suggests: It shows off how symmetric a parabola is. A parabola is a mirror image on either side of its axis. Another feature is the parabola's *vertex.* The vertex is the curve's extreme point — the lowest or highest point, or the point on the curve farthest right or farthest left. The vertex is also the point where the axis of symmetry crosses the curve.

Generalizing the form of a parabola's equation

The curves of parabolas can open upward, downward, to the left, or to the right; they also can be steep (tight) or widespread. The vertex can be anywhere in the coordinate plane. So, how do you track the curves down to pin them on a graph? You look to their equations, which give you all the information you need to find out where they've wandered to.

Opening left or right

When the vertex of a parabola is at the point (h, k), and the general form for the equation is as follows, the parabola opens left or right:

$$(y - k)^2 = 4a(x - h)$$

When the y variable is squared, the parabola opens left or right. From this equation, you can extract information about the elements:

- If $4a$ is positive, the curve opens right; if $4a$ is negative, the curve opens left.
- If $|4a| > 1$, the parabola is relatively wide; if $|4a| < 1$, the parabola is relatively narrow.
- The focus is at the point $(h + a, k)$.
- The directrix is $x = h - a$.

Opening up or down

When the vertex of a parabola is at the point (h, k), and the general form for the equation is as follows, the parabola opens up or down:

$$(x - h)^2 = 4a(y - k)$$

When the x variable is squared, the parabola opens up or down. Here's the info you can extract from this equation:

- If $4a$ is positive, the parabola opens upward; if $4a$ is negative, the curve opens downward.
- If $|4a| > 1$, the parabola is wide; if $|4a| < 1$, the parabola is narrow.
- The focus is at the point $(h, k + a)$.
- The directrix is $y = k - a$.

Making short work of a parabola's sketch

Parabolas have distinctive U-shaped graphs, and with just a little information, you can make a relatively accurate sketch of the graph of a particular parabola. The first step is to think of all parabolas as being in one of the general forms I list in the previous section.

Here's the full list of steps to follow when sketching the graph of a parabola — either $(x - h)^2 = 4a(y - k)$ or $(y - k)^2 = 4a(x - h)$:

1. **Determine the coordinates of the vertex (h, k) and plot that vertex.**

   If the equation contains $(x + h)$ or $(y + k)$, change the forms to $(x - [-h])$ or $(y - [-k])$, respectively, to determine the correct signs. Actually, you're just reversing the sign that's already there.

2. **Determine the direction the parabola opens, and decide if it's wide or narrow, by looking at the $4a$ portion of the general parabola equation.**

3. **Lightly sketch in the axis of symmetry that goes through the vertex.**

   $x = h$ when the parabola opens up or down and $y = k$ when it opens left or right.

4. **Choose a couple other points on the parabola and find each of their partners on the other side of the axis of symmetry to help you with the sketch.**

For example, if you want to graph the parabola $(y + 2)^2 = 8(x - 1)$, you first note that this parabola has its vertex at the point (1, –2) and opens to the right, because the y is squared (if the x had been squared, it would open up or down) and a, being 2, is positive. The graph is relatively wide about the axis of symmetry, $y = -2$, because $a = 2$, which is greater than 1. Figure 10-2a shows the vertex, axis of symmetry, and two points that satisfy the equation of the parabola. You find the points by substituting in a value for y and solving for x.

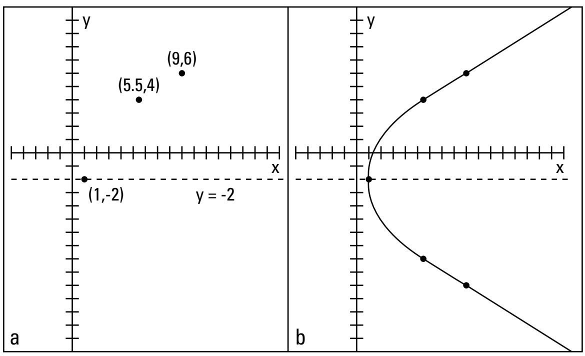

Figure 10-2: A parabola sketched from points and lines deduced from the standard equation.

