

Chapter 5

Broadening the Factoring Horizon

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In This Chapter

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 - ▶ Tackling quadratic-like expressions
 - ▶ Employing synthetic division
 - ▶ Making rational expressions behave rationally when factoring
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This chapter has some mighty helpful factoring information that doesn't belong under linear or quadratic factoring rules. Half of the factoring process is knowing how to use the rules, and the other half is recognizing when to use what rule. These skills are equally important — you need both to be successful.

Grabbing Onto Grouping

Factoring by grouping is a way of dealing with particular four-term or six-term expressions. You identify common factors in *groups* of terms — not all of them at once. Then, if grouping is to work, you find a new common factor in the newly created terms.

Getting the groups together

The easiest way to explain grouping is to go through an example, step by step, so you see how it works.



Factor the expression $x^5 + 7x - 14 - 2x^4$ using grouping.

- 1. Write your expression in decreasing powers of a variable.**

The expression becomes $x^5 - 2x^4 + 7x - 14$.

- 2. Determine a common factor for the first two terms and then a common factor for the second two terms. Do the factorization.**

The first two terms have a common factor of x^4 , and the second two terms have a common factor of 7.

$$x^5 - 2x^4 + 7x - 14 = x^4(x - 2) + 7(x - 2)$$

- 3. Determine if the newly written version of the expression contains terms with a common factor. Do the factorization.**

The new expression has two terms, each with a factor of $(x - 2)$.

$$x^4(x - 2) + 7(x - 2) = (x - 2)(x^4 + 7)$$

You should always check to see if one or both of the factors can themselves be factored. In this case, both binomials are prime, so you're done.

Grouping and unFOILing in the same package

In the next example, you see six terms — a type of expression that uses grouping to factor it. And a big surprise comes after the grouping is finished.

Grouping isn't used on the three-term quadratic. It's usually used on expressions with four or six or some other even number of terms. When grouping is employed, you divide up the expression and factor just one part at a time. The next example walks you through the steps.



Factor: $3x^2y - 24xy - 27y - 5x^2z + 40xz + 45z$.

You see three different variables, and the only variable with a power greater than 1 is the x variable. You rewrite the expression as

$$3x^2y - 5x^2z - 24xy + 40xz - 27y + 45z$$

You see a common factor of x^2 in the first two terms, a common factor of $8x$ in the third and fourth terms, and a common factor of -9 in the last two. (By the way, I'm purposely leading you astray for a moment and not factoring out $-8x$ in the middle grouping — just to show you how to *repair* situations when needed.) Do the three factorizations next.

$$x^2(3y - 5z) + 8x(-3y + 5z) - 9(3y - 5z)$$

You see that the three binomials are not the same. In order for grouping to work, you have to create a fewer number of terms — each with some factor in common. The problem here is that you didn't factor out $-8x$. Changing the $+8x$ to $-8x$, you factor -1 out of each term in the second binomial and basically just change each sign to its opposite:

$$= x^2(3y - 5z) - 8x(3y - 5z) - 9(3y - 5z)$$

Now you see that each of the three terms has a common factor of the binomial $(3y - 5z)$. Now factor that GCF out of the three terms:

$$= (3y - 5z)(x^2 - 8x - 9)$$

Finally, the trinomial itself can be factored using unFOIL. You want the difference of the cross-products to be 8, so you use the factors $1 \cdot 1$ and $9 \cdot 1$:

$$= (3y - 5z)(x - 9)(x + 1)$$

Yes, that's my big surprise — getting to unFOIL after grouping and factoring out several greatest common factors.

Tackling Multiple Factoring Methods

Any factoring problem is a matter of recognizing what you have so you know what method to apply. With trinomials, you can use unFOIL if the trinomial is of the form $ax^2 + bx + c$. You can find the GCF of a trinomial if a common factor is available. When you have a binomial, you look for sums or differences of cubes and differences of squares. What I show you in this section is how the different methods often appear together, and what to do when the problem needs more than one method of factoring.

When factoring, determine what type of expression you have — binomial, trinomial, squares, cubes, and so on. This helps you decide what method to use. Keep going, checking inside all parentheses for more factoring opportunities, until you're done.

Beginning with binomials

This first example starts with a binomial. You see two squared factors in amongst the others, and that's it, so you don't expect anything exciting to happen. Oh, foolish you. Take the GCF out, and you find the difference of perfect cubes.



