Chapter 3

Ordering and Distributing: The Business of Algebra

In This Chapter

- Applying the order of operations
- Considering the operations with constants and variables
- Distributing over two terms or many

The *order of operations* is a biggie that you use frequently when working in algebra. It tells you what to do first, next, and last in a problem, whether terms are in grouping symbols or raised to a power.

And then, after paying attention to the order of operations, you find that algebra is full of converse actions. First, you're asked to factor, and then to *distribute* or "unfactor." Distributing is a way of changing a product into a sum or difference, which allows you to combine terms and do other exciting algebraic processes.

Taking Orders for Operations

In algebra, the order used in expressions with multiple operations depends on which mathematical operations are performed. If you're doing only addition or you're doing only multiplication, you can use any order you want. But as soon as you mix things up with addition and multiplication in the same expression, you have to pay close attention to the correct order. Mathematicians designed rules so that anyone reading a mathematical expression would do it the same way as everyone else and get the same *correct* answer. In the case of multiple signs and operations, working out the problems needs to be done in a specified *order,* from the first to the last. This is the *order of operations.*

According to the order of operations, work out the operations and signs in the following order:

1. **Powers and roots**
2. **Multiplication and division**
3. **Addition and subtraction**

If you have more than two operations of the same level, do them in order from left to right, following the order of operations. Also, if you have any grouping symbols, perform the operations inside the grouping symbols before using the result in the order of operations.

Simplify the following expression using the order of operations: $6^2 - 5 \cdot 4 + 2\sqrt{16} + 24 \div 6 - 5$.

Perform the power and root first:

$$6^2 - 5 \cdot 4 + 2\sqrt{16} + 24 \div 6 - 5 =$$
$$36 - 5 \cdot 4 + 2 \cdot 4 + 24 \div 6 - 5$$

A multiplication symbol is introduced when the radical is removed — to show that the 2 multiplies the result. Two multiplications and a division are performed to get

$$36 - 20 + 8 + 4 - 5$$

Now subtract and add:

$$16 + 8 + 4 - 5 = 23$$

When you have several operations of the same "level," you perform them moving from left to right through the expression.

Simplify the expression: $[8 \div (5 - 3)] \cdot 5$.

$$[8 \div (5 - 3)] \cdot 5 =$$

You have to perform the operations inside the parentheses and then the bracket before multiplying by 5:

$$[8 \div 2] \cdot 5 = 4 \cdot 5 = 20$$

Dealing with Distributing

Distributing items is the act of spreading them out equally. Algebraic distribution means to multiply each of the terms within the parentheses by another term that is outside the parentheses. Each term gets multiplied by the same amount.

To distribute a term over several other terms, multiply each of the other terms by the first. Distribution is multiplying each individual term in a grouped series of terms by a value outside the grouping.

$$a(b + c + d + e + \ldots) = ab + ac + ad + ae + \ldots$$

The addition signs could just as well be subtraction; and a is any real number: positive, negative, integer, or fraction.

Distribute the number 2 over the terms $4x + 3y - 6$.

1. **Multiply each term by the number(s) and/or variable(s) outside the parentheses.**

   $$2(4x + 3y - 6)$$

   $$2(4x) + 2(3y) - 2(6)$$

2. **Perform the multiplication operation in each term.**

   $$8x + 6y - 12$$

When a number is distributed over terms within parentheses, you multiply each term by that number. And then there are the signs: Positive (+) and negative (–) signs are simple to distribute, but distributing a negative sign can create errors.

When distributing a negative sign, each term has a change of sign: from negative to positive or from positive to negative.

Distribute –1 over the terms in the parentheses: $-(4x + 2y - 3z + 7)$ is the same as multiplying through by –1:

$$-1(4x + 2y - 3z + 7) =$$

$$-1(4x) - 1(2y) - 1(-3z) - 1(7) =$$

$$-4x - 2y + 3z - 7$$

Each term was changed to a term with the opposite sign.

Simplify the expression by distributing and combining like terms: $4x(x - 2) - (5x + 3)$. Treat the subtraction symbol as a distribution of –1 over the terms in the parentheses.

