Chapter 7

Tackling Second-Degree Quadratic Equations

In This Chapter

- Solving quadratic equations by factoring
- Finding solutions using the quadratic formula
- Investigating imaginary answers to quadratics

A *quadratic equation* is a *quadratic expression* (a grouping of terms whose highest variable power is 2) with an equal sign attached. As with linear equations, specific methods or processes, given in detail in this chapter, are employed to successfully solve quadratic equations. The most commonly used technique for solving these equations is factoring, but there's also a quick and dirty rule for one of the special types of quadratic equations.

Quadratic equations are important to algebra and many other sciences. Some quadratic equations say that what goes up must come down. Other equations describe the paths that planets and comets take. In all, quadratic equations are fascinating — and just dandy to work with.

Recognizing Quadratic Equations

A quadratic equation contains a variable term with an exponent of 2 and no variable term with a higher power.

A quadratic equation has a general form that looks like this: $ax^2 + bx + c = 0$. The constants a, b, and c in the equation are real numbers, and a cannot be equal to 0. (If a were 0, you wouldn't have a quadratic equation anymore.)

Here are some examples of quadratic equations and their solutions:

- $\mathbf{4x^2 + 5x - 6 = 0}$**:** In this equation, none of the coefficients is 0. The two solutions are $x = -2$ and $x = \frac{3}{4}$.
- $\mathbf{2x^2 - 18 = 0}$**:** In this equation, the b is equal to 0. The solutions are $x = 3$ and $x = -3$.
- $\mathbf{x^2 + 3x = 0}$**:** In this equation, the c is equal to 0. The solutions are $x = 0$ and $x = -3$.
- $\mathbf{x^2 = 0}$**:** In this equation, both b and c are equal to 0. The equation has only one solution, $x = 0$.

A special feature of quadratic equations is that they can, and often do, have two completely different answers. As you see in the preceding examples, three of the equations have different solutions. The last equation has just one solution, but, technically, you count that solution twice, calling it a *double root.* Some quadratic equations have no solutions if you're only considering real numbers, but get real! I stick to real solutions for now.

Finding Solutions for Quadratic Equations

The general quadratic equation has the form $ax^2 + bx + c = 0$, and b or c or both of them can be equal to 0. This section shows you how nice it is — and how easy it is to solve equations — when b is equal to 0.

The following is the rule for some special quadratic equations — the ones where $b = 0$. They start out looking like $ax^2 + c = 0$, but the c is usually negative, giving you $ax^2 - c = 0$, and the equation is rewritten as $ax^2 = c$.

If $x^2 = k$, then $x = \pm\sqrt{k}$ or if $ax^2 = c$, then $x = \pm\sqrt{\frac{c}{a}}$. If the square of a variable is equal to the number k, then the variable is equal to either a positive or negative number — both the positive and negative roots of k.

The following examples show you how to use this square-root rule on quadratic equations where $b = 0$.

Solve for x in $x^2 = 49$.

Using the square-root rule, $x = \pm\sqrt{49} = \pm 7$. Checking, $(7)^2 = 49$ and $(-7)^2 = 49$.

Solve for m in $3m^2 + 4 = 52$.

This equation isn't quite ready for the square-root rule. Add –4 to each side:

$$3m^2 = 48$$

Now divide each side by 3:

$$m^2 = 16$$

So $m = \pm\sqrt{16} = \pm 4$.

Solve for q in $(q + 3)^2 = 25$.

In this case, you end up with two completely different answers, not one number and its opposite. Use the square-root rule, first, to get $q + 3 = \pm\sqrt{25} = \pm 5$.

Now you have two different linear equations to solve:

$$q + 3 = +5 \qquad q + 3 = -5$$

Subtracting 3 from each side of each equation, the two answers are

$$q = 2 \qquad q = -8$$

Applying Factorizations

This section is where running through all the factoring methods can really pay off. In most quadratic equations, factoring is used rather than the square-root rule method covered in the preceding section. The square-root rule is used only when $b = 0$ in the quadratic equation $ax^2 + bx + c = 0$. Factoring is used when $c = 0$ or when neither b nor c is 0.

A very important property used along with the factoring to solve these equations is the multiplication property of zero. This is a very straightforward rule — and it even makes sense. Use the greatest common factor (GCF) and the multiplication property of zero when solving quadratic equations that aren't in the form for the square-root rule.

Zeroing in on the multiplication property of zero

Before you get into factoring quadratics for solutions, you need to know about the multiplication property of zero. By itself, 0 is nothing. Put it as the result of a multiplication problem, and you really have something: the *multiplication property of zero.*

The *multiplication property of zero* (MPZ) states that if $pq = 0$, then either $p = 0$ or $q = 0$. One of them must be equal to 0 (or both could be 0).