Factor $4x^4y - 108xy$.

The GCF of the two terms is $4xy$. Factor that out of each term first:

$$4x^4y - 108xy = 4xy(x^3 - 27)$$

Now you see that the binomial in the parentheses is the difference of two perfect cubes and can be factored using the rule from earlier in this chapter:

$$4xy(x^3 - 27) = 4xy(x - 3)(x^2 + 3x + 9)$$

Even though the last factor, the trinomial, seems to be a candidate for unFOIL, you don't have to bother. When you get a trinomial from factoring cubes, it's almost always prime. The only thing that may factor them is finding a GCF.

EXAMPLE



Factor $16x^4y^5(81 - z^4) - 54xy^2(81 - z^4)$.

The first thing that should jump out at you is that you see a common binomial factor of $(81 - z^4)$. Then, looking closer, you see that both terms contain factors of 2 and powers of x and y . Factoring out the GCF,

$$\begin{aligned} &16x^4y^5(81 - z^4) - 54xy^2(81 - z^4) = \\ &2xy^2(81 - z^4)(8x^3y^3 - 27) \end{aligned}$$

Now the factored form contains two binomial factors that can be factored. The first binomial is the difference of two squares, and the second binomial is the difference of cubes. Factoring,

$$= 2xy^2(9 - z^2)(9 + z^2)(2xy - 3)(4x^2y^2 + 6xy + 9)$$

And, of course, you realize that you're not finished. That first binomial can be factored as the difference of squares. Finishing the factoring,

$$= 2xy^2(3 - z)(3 + z)(9 + z^2)(2xy - 3)(4x^2y^2 + 6xy + 9)$$

Whew!

Finishing with binomials

In this section, I show you how you can start with four terms, and apply a form of grouping. The result is the difference of two squares.

EXAMPLE



Factor $x^2 + 8x + 16 - y^2$.

When using grouping, you usually divide the four or six terms into equal-size groups. Sometimes four terms can be separated into unequal groupings with three terms in one group and one term in the other. The way to spot these special types of factoring situations is to look for squares. Of course, you usually don't even look for unequal groupings unless other grouping methods have failed you.

This expression has four terms, but there's no good equal pairing of terms that will give you a set of useful common factors. Another option is to group unevenly. Group the first

three terms together because they form a trinomial that can be factored. That leaves the last term by itself:

$$x^2 + 8x + 16 - y^2 = (x^2 + 8x + 16) - y^2$$

Now you can factor the trinomial in the parentheses using unFOIL:

$$(x + 4)^2 - y^2$$

Notice that there are now two terms and that each is a perfect square.

Using the rule $a^2 - b^2 = (a + b)(a - b)$, finish this example:

$$(x + 4)^2 - y^2 = [(x + 4) + y][(x + 4) - y]$$

There's no big advantage to dropping the parentheses inside the brackets, so leave the answer the way it is.

Recognizing when you have a quadratic-like expression

This next example actually incorporates a factoring technique called *quadratic-like factorization*. The trinomials you factor using this method have a resemblance to the $ax^2 + bx + c$ format with one difference: Instead of exponents of 2 and 1 on the x variables, you have exponents of $2n$ and n — the first exponent is twice the second.

The quadratic-like equation $ax^{2n} + bx^n + c$ may factor into the product of two binomials of the form $(dx^n + e)(fx^n + g)$.

Factor $x^4 - 104x^2 + 400$.

There's no GCF, but you recognize that the first exponent is twice the second, so it fits into the pattern of a quadratic-like expression. It factors as follows:

$$x^4 - 104x^2 + 400 = (x^2 - 4)(x^2 - 100)$$



There are now two factors, but each of them is the difference of perfect squares:

$$(x^2 - 4)(x^2 - 100) = (x + 2)(x - 2)(x + 10)(x - 10)$$

You're finished!

Now I'll show you one more quadratic-like factorization — this time with exponents that are both negatives and fractions!



Factor the trinomial $3z^{-\frac{2}{3}} + 5z^{-\frac{1}{3}} - 2$.