Distribute the $4x$ over the x and the –2 by multiplying both terms by $4x$:

$$4x(x - 2) = 4x(x) - 4x(2)$$

Distribute the negative sign over the $5x$ and the 3 by changing the sign of each term. Be careful — you can easily make a mistake if you stop after only changing the $5x$.

$$-(5x + 3) = -(+5x) - (+3)$$

Multiply and combine the like terms:

$$4x(x) - 4x(2) - (+5x) - (+3) =$$

$$4x^2 - 8x - 5x - 3 = 4x^2 - 13x - 3$$

Making Numbers and Variables Cooperate

Distributing variables over the terms in an algebraic expression involves multiplication rules and the rules for exponents. When different variables are multiplied together, they can be written side by side without using any multiplication symbols between them. If the same variable is multiplied as part of the distribution, then the exponents are added together. Let me show you a couple of distribution problems involving factors with exponents.

Distribute the a through the terms in the parentheses: $a(a^4 + 2a^2 + 3)$.

Multiply a times each term:

$$a(a^4 + 2a^2 + 3) = a \cdot a^4 + a \cdot 2a^2 + a \cdot 3$$

Use the rules of exponents to simplify:

$$a^5 + 2a^3 + 3a$$

Distribute z^4 over the terms in the expression $2z^2 - 3z^{-2} + z^{-4} + 5z^{\frac{1}{3}}$.

Distribute the z^4 by multiplying it times each term:

$$z^4\left(2z^2 - 3z^{-2} + z^{-4} + 5z^{\frac{1}{3}}\right) =$$

$$z^4 \cdot 2z^2 - z^4 \cdot 3z^{-2} + z^4 \cdot z^{-4} + z^4 \cdot 5z^{\frac{1}{3}}$$

Simplify by adding the exponents:

$$2z^{4+2} - 3z^{4-2} + z^{4-4} + 5z^{4+\frac{1}{3}} =$$

$$2z^6 - 3z^2 + z^0 + 5z^{\frac{13}{3}} = 2z^6 - 3z^2 + 1 + 5z^{\frac{13}{3}}$$

The exponent 0 means the value of the expression is 1. $x^0 = 1$ for any real number x except 0.

You combine exponents with different signs by using the rules for adding and subtracting signed numbers. Fractional exponents are combined after finding common denominators. Exponents that are improper fractions are left in that form.

Relating negative exponents to fractions

As the heading suggests, a base that has a negative exponent can be changed to a fraction. The base and the exponent become part of the denominator of the fraction, but the exponent loses its negative sign in the process. Then you cap it all off with a 1 in the numerator.

The formula for changing negative exponents to fractions is $a^{-n} = \frac{1}{a^n}$. (See Chapter 2 for more details on negative exponents.)

In the following example, I show you how a negative exponent leads to a fractional answer.

Distribute the $5a^{-3}b^{-2}$ over each term in the parentheses:

$$5a^{-3}b^{-2}(2ab^3 - 3a^2b^2 + 4a^4b - ab) =$$

$$5a^{-3}b^{-2}(2ab^3) - (5a^{-3}b^{-2})(3a^2b^2) + (5a^{-3}b^{-2})(4a^4b) - (5a^{-3}b^{-2})(ab)$$

Multiplying the numbers and adding the exponents:

$$10a^{-3+1}b^{-2+3} - 15a^{-3+2}b^{-2+2} + 20a^{-3+4}b^{-2+1} - 5a^{-3+1}b^{-2+1}$$

The factor of b with the 0 exponent becomes 1:

$$10a^{-2}b^{1} - 15a^{-1}b^{0} + 20a^{1}b^{-1} - 5a^{-2}b^{-1}$$

This next step shows the final result without negative exponents — using the formula for changing negative exponents to fractions (see earlier in this section):

$$\frac{10b}{a^2} - \frac{15}{a} + \frac{20a}{b} - \frac{5}{a^2b}$$

Creating powers with fractions

Exponents that are fractions work the same way as exponents that are integers. When multiplying factors with the same base, the exponents are added together. The only hitch is that the fractions must have the same denominator to be added. (The rules don't change just because the fractions are exponents.)