The two randomly chosen points have counterparts on the opposite side of the axis of symmetry. The point (9, 6) is 8 units above the axis of symmetry, so 8 units below the axis puts you at (9, –10). The point (5.5, 4) is 6 units above the axis of symmetry, so its partner is the point (5.5, –8). Figure 10-2b shows the two new points and the parabola sketched in.

Changing a parabola's equation to the standard form

When the equation of a parabola appears in standard form, you have all the information you need to graph it or to determine some of its characteristics, such as direction or size. Not all equations come packaged that way, though. You may have to do some work on the equation first to be able to identify anything about the parabola.

The standard form of a parabola is $(x - h)^2 = 4a(y - k)$ or $(y - k)^2 = 4a(x - h)$, where (h, k) is the vertex.

The methods used here to rewrite the equation of a parabola into its standard form also apply when rewriting equations of circles, ellipses, and hyperbolas. The standard forms for conic sections are factored forms that allow you to immediately identify needed information. Different algebra situations call for different standard forms — the form just depends on what you need from the equation.

For example, if you want to convert the equation $x^2 + 10x - 2y + 23 = 0$ into the standard form, you act out the following steps, which contain a method called *completing the square,* which I show you here.

1. **Rewrite the equation with the x^2 and x terms (or the y^2 and y terms) on one side of the equation and the rest of the terms on the other side.**

   $x^2 + 10x = 2y - 23$

2. **Add a number to each side to make the side with the squared term into a perfect square trinomial (thus, completing the square).**

   $x^2 + 10x + 25 = 2y - 23 + 25$

3. **Rewrite the perfect square trinomial in factored form, and factor the terms on the other side by the coefficient of the variable.**

   $(x + 5)^2 = 2y + 2$

   $(x + 5)^2 = 2(y + 1)$

You now have the equation in standard form. The vertex is at $(-5, -1)$; it opens upward and is fairly wide.

Circling Around a Conic

A *circle,* probably the most recognizable of the conic sections, is defined as all the points plotted at the same distance from a fixed point — the circle's center, (h, k). The fixed distance is the radius, r, of the circle.

ALGEBRA RULES

The standard form for the equation of a circle with radius r and with its center at the point (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.

When the equation of a circle appears in the standard form, it provides you with all you need to know about the circle: its center and radius. With these two bits of information, you can sketch the graph of the circle. The equation $x^2 + y^2 + 6x - 4y - 3 = 0$, for example, is the equation of a circle. You can change this equation to the standard form by *completing the square* for each of the variables. Just follow these steps:

1. **Change the order of the terms so that the x's and y's are grouped together and the constant appears on the other side of the equal sign.**

   Leave a space after the groupings for the numbers that you need to add:

   $x^2 + 6x \quad + y^2 - 4y \quad = 3$

2. **Complete the square for each variable, adding the numbers that create perfect square trinomials.**

   $x^2 + 6x + 9 + y^2 - 4y + 4 = 3 + 9 + 4$

3. **Factor each perfect square trinomial.**

   $(x + 3)^2 + (y - 2)^2 = 16$

The example circle has its center at the point (–3, 2) and has a radius of 4 (the square root of 16). To sketch this circle, you locate the point (–3, 2) and then count 4 units up, down, left, and right; sketch in a circle that includes those points.

Getting Eclipsed by Ellipses

The ellipse is considered the most aesthetically pleasing of all the conic sections. It has a nice oval shape often used for mirrors, windows, and art forms.

The definition of an *ellipse* is all the points where the sum of the distances from the points to two fixed points is a constant. The two fixed points are the *foci* (plural of *focus*), denoted by *F*. Figure 10-3 illustrates this definition. You can pick a point on the ellipse, and the two distances from that point to the two foci add up to the same number as the sum of the distances from any other point on the ellipse to the foci. In Figure 10-3, the distances from point *A* to the two foci are 3.2 and 6.8, which add up to 10. The distances from point *B* to the two foci are 5 and 5, which also add up to 10.

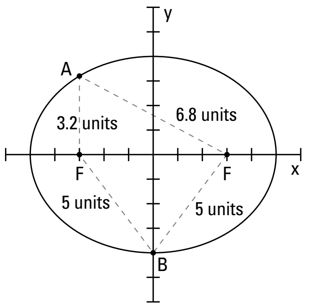

Figure 10-3: The sum of the two distances to the foci are the same.