This may seem obvious, but think about it. A product of 0 leads to one conclusion: One of the multipliers must be 0. No other means of arriving at a 0 product exists. Why is this such a big deal? Let me show you a few equations and how the MPZ works.

Find the value of x if $3x = 0$.

$x = 0$ because 3 can't be 0. Using the MPZ, if the one factor isn't 0, then the other must be 0.

Find the value of x and y if $xy = 0$.

You have two possibilities in this equation. If $x = 0$, then y can be any number, even 0. If $x \neq 0$, then y must be 0, according to the MPZ.

Solve for x in $x(x - 5) = 0$.

Again, you have two possibilities. If $x = 0$, then the product of $0(-5) = 0$. The other choice is when $x = 5$. Then you have $5(0) = 0$.

Solving quadratics by factoring and applying the multiplication property of zero

Factoring is relatively simple when there are only two terms and they have a common factor. This is true in quadratic equations of the form $ax^2 + bx = 0$ (where $c = 0$). The two terms left have the common factor of x, at least. You find the GCF and factor that out, and then use the MPZ to solve the equation.

The following example makes use of the fact that the constant term is 0, and there's a common factor of at least an x in the two terms.

Solve for x in $6x^2 + 18x = 0$.

The GCF of the two terms is $6x$, so write the left side in factored form:

$$6x(x + 3) = 0$$

Use the MPZ to say that $6x = 0$ or $x + 3 = 0$, which gives you the two solutions $x = 0$ and $x = -3$.

Technically, I could have written three different equations from the factored form:

$$6 = 0 \qquad x = 0 \qquad x + 3 = 0$$

The first equation, $6 = 0$, makes no sense — it's an impossible statement. So you either ignore setting the constants equal to 0 or combine them with the factored-out variable, where they'll do no harm.

Missing the $x = 0$, a full half of the solution, is an amazingly frequent occurrence. You don't notice the lonely little x in the front of the parentheses and forget that it gives you one of the two answers. Be careful.

Solving Three-Term Quadratics

In the two previous sections, either b or c has been equal to 0 in the quadratic equation $ax^2 + bx + c = 0$. Now I won't let anyone skip out. In this section, each of the letters, a, b, and c is a number that is not 0.

To solve a quadratic equation, moving everything to one side with 0 on the other side of the equal sign is the most efficient method. Factor the equation if possible, and use the MPZ after you factor. If there aren't three terms in the equation, then refer to the previous sections.

In the following example, I list the steps you use for solving a quadratic trinomial by factoring.

Solve for x in $x^2 - 3x = 28$. Follow these steps:

1. **Move all the terms to one side. Get 0 alone on the right side.**

   In this case, you can subtract 28 from each side:

   $$x^2 - 3x - 28 = 0$$

2. **Determine all the ways you can multiply two numbers to get a.**

   In $x^2 - 3x - 28 = 0$, $a = 1$, which can only be 1 times itself.

3. **Determine all the ways you can multiply two numbers to get c (ignore the sign for now).**

   Twenty-eight can be $1 \cdot 28$, $2 \cdot 14$, or $4 \cdot 7$.

4. **Factor.**

   If c is positive, find an operation from your Step 2 list and an operation from your Step 3 list that match so that the sum of their cross-products is the same as b.

If c is negative, find an operation from your Step 2 list and an operation from your Step 3 list that match so that the difference of their cross-products is the same as b.

In this problem, c is negative, and the difference of 4 and 7 is 3. Factoring, you get $(x - 7)(x + 4) = 0$.

5. **Use the MPZ.**

   Either $x - 7 = 0$ or $x + 4 = 0$; now try solving for x by getting x alone to one side of the equal sign.

   - $x - 7 + 7 = 0 + 7$ gives you that $x = 7$.
   - $x + 4 - 4 = 0 - 4$ gives you that $x = -4$.

   So the two solutions are $x = 7$ and $x = -4$.

6. **Check your answer.**

   If $x = 7$, then $(7)^2 - 3(7) = 49 - 21 = 28$.

   If $x = -4$, then $(-4)^2 - 3(-4) = 16 + 12 = 28$.

   They both check.

Factoring to solve quadratics sounds pretty simple on the surface. But factoring *trinomial equations* — those with three terms — can be a bit less simple. If a quadratic with three terms can be factored, then the product of two binomials is that trinomial. If the quadratic equation with three terms can't be factored, then use the quadratic formula (see "Calling on the Quadratic Formula" later in this chapter).