The trinomial has the characteristic $2n$ and n format for the exponents. You may be overwhelmed a bit, though, with coming up with the two binomials needed. To get rid of the distraction of the negative fractional exponents, rewrite the expression using a different letter and the normal exponents:

$$3y^2 + 5y - 2$$

Now it may be more clear that you can factor the basic trinomial $3y^2 + 5y - 2 = (3y - 1)(y + 2)$ and then apply that pattern to the beginning trinomial:

$$3z^{-\frac{2}{3}} + 5z^{-\frac{1}{3}} - 2 = \left(3z^{-\frac{1}{3}} - 1\right)\left(z^{-\frac{1}{3}} + 2\right)$$

Knowing When Enough Is Enough

One of my favorite scenes from the movie *The Agony and the Ecstasy*, which chronicles Michelangelo's painting of the Sistine Chapel, comes when the pope enters the Sistine Chapel, looks up at the scaffolding, dripping paint, and Michelangelo perched up near the ceiling, and yells, "When will it be done?" Michelangelo's reply: "When I'm finished!"

The pope's lament can be applied to factoring problems: "When is it done?"

Factoring is done when no more parts can be factored. If you refer to the listing of ways to factor two, three, four, or more terms, then you can check off the options, discard those that

don't fit, and stop when none works. After doing one type of factoring, you should then look at the values in parentheses to see if any of them can be factored.



Factor $3x^5 - 18x^3 - 81x$.

The GCF of the terms is $3x$.

$$3x^5 - 18x^3 - 81x = 3x(x^4 - 6x^2 - 27)$$

The trinomial can be unFOILED:

$$3x(x^4 - 6x^2 - 27) = 3x(x^2 - 9)(x^2 + 3)$$

The first binomial is the difference of squares:

$$3x(x^2 - 9)(x^2 + 3) = 3x(x - 3)(x + 3)(x^2 + 3)$$

You're finished!

Recruiting the Remainder Theorem

The *remainder theorem* is used heavily when you're dealing with polynomials of high degrees and you want to graph them or find solutions for equations involving the polynomials. I go into these processes in great detail in *Algebra II For Dummies* (Wiley). For now, I pick out just the best part (lucky you) and show you how to make use of the remainder theorem and synthetic division to help you with your factoring chores.

The remainder theorem of algebra says that when you divide a polynomial by some linear binomial, the remainder resulting from the division is the same number as you'd get if you evaluated the polynomial using the opposite of the constant in the binomial.



The remainder theorem states that the remainder, R , resulting from dividing $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x^1 + a_0$ by $x + a$, is equal to $P(-a)$.

So, if you were to divide $x^3 + x^2 - 3x + 4$ by $x + 1$, the remainder would be $P(-1) = (-1)^3 + (-1)^2 - 3(-1) + 4 = -1 + 1 + 3 + 4 = 7$. This is what the long division looks like (and why you want to avoid it here):

$$\begin{array}{r} x^2 \qquad -3x \\ x+1 \overline{) x^3 + x^2 - 3x + 4} \\ \underline{-(x^3 + x^2)} \\ 0 - 3x + 4 \\ \underline{-(-3x - 3)} \\ 7 \end{array}$$

What you want, in factoring polynomials, is for the remainder to be a 0. No remainder means that the factor divided evenly into the polynomial. Long division can be tedious, and even the evaluation of polynomials can be a bit messy. So, *synthetic division* comes to the rescue.

Getting real with synthetic division

Synthetic division is a way of dividing a polynomial by a first-degree binomial without all the folderol. In this case, the folderol is all the variables — you just use coefficients and constants. To divide $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x^1 + a_0$ by $x + a$, you list all the coefficients, a_i , putting in zeros for missing terms in the decreasing powers, and then put an upside-down division sign in front of your work. You change the a in the binomial to its opposite and place it in the division sign. Then you multiply, add, multiply, add, and so on until all the coefficients have been added. The last number is your remainder.



Divide $x^4 + 5x^3 - 2x^2 - 28x - 12$ by $x + 3$ using synthetic division.