You can think of the ellipse as a sort of squished circle. Of course, there's much more to ellipses than that, but the label sticks because the standard equation of an ellipse has a vague resemblance to the equation for a circle (see the previous section).

ALGEBRA RULES

The standard equation for an ellipse with its center at the point (h, k) is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where

- (x, y) is a point on the ellipse.
- a is half the length of the ellipse from left to right at its widest point.
- b is half the distance up or down the ellipse at its tallest point.

The standard equation tells you about the center, whether the ellipse is long and narrow or tall and slim. The equation tells you how long across, and how far up and down. You may even want to know the coordinates of the foci. You can determine all these elements from the equation.

Determining the shape

An ellipse is crisscrossed by a *major axis* and a *minor axis*. Each axis divides the ellipse into two equal halves, with the *major axis* being the longer of the segments. The two axes intersect at the center of the ellipse. At the ends of the major axis, you find the *vertices* of the ellipse. Figure 10-4 shows two ellipses with their axes and vertices identified.

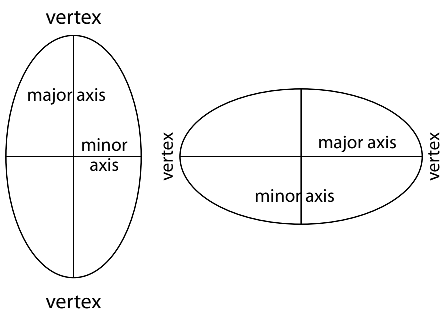

Figure 10-4: Ellipses with their axis properties identified.

REMEMBER

To determine the shape of an ellipse, you need to pinpoint two characteristics:

- **Lengths of the axes:** You can determine the lengths of the two axes from the standard equation of the ellipse. You take the square root of the numbers in the denominators of the fractions. Whichever value is larger, a^2 or b^2, tells you which one is the major axis. The square roots of these numbers represent the distances from the center to the points on the ellipse along their respective axes. In other words, a is half the length of one axis, and b is half the length of the other. Therefore, $2a$ and $2b$ are the lengths of the axes.
- **Assignment of the axes:** The positioning of the axes is significant. The denominator that falls under the x's signifies the axis that runs parallel to the x-axis. The denominator that falls under the y factor signifies the axis that runs parallel to the y-axis.

Finding the foci

You can find the two foci of an ellipse by using information from the standard equation. The foci, for starters, always lie on the major axis. They lie c units from the center. To find the value of c, you use parts of the ellipse equation to form the equation $c^2 = a^2 - b^2$ or $c^2 = b^2 - a^2$, depending on which is larger, a^2 or b^2. The value of c^2 has to be positive.

In the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, for example, the major axis runs across the ellipse, parallel to the x-axis. Actually, the major axis *is* the x-axis, because the center of this ellipse is the origin. You know this because the h and k are missing from the equation (actually, they're both equal to 0). You find the foci of this ellipse by solving the foci equation:

$$\begin{aligned} c^2 &= a^2 - b^2 \\ c^2 &= 25 - 9 \\ c^2 &= 16 \\ c &= \pm\sqrt{16} \\ &= \pm 4 \end{aligned}$$

So, the foci are 4 units on either side of the center of the ellipse. In this case, the coordinates of the foci are (–4, 0) and (4, 0).

Getting Hyped for Hyperbolas

The hyperbola is a conic section that features two completely disjoint curves, or *branches,* that face away from one another but are mirror images across a line that runs halfway between them.

A *hyperbola* is defined as all the points such that the difference of the distances from the point to two fixed points (called *foci*) is a positive constant value. In other words, you pick a value, such as the number 6; you find two distances whose difference is 6, such as 10 and 4; and then you find a point that rests 10 units from the one point and 4 units from the other point. The hyperbola has two axes, just as the ellipse has two axes (see the previous section). The axis of the hyperbola that goes through its two foci is called the *transverse axis.* The other axis, the *conjugate axis,* is perpendicular to the transverse axis, goes through the center of the hyperbola, and acts as the mirror line for the two branches.