REMEMBER

The product of the two binomials $(ax + b)(cx + d)$ is equal to the trinomial $acx^2 + (ad + bc)x + bd$. This is a fancy way of showing what you get from using FOIL when multiplying the two binomials together.

Now, on to using unFOIL. If you need more of a review of FOIL and unFOIL, check out Chapter 4.

The following examples all show how factoring and the MPZ allow you to find the solutions of a quadratic equation with all three terms showing.

Solve for x in $x^2 - 5x - 6 = 0$.

1. **Check whether the equation is in standard form.**

   The equation is in standard form, so you can proceed.

2. **Determine all the ways you can multiply to get *a*.**

   $a = 1$, which can only be 1 times itself. If there are two binomials that the left side factors into, then they must each start with an x because the coefficient of the first term is 1.

   $$(x\quad)(x\quad) = 0$$

3. **Determine all the ways you can multiply to get *c*.**

   $c = -6$, so, looking at just the positive factors, you have $1 \cdot 6$ or $2 \cdot 3$.

4. **Factor.**

   To decide which combination should be used, look at the sign of the last term in the trinomial, the 6, which is negative. This tells you that you have to use the *difference* of two numbers in the list (think of the numbers without their signs) to get the middle term in the trinomial, the –5. In this case, one of the 1 and 6 combinations work, because their difference is 5. If you use the +1 and –6, then you get the –5 immediately from the cross-product in the FOIL process. So $(x - 6)(x + 1) = 0$.

5. **Use the MPZ.**

   Using the MPZ, $x - 6 = 0$ or $x + 1 = 0$. This tells you that $x = 6$ or $x = -1$.

6. **Check.**

   If $x = 6$, then $(6)^2 - 5(6) - 6 = 36 - 30 - 6 = 0$.

   If $x = -1$, then $(-1)^2 - 5(-1) - 6 = 1 + 5 - 6 = 0$.

   They both work!

Solve for x in $6x^2 + x = 12$.

1. **Put the equation in the standard form.**

   The first thing to do is to add –12 to each side to get the equation into the standard form for factoring and solving:

   $$6x^2 + x - 12 = 0$$

This one will be a bit more complicated to factor because the 6 in the front has a couple of choices of factors, and the 12 at the end also has several choices. The trick is to pick the correct combination of choices.

2. **Find all the combinations that can be multiplied to get *a*.**

   You can get 6 with $1 \cdot 6$ or $2 \cdot 3$.

3. **Find all the combinations that can be multiplied to get *c*.**

   You can get 12 with $1 \cdot 12$, $2 \cdot 6$, or $3 \cdot 4$.

4. **Factor.**

   You have to choose the factors to use so that the difference of their cross-products (outer and inner) is 1, the coefficient of the middle term. How do you know this? Because the 12 is negative, in this standard form, and the value multiplying the middle term is assumed to be 1 when there's nothing showing.

   Looking this over, you can see that using the 2 and 3 from the 6 and using the 3 and 4 from the 12 will work: $2 \cdot 4 = 8$ and $3 \cdot 3 = 9$. The difference between the 8 and the 9 is, of course, 1. You can worry about the sign later.

   Fill in the binomials and line up the factors so that the 2 multiplies the 4 and the 3 multiplies the 3, and you get a 6 in the front and 12 at the end. Whew!

   $$(2x \quad 3)(3x \quad 4) = 0$$

   The quadratic has a + on the term in the middle, so I need the bigger product of the outer and inner to be positive. I get this by making the $9x$ positive, which happens when the 3 is positive and the 4 is negative.

   $$(2x + 3)(3x - 4) = 0$$

5. **Use the MPZ to solve the equation.**

   The trinomial has been factored. The MPZ tells you that either $2x + 3 = 0$ or $3x - 4 = 0$. If $2x + 3 = 0$ then $2x = -3$ or $x = -\frac{3}{2}$. If $3x - 4 = 0$ then $3x = 4$ or $x = \frac{4}{3}$.

Solve for z in $12z^2 - 4z - 8 = 0$.