Write the coefficients in a row, with a -3 in front.

$$\begin{array}{r} -3 \mid 1 \quad 5 \quad -2 \quad -28 \quad -12 \end{array}$$

Now bring the 1 down, multiply it times -3 , put the result under the 5, and add. Multiply the sum by the -3 , put it under

the -2 and add. Multiply the sum times the -3 , put the product under the -28 , and so on. Yes!

$$\begin{array}{r|rrrrr} -3 & 1 & 5 & -2 & -28 & -12 \\ & & -3 & -6 & 24 & 12 \\ \hline & 1 & 2 & -8 & -4 & 0 \end{array}$$

The first four numbers along the bottom are the coefficients of the quotient, and the 0 is the remainder. When using synthetic division to help you with factoring, the 0 remainder is what you're looking for. It means that the binomial divides evenly and is a factor. The polynomial can now be written:

$$= (x + 3)(x^3 + 2x^2 - 8x - 4)$$

Next, you can see if the third-degree polynomial in the parentheses can be factored. (As it turns out, the trinomial is prime. Read the next section to see how that's determined.)



When rewriting a polynomial in factored form after applying synthetic division, be sure to change the sign of the number you used in the division to its opposite in the binomial.

Making good choices for synthetic division

Synthetic division is quick, neat, and relatively painless. But even quick, neat, and painless becomes tedious when you apply it without good results. When determining what might factor a particular polynomial, you need some clues. For example, you might be wondering if $(x - 1)$, $(x + 4)$, $(x - 3)$, or some other binomials are factors of $x^4 - x^3 - 7x^2 + x + 6$. I can tell just by looking that the binomial $(x + 4)$ won't work and that the other two factors are possibilities. How can I do that?



The *rational root theorem* says that if a *rational number* (a number that can be written as a fraction) is a solution, r , of the equation $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x^1 + a_0 = 0$, then

$$r = \frac{\text{some factor of } a_0}{\text{some factor of } a_n}.$$

Using the rational root theorem for my factoring, I just find these possible solutions of the equation and do the synthetic division using only these possibilities.

EXAMPLE



Factor $x^4 - x^3 - 7x^2 + x + 6$ using synthetic division, the rational root theorem, and the factor theorem.

REMEMBER



The *factor theorem* says that if $x = a$ is a root of a polynomial equation, then $x - a$ is a factor of the polynomial.

First, I make a list of the possible solutions if this were an equation. All the factors of the constant, a_0 , are ± 1 , ± 2 , ± 3 , and ± 6 .

Next, I divide each of the factors by the factors of the lead coefficient, a_n . I caught a break here. The lead coefficient is a 1, so the divisions are just the original numbers.

Now I use synthetic division to see if I get a remainder of 0 using any of these numbers:

$$\begin{array}{r|rrrrrr} 1 & 1 & -1 & -7 & 1 & 6 \\ & & & 1 & 0 & -7 & -6 \\ \hline & 1 & 0 & -7 & -6 & 0 & \end{array}$$

The number 1 is a solution, so $(x - 1)$ is a factor. Dividing again, into the result:

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

The number -1 is a solution, so $(x + 1)$ is a factor. The numbers across the bottom are the coefficients of the trinomial factor multiplying the two binomial factors, so you can write

$$x^4 - x^3 - 7x^2 + x + 6 = (x - 1)(x + 1)(x^2 - x - 6)$$

What's even nicer is that the trinomial is easily factored, giving you an end result of

$$(x - 1)(x + 1)(x - 3)(x + 2)$$

Factoring Rational Expressions

A rational expression consists of a fraction with algebraic terms in the numerator or denominator or both. Because the fraction line acts like a grouping symbol, you have to factor



the numerator and denominator separately before doing any reducing of fractions or other operations.

Reduce the fraction using a GCF: $\frac{4x^2 - 36x}{x^2 - 81}$

First, you factor the numerator by pulling out the greatest common factor, $4x$. The denominator is the difference of perfect squares.

$$\frac{4x^2 - 36x}{x^2 - 81} = \frac{4x(x - 9)}{(x - 9)(x + 9)}$$

The numerator and denominator have the common factor $(x - 9)$, so that can be divided out.

$$\frac{4x(\cancel{x - 9})}{(\cancel{x - 9})(x + 9)} = \frac{4x}{x + 9}$$



A *cardinal sin* in algebra is to reduce rational expressions incorrectly. You can only divide out factors — multipliers — but never terms. The error I show you here is a *huge* no-no:

$$\frac{4x^2 - 36x}{x^2 - 81}$$

~~$$\frac{4x^2 - 36x}{x^2 - 81}$$~~