

Chapter 1

Making Advances in Algebra

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In This Chapter

- ▶ Making algebra orderly with the order of operations and other properties
 - ▶ Enlisting rules of exponents
 - ▶ Focusing on factoring
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Algebra is a branch of mathematics that people study before they move on to other areas or branches in mathematics and science. Algebra all by itself is esthetically pleasing, but it springs to life when used in other applications.

Any study of science or mathematics involves rules and patterns. You approach the subject with the rules and patterns you already know, and you build on those rules with further study. In this chapter, I recap for you the basic rules from Algebra I so that you work from the correct structure. I present these basics so you can further your study of algebra and feel confident in your algebraic ability.

Bringing Out the Best in Algebraic Properties

Mathematicians developed the rules and properties you use in algebra so that every student, researcher, curious scholar, and bored geek working on the same problem would get the same answer — no matter the time or place.

Making short work of the basic properties

The commutative, associative, and other such properties are not only basic to algebra, but also to geometry and many other mathematical topics. I present the properties here so that I can refer to them as I solve equations and simplify expressions in later chapters.

The commutative property

The *commutative property* applies to the operations of addition and multiplication. It states that you can change the order of the values in an operation without changing the final result:

$$a + b = b + a \quad \text{Commutative property of addition}$$

$$a \cdot b = b \cdot a \quad \text{Commutative property of multiplication}$$

So you can be sure that $2 + 4 = 4 + 2$ and $8 \cdot 7 = 7 \cdot 8$.

The associative property

Like the commutative property (see the preceding section), the *associative property* applies to the operations of addition and multiplication. The *associative property* states that you can change the grouping of operations without changing the result:

$$a + (b + c) = (a + b) + c \quad \text{Associative property of addition}$$

$$a(b \cdot c) = (a \cdot b)c \quad \text{Associative property of multiplication}$$

This property tells you that $3 + (8 + 5) = (3 + 8) + 5$ and that $-4 \cdot (8 \cdot 3) = (-4 \cdot 8) \cdot 3$.

The distributive property

The *distributive property* states that you can multiply each term in an expression within parentheses by the factor outside the parentheses and not change the value of the expression. It takes one operation — multiplication — and spreads it out over terms that you add to and subtract from one another:



$$a(b + c - d) = a \cdot b + a \cdot c - a \cdot d$$

Distributing multiplication over addition and subtraction

For example, you can use the distributive property on the problem $12\left(\frac{1}{2} + \frac{2}{3} - \frac{3}{4}\right)$ to make your life easier. You distribute the 12 over the fractions by multiplying each fraction by 12 and then combining the results: $= 12 \cdot \frac{1}{2} + 12 \cdot \frac{2}{3} - 12 \cdot \frac{3}{4} = 6 + 8 - 9 = 5$.



Identities

The numbers 0 and 1 have special roles in algebra — as *identities*.

$a + 0 = 0 + a = a$ The *additive identity* is 0. Adding 0 to a number doesn't change that number; the number keeps its identity.

$a \cdot 1 = 1 \cdot a = a$ The *multiplicative identity* is 1. Multiplying a number by 1 doesn't change that number; the number keeps its identity.



Inverses

You find two types of *inverses* in algebra — additive inverses and multiplicative inverses:

- ✓ A number and its *additive inverse* add up to 0.
- ✓ A number and its *multiplicative inverse* have a product of 1.

The additive inverse of 6 is -6 , so $6 + (-6) = 0$. And the multiplicative inverse of 6 is $\frac{1}{6}$, so $6 \cdot \frac{1}{6} = 1$.

The multiplication property of zero

The *multiplication property of zero* (MPZ) states that if the product of $a \cdot b \cdot c \cdot d \cdot e \cdot f = 0$, at least one of the terms has to represent the number 0. The only way the product of two or more values can be 0 is for at least one of the values to actually be 0. If you multiply $(16)(467)(11)(9)(0)$, the result is 0. It doesn't really matter what the other numbers are — the 0 always wins.



Organizing your operations

When mathematicians switched from words to symbols to describe mathematical processes, their goal was to make dealing with problems as simple as possible; however, at the same time, they wanted everyone to know what was meant by an expression and for everyone to get the same answer to a problem. Along with the special notation came a special set of rules on how to handle more than one operation in an expression.



The *order of operations* dictates that you follow this sequence:

1. Raise to powers or find roots.
2. Multiply or divide.
3. Add or subtract.



If you have to perform more than one operation from the same level, work those operations moving from left to right. If any grouping symbols appear, perform the operation inside the grouping symbols first.

So, to do the problem $4 + 3^2 - 5 \cdot 6 + \sqrt{23 - 7} + \frac{14}{2}$, follow the order of operations:

1. The radical acts like a grouping symbol, so you subtract what's in the radical first to get $4 + 3^2 - 5 \cdot 6 + \sqrt{16} + \frac{14}{2}$.
2. Raise the power and find the root: $4 + 9 - 5 \cdot 6 + 4 + \frac{14}{2}$.
3. Do the multiplication and then the division: $4 + 9 - 30 + 4 + 7$.
4. Add and subtract, moving from left to right: $4 + 9 - 30 + 4 + 7 = 13 - 30 + 4 + 7 = -17 + 4 + 7 = -13 + 7 = -6$.