ALGEBRA RULES

There are two basic equations for hyperbolas. You use one when the hyperbola opens to the left and right: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$. You use the other when the hyperbola opens up and down: $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.

In both cases, the center of the hyperbola is at (h, k), and the foci are c units away from the center, where the relationship $b^2 = c^2 - a^2$ describes the relationship between the different parts of the equation.

Including the asymptotes

TIP

A very helpful tool you can use to sketch hyperbolas is to first lightly sketch in the two diagonal asymptotes of the hyperbola. *Asymptotes* aren't actual parts of the graph; they just help you determine the shape and direction of the curves. The asymptotes of a hyperbola intersect at the center of the hyperbola. You find the equations of the asymptotes by replacing the 1 in the equation of the hyperbola with a 0 and simplifying the resulting equation into the equations of two lines.

EXAMPLE

Find the equations of the asymptotes of the hyperbola $\frac{(x-3)^2}{9} - \frac{(y+4)^2}{16} = 1$.

Change the 1 to 0, set the two fractions equal to one another, and take the square root of each side:

$$\frac{(x-3)^2}{9} - \frac{(y+4)^2}{16} = 0$$

$$\frac{(x-3)^2}{9} = \frac{(y+4)^2}{16}$$

$$\sqrt{\frac{(x-3)^2}{9}} = \pm\sqrt{\frac{(y+4)^2}{16}}$$

$$\frac{x-3}{3} = \pm\frac{y+4}{4}$$

Then you multiply each side by 12 to get the equations of the asymptotes in better form and consider the two cases — one using the positive sign, and the other using the negative sign. The equations of the two asymptotes that result are $y = \frac{4}{3}x - 8$ and $y = -\frac{4}{3}x$. Notice that the slopes of the lines are the opposites of one another.

Graphing hyperbolas

Hyperbolas are relatively easy to sketch, *if* you pick up the necessary information from the equations. To graph a hyperbola, use the following steps as guidelines:

1. **Determine if the hyperbola opens to the sides or up and down by noting whether the *x* term is first or second.**

   The x term first means it opens to the sides.

2. **Find the center of the hyperbola by looking at the values of *h* and *k*.**

3. **Lightly sketch in a rectangle twice as wide as the square root of the denominator under the *x* value and twice as high as the square root of the denominator under the *y* value.**

   The rectangle's center is the center of the hyperbola.

4. **Lightly sketch in the asymptotes through the vertices of the rectangle (see the preceding section).**

5. **Draw in the hyperbola, making sure it touches the midpoints of the sides of the rectangle.**

You can use these steps to graph the hyperbola $\frac{(x+2)^2}{9} - \frac{(y-3)^2}{16} = 1$. First, note that this hyperbola opens to the left and right because the x value comes first in the equation. The center of the hyperbola is at (–2, 3).

Now comes the mysterious rectangle. Starting at the center at (–2, 3), you count 3 units to the right and left of center (totaling 6), because twice the square root of 9 is 6. Now you count 4 units up and down from center, because twice the square root of 16 is 8. When the rectangle is in place, you draw in the asymptotes of the hyperbola, diagonally through the vertices (corners) of the rectangle. Lastly, with the asymptotes in place, you draw in the hyperbola, making sure it touches the sides of the rectangle at the midpoints and slowly gets closer and closer to the asymptotes as they get farther from the center. You can see the full hyperbola in Figure 10-5.

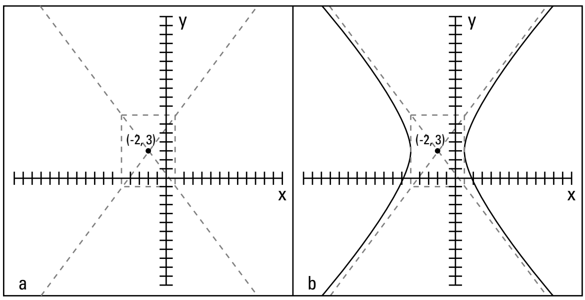

Figure 10-5: The hyperbola takes its shape with the asymptotes in place.