1. **Check to see whether this quadratic is in standard form.**

   You can start out by looking for combinations of factors for the 12 and the 8, but you may notice that all three terms are divisible by 4. To make things easier, take out that GCF first, and then work with the smaller numbers in the parentheses.

   $$12z^2 - 4z - 8 = 4(3z^2 - z - 2) = 0$$

2. **Find the numbers that multiply to get 3.**

   $$3 = 1 \cdot 3$$

3. **Find the numbers that multiply to get 2.**

   $$2 = 1 \cdot 2$$

4. **Factor.**

   This is really wonderful, especially because the 3 and 2 are both prime and can be factored only one way. Your only chore is to line up the factors so there will be a difference of 1 between the cross-products.

   $$4(3z^2 - z - 2) = 4(3z \quad 2)(z \quad 1) = 0$$

   Because the middle term is negative, you need to make the larger product negative, so put the negative sign on the 1.

   $$4(3z + 2)(z - 1) = 0$$

5. **Use the MPZ to solve for the value of *z*.**

   This time, when you use the MPZ, there are three factors to consider. Either $4 = 0$, $3z + 2 = 0$, or $z - 1 = 0$. The first equation is impossible; 4 doesn't ever equal 0. But the other two equations give you answers. If $3z + 2 = 0$, then $z = -\frac{2}{3}$. If $z - 1 = 0$, then $z = 1$.

Applying Quadratic Solutions

Quadratic equations are found in many mathematics, science, and business applications; that's why they're studied so much. The graphs of quadratic equations are always U-shaped, with an extreme point that's highest, lowest, farthest left, or farthest right. That extreme point is often the answer to a question about the situation being modeled by the quadratic. In other applications, you want the point(s) at which the U-shaped curve crosses an axis; those points are found by finding solutions to setting the quadratic equal to 0. In this section, I show you an example of how a quadratic equation is used in an application.

In physics, an equation that tells you how high an object is after a certain amount of time can be written

$$h = -16t^2 + v_0 t + h_0$$

In this equation, the $-16t^2$ part accounts for the pull of gravity on the object. The number representing v_0 is the initial velocity — what the speed is at the very beginning. The h_0 is the starting height — the height in feet of the building, cliff, or stool from which the object is thrown, shot, or dropped. The variable t represents time — how many seconds have passed.

A stone was thrown upward from the top of a 40-foot building with a beginning speed of 128 feet per second. When was the stone 296 feet up in the air?

Replacing the height, h, with the 296, the v_0 with 128, and the h_0 with 40, the equation now reads: $296 = -16t^2 + 128t + 40$. You can solve it using the following steps:

1. **Put the equation in standard form.**

   Add –296 to each side.

   $$0 = -16t^2 + 128t - 256$$

2. **Factor out the GCF.**

   In this case, the GCF is –16.

   $$0 = -16(t^2 - 8t + 16)$$

3. **Factor the quadratic trinomial inside the parentheses.**

   $$0 = -16(t - 4)^2$$

4. **Use the MPZ to solve for the variable.**

   $$t - 4 = 0, t = 4$$

   After 4 seconds, the stone will be 296 feet up in the air.

Calling On the Quadratic Formula

The quadratic formula is special to quadratic equations. A quadratic equation, $ax^2 + bx + c = 0$, can have as many as two solutions, but there may be only one solution or even no solutions at all.

a, b, and c are any real numbers. The a can't equal 0, but the b or c can equal 0.

The quadratic formula allows you to find solutions when the equations aren't very nice. Numbers aren't *nice* when they're funky fractions, indecent decimals with no end, or raucous radicals.

The quadratic formula says that if an equation is in the form $ax^2 + bx + c = 0$, then its solutions, the values of x, can be found with the following:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You see an operation symbol, ±, in the formula. The symbol is shorthand for saying that the equation can be broken into two separate equations, one using the plus sign and the other using the minus sign.

You can apply this formula to *any* quadratic equation to find the solutions — whether it factors or not. Let me show you some examples of how the formula works.

Use the quadratic formula to solve $2x^2 + 7x - 4 = 0$.

Refer to the standard form of a quadratic equation where the coefficient of x^2 is a, the coefficient of x is b, and the constant is c. In this case, $a = 2$, $b = 7$, and $c = -4$. Inserting those numbers into the formula, you get

$$x = \frac{-7 \pm \sqrt{7^2 - 4(2)(-4)}}{2(2)}$$

Now, simplifying, and paying close attention to the order of operations, you get

$$x = \frac{-7 \pm \sqrt{49-(-32)}}{4} = \frac{-7 \pm \sqrt{81}}{4} = \frac{-7 \pm 9}{4}$$

The two solutions are found by applying the + in front of the 9 and then the – in front of the 9.

$$x = \frac{-7+9}{4} = \frac{2}{4} = \frac{1}{2}$$
$$x = \frac{-7-9}{4} = \frac{-16}{4} = -4$$

Whenever the answers you get from using the quadratic formula come out as integers or fractions, it means that the trinomial could've been factored. It doesn't mean, though, that you shouldn't use the quadratic formula on factorable problems.

TIP: Sometimes using the quadratic formula is easier if the equation has really large or nasty numbers. In general, though, when you can, factoring using unFOIL and then the MPZ is quicker.