Enumerating Exponential Rules

Several hundred years ago, mathematicians introduced powers of variables and numbers called *exponents*. Instead of writing $xxxxxxx$, you use the exponent 8 by writing x^8 . This form is easier to read and much quicker. The use of exponents

expanded to being able to write fractions with negative exponents and radicals with fractional exponents. You find all the details in *Algebra I For Dummies*, 2nd Edition (Wiley).

Multiplying and dividing exponents



When two numbers or variables have the same base, you can multiply or divide those numbers or variables by adding or subtracting their exponents:

- ✓ $a^m \cdot a^n = a^{m+n}$: When multiplying numbers with the same base, you add the exponents.
- ✓ $\frac{a^m}{a^n} = a^{m-n}$: When dividing numbers with the same base, you subtract the exponents (numerator minus denominator).

To multiply $x^4 \cdot x^5$, for example, you add: $x^{4+5} = x^9$. When dividing x^8 by x^5 , you subtract: $\frac{x^8}{x^5} = x^{8-5} = x^3$.

You have to be sure that the bases of the expressions are the same. You can multiply or divide 3^2 and 3^4 , but you can't use the multiplication or division rules for exponents to multiply or divide 3^2 and 4^3 .

Rooting out exponents



Radical expressions — such as square roots, cube roots, fourth roots, and so on — appear with a radical to show the root. Another way you can write these values is by using fractional exponents. You'll have an easier time combining variables with the same base if they have fractional exponents in place of radical forms:

- ✓ $\sqrt[n]{x} = x^{\frac{1}{n}}$: The root goes in the denominator of the fractional exponent.
- ✓ $\sqrt[n]{x^m} = x^{\frac{m}{n}}$: The root goes in the denominator of the fractional exponent, and the power goes in the numerator.

To simplify a radical expression such as $\frac{\sqrt[4]{x^6}\sqrt{x^5}}{\sqrt[3]{x}}$, you change the radicals to exponents and apply the rules for multiplication and division of values with the same base (see the preceding section):

$$\begin{aligned}\frac{\sqrt[4]{x^6}\sqrt{x^5}}{\sqrt[3]{x}} &= x^{\frac{1}{4}} \cdot x^{\frac{5}{6}} \\ &= x^{\frac{1}{4} + \frac{5}{6} - \frac{1}{3}} \\ &= x^{\frac{3}{12} + \frac{10}{12} - \frac{4}{12}} \\ &= x^{\frac{9}{12}} \\ &= x^{\frac{3}{4}} \\ &= \sqrt[4]{x^3}\end{aligned}$$

Powering up exponents



When raising a power to a power, you multiply the exponents. When taking the root of a power, you divide the exponent by the root:

- ✓ $(a^m)^n = a^{(m)(n)}$: Raise a power to a power by multiplying the exponents.
- ✓ $\sqrt[n]{a^m} = a^{\frac{m}{n}}$: Reduce the power when taking a root by dividing the exponents.

The second rule may look familiar — it's one of the rules that govern changing from radicals to fractional exponents from the preceding section. Here's an example of how you apply the two rules when simplifying an expression:

$$\sqrt[3]{(x^4)^6} \cdot x^9 = \sqrt[3]{x^{24}} \cdot x^9 = \sqrt[3]{x^{33}} = x^{\frac{33}{3}} = x^{11}$$

Working with negative exponents



You use negative exponents to indicate that a number or variable belongs in the denominator of the term:

$$\frac{1}{a} = a^{-1} \qquad \frac{1}{a^n} = a^{-n} \qquad \frac{1}{a^{-n}} = a^{-(-n)} = a^n$$

Writing variables with negative exponents allows you to combine those variables with other factors that share the same base. You can rewrite the fractions by using negative exponents and then simplify by using the rules for multiplying factors with the same base:

$$\frac{1}{x^4} \cdot x^7 \cdot \frac{3}{x} = x^{-4} \cdot x^7 \cdot 3x^{-1} = 3x^{-4+7-1} = 3x^2$$

Assigning Factoring Techniques

When you *factor* an algebraic expression, you rewrite the sums and differences of the terms as a product. The factored form comes in handy when you set an expression equal to 0 to solve an equation. Factored numerators and denominators in fractions also make it possible to reduce the fractions.

Making two terms factor

When an algebraic expression has two terms, you have four different choices for its factorization — if you can factor the expression at all. If you try the following four methods and none of them works, you can stop your attempt; you just can't factor the expression:

$ax + ay = a(x + y)$	Greatest common factor (GCF)
$x^2 - a^2 = (x - a)(x + a)$	Difference of two perfect squares
$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$	Difference of two perfect cubes
$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$	Sum of two perfect cubes



In general, you check for a GCF before attempting any of the other methods. By taking out the common factor, you often make the numbers smaller and more manageable, which helps you see clearly whether any other factoring is necessary.