Just to illustrate this point, look at the previous example when it's solved using factoring and the MPZ:

$$2x^2 + 7x - 4 = (2x-1)(x+4) = 0$$

Then, using the MPZ, you get $2x - 1 = 0$ or $x + 4 = 0$, so $x = \frac{1}{2}$ or $x = -4$.

So, what do the results look like when the equation can't be factored? The next example shows you.

WARNING! Here are two things to watch out for when using the quadratic formula:

- **Don't forget that –*b* means to use the *opposite* of *b*.** If the coefficient *b* in the standard form of the equation is a positive number, change it to a negative number before inserting into the formula. If *b* is negative, then change it to positive in the formula.

- **Be careful when simplifying under the radical.** The order of operations dictates that you square the value of *b* first, and then multiply the last three factors together before subtracting them from the square of *b*. Some sign errors can occur if you're not careful.

Solve for x using the quadratic formula in $2x^2 + 8x + 7 = 0$.

In this problem, you let $a = 2$, $b = 8$ and $c = 7$ when using the formula:

$$x = \frac{-8 \pm \sqrt{8^2 - 4(2)(7)}}{2(2)} = \frac{-8 \pm \sqrt{64 - 56}}{4} = \frac{-8 \pm \sqrt{8}}{4}$$

The radical can be simplified because $\sqrt{8} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$, so

$$x = \frac{-8 \pm 2\sqrt{2}}{4} = \frac{-{}^{4}\cancel{8} \pm \cancel{2}\sqrt{2}}{{}^{2}\cancel{4}} = \frac{-4 \pm \sqrt{2}}{2}$$

Be careful when simplifying this expression: $\frac{\left(-4 + \sqrt{2}\right)}{2} \neq -2 + \sqrt{2}$. Both terms in the numerator of the fraction have to be divided by the 2.

Here are the decimal equivalents of the answers:

$$\frac{-4 + \sqrt{2}}{2} \approx \frac{-4 + 1.414}{2} = \frac{-2.586}{2} = -1.293$$
$$\frac{-4 - \sqrt{2}}{2} \approx \frac{-4 - 1.414}{2} = \frac{-5.414}{2} = -2.707$$

When you check these answers, what do the estimates do? If $x = -1.293$, then $2(-1.293)^2 + 8(-1.293) + 7 = 3.343698 - 10.344 + 7 = -0.000302$.

That isn't 0! What happened? Is the answer wrong? No, it's okay. The rounding caused the error — it didn't come out exactly right. This happens when you use a rounded value for the answer, rather than the exact radical form. An estimate was used for the answer because the square root of a number that is not a perfect square is an irrational number, and the decimal never ends. Rounding the decimal value to three decimal places seemed like enough decimal places.

You shouldn't expect the check to come out to be *exactly* 0. In general, if you round the number you get from your check to the same number of places that you rounded your estimate of the radical, then you should get the 0 you're aiming for.

Ignoring Reality with Imaginary Numbers

An imaginary number is something that doesn't exist — well, at least until some enterprising mathematicians had their way. Not being happy with having to halt progress in solving some equations because of negative numbers under the radical, mathematicians came up with the imaginary number i.

The square root of –1 is designated as i. $\sqrt{-1} = i$ and $i^2 = -1$.

Since the declaration of the value of i, all sorts of neat mathematics and applications have cropped up. Sorry, I can't cover all that good stuff in this book, but I at least give you a little preview of what *complex numbers* are all about.

You're apt to run into these imaginary numbers when using the quadratic formula. In the following example, the quadratic equation doesn't factor and doesn't have any *real* solutions — the only possible answers are *imaginary*.

Use the quadratic formula to solve $5x^2 - 6x + 5 = 0$.

In this quadratic, $a = 5$, $b = -6$, and $c = 5$. Putting the numbers into the formula:

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(5)(5)}}{2(5)} \\ &= \frac{6 \pm \sqrt{36 - 100}}{10} \\ &= \frac{6 \pm \sqrt{-64}}{10} \end{aligned}$$

You see a –64 under the radical. Only positive numbers and 0 have square roots. So you use the definition of the imaginary number where $i = \sqrt{-1}$ and apply it after simplifying the radical:

$$\frac{6 \pm \sqrt{-64}}{10} = \frac{6 \pm \sqrt{-1}\sqrt{64}}{10} = \frac{6 \pm i \cdot 8}{10} = \frac{3 \pm 4i}{5}$$

Applying this new *imaginary* number allowed mathematicians to finish their problems. You have two answers — although both are imaginary.