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EXAMPLE



Factor the expression $6x^4 - 6x$.

First factor out the common factor, $6x$, and then use the pattern for the difference of two perfect cubes:

$$6x^4 - 6x = 6x(x^3 - 1) = 6x(x - 1)(x^2 + x + 1)$$

REMEMBER



A *quadratic trinomial* is a three-term polynomial with a term raised to the second power (and no higher powers). When you see something like $x^2 + x + 1$ (as in this case), you immediately run through the possibilities of factoring it into the product of two binomials (see the next section). You can just stop. These trinomials that crop up with factoring cubes just don't cooperate.

EXAMPLE



Factor $48x^3y^2 - 300x^3$.

When you factor the expression, first divide out the common factor, $12x^3$, to get $12x^3(4y^2 - 25)$. Then factor the difference of perfect squares in the parentheses: $48x^3y^2 - 300x^3 = 12x^3(2y - 5)(2y + 5)$.

Factoring three terms



When a quadratic expression has three terms, making it a *trinomial*, you have two different ways to factor it. One method is factoring out a GCF, and the other is finding two binomials whose product is identical to the sum and/or difference of the original three terms:

$$ax + ay + az = a(x + y + z) \quad \text{GCF}$$

$$ax^{2n} + bx^n + c = (dx^n + e)(fx^n + g) \quad \text{Two binomials}$$

When you factor a trinomial that results from multiplying two binomials, you have to play detective and piece together the parts of the puzzle. Look at the following generalized product of binomials and the pattern that appears:

$$\begin{aligned}(dx + e)(fx + g) &= dfx^2 + dgx + efx + eg = dfx^2 + (dg + ef)x + eg \\ &= ax^2 + bx + c\end{aligned}$$

The F in FOIL stands for *first*; the *first* terms are the dx and fx . The O in FOIL stands for *outer*; the *outer* terms are dx and g . The I in FOIL stands for *inner*; the *inner* terms are e and fx .

Their products are dgx (outer) and efx (inner). You add these two values. The L in FOIL stands for *last*; the *last* terms, e and g , have a product of eg .

Now, think of every quadratic trinomial as being of the form $ax^2 + bx + c = dx^2 + (dg + ef)x + eg$. The coefficient of the x^2 term, df , is the product of the coefficients of the two x terms in the parentheses; the last term, eg , is the product of the two second terms in the parentheses; and the coefficient of the middle term is the sum of the outer and inner products. To factor these trinomials into the product of two binomials, you use the opposite of FOIL and figure out which factorizations to use.



Here are the basic steps you take to unFOIL a quadratic trinomial:

1. Determine all the ways you can multiply two numbers to get a , the coefficient of the squared term.
2. Determine all the ways you can multiply two numbers to get c , the constant term.
3. If the last term is positive, find the combination of factors from steps 1 and 2 whose *sum* is that middle term; if the last term is negative, you want the combination to be a difference.
4. Arrange your choices as binomials so that the factors line up correctly.
5. Insert the $+$ and $-$ signs to finish off the factoring and make the sign of the middle term come out right.



Factor $x^2 + 9x + 20$.

You find two terms whose product is 20 and whose sum is 9. The coefficient of the squared term is 1, so you don't have to take any other factors into consideration. You choose 4 and 5 as the factors of 20, because $4 + 5 = 9$. Arranging the factors and x 's, you get $x^2 + 9x + 20 = (x + 4)(x + 5)$.



Factor $6x^2 - x - 12$.

You have to consider both the factors of 6 and the factors of 12. Start 2 and 3 for the factors of 6 and write: $(2x \quad)(3x \quad)$. Don't insert any signs until the end of the process.

Now, using the factors of 12, you look for a pairing that gives you a difference of 1 between the outer and inner products. Try the product of $3 \cdot 4$, matching (multiplying) the 3 with the $3x$ and the 4 with the $2x$. Bingo! Write $(2x - 3)(3x + 4)$. You'll multiply the 3 and $3x$ because they're in different parentheses — not the same one. The difference has to be negative, so you can put the negative sign in front of the 3 in the first binomial: $6x^2 - x - 12 = (2x - 3)(3x + 4)$.

Factoring four or more terms by grouping

When four or more terms come together to form an expression, you look for a GCF first. If you can't find a factor common to all the terms at the same time, your other option is *grouping*. To group, you take the terms two at a time and look for common factors for each of the pairs on an individual basis. After factoring, you see if the new groupings have a common factor.



Factor $x^3 - 4x^2 + 3x - 12$.

The four terms of $x^3 - 4x^2 + 3x - 12$ don't have any common factor. However, the first two terms have a common factor of x^2 , and the last two terms have a common factor of 3:

$$x^3 - 4x^2 + 3x - 12 = x^2(x - 4) + 3(x - 4)$$

Notice that you now have two terms, not four, and they both have the factor $(x - 4)$. Now, factoring $(x - 4)$ out of each term, you have $(x - 4)(x^2 + 3)$.