Distribute and simplify: $x^{\frac{1}{4}}y^{\frac{2}{3}}\left(x^{\frac{1}{2}} + x^{\frac{3}{4}}y^{\frac{1}{3}} - y^{-\frac{1}{3}}\right)$.

Multiply the factor times each term:

$$x^{\frac{1}{4}}y^{\frac{2}{3}} \cdot x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{2}{3}} \cdot x^{\frac{3}{4}}y^{\frac{1}{3}} - x^{\frac{1}{4}}y^{\frac{2}{3}} \cdot y^{-\frac{1}{3}}$$

Rearrange the variables and add the exponents:

$$x^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{2}{3}} + x^{\frac{1}{4}}x^{\frac{3}{4}}y^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{4}}y^{\frac{2}{3}}y^{-\frac{1}{3}} =$$

$$x^{\frac{1}{4}+\frac{1}{2}}y^{\frac{2}{3}} + x^{\frac{1}{4}+\frac{3}{4}}y^{\frac{2}{3}+\frac{1}{3}} - x^{\frac{1}{4}}y^{\frac{2}{3}-\frac{1}{3}}$$

Finish up by adding the fractions:

$$x^{\frac{3}{4}}y^{\frac{2}{3}} + x^{1}y^{1} - x^{\frac{1}{4}}y^{\frac{1}{3}}$$

Simplify by distributing: $\sqrt{xy^3}\left(\sqrt{x^5y} - \sqrt{xy^7}\right)$.

Change the radical notation to fractional exponents:

$$\sqrt{xy^3}\left(\sqrt{x^5y} - \sqrt{xy^7}\right) = \left(xy^3\right)^{\frac{1}{2}}\left[\left(x^5y\right)^{\frac{1}{2}} - \left(xy^7\right)^{\frac{1}{2}}\right]$$

Raise the powers of the products inside the parentheses:

$$x^{\frac{1}{2}}y^{\frac{3}{2}}\left[x^{\frac{5}{2}}y^{\frac{1}{2}} - x^{\frac{1}{2}}y^{\frac{7}{2}}\right]$$

Distribute the outside term over each term within the parentheses:

$$x^{\frac{1}{2}}y^{\frac{3}{2}}\left(x^{\frac{5}{2}}y^{\frac{1}{2}}\right) - x^{\frac{1}{2}}y^{\frac{3}{2}}\left(x^{\frac{1}{2}}y^{\frac{7}{2}}\right)$$

Add the exponents of the variables:

$$x^{\frac{6}{2}}y^{\frac{4}{2}} - x^{\frac{2}{2}}y^{\frac{10}{2}}$$

Simplify the fractional exponents:

$$x^3y^2 - x^1y^5$$

Making Distributions Over More Than One Term

The preceding sections in this chapter describe how to distribute one term over several others. This section shows you how to distribute a *binomial* (a polynomial with two terms). This same procedure can be used to distribute polynomials with three or more terms.

Distributing two terms (a *binomial*) over several terms amounts to just applying the distribution process twice. Following is an example with the steps telling you how to distribute a binomial over some polynomial.

Multiply using distribution: $(x^2 + 1)(y - 2)$.

1. **Break the binomial into its two terms.**

   In this case, $(x^2 + 1)(y - 2)$, break the first binomial into its two terms, x^2 and 1.

2. **Distribute each term over the other factor.**

   Multiply the first term, x^2, times the second binomial, and multiply the second term, 1, times the second binomial:

   $x^2(y - 2) + 1(y - 2)$

3. **Do the two distributions.**

   $x^2(y - 2) + 1(y - 2) = x^2y - 2x^2 + y - 2$

4. **Simplify and combine any like terms.**

   In this case, nothing can be combined; none of the terms is like any other.

When distributing a polynomial (many terms) over any number of other terms, multiply each term in the first factor times each term in the second factor. When the distribution is done, combine anything that goes together to simplify.

$$(a + b + c + d + \ldots)(z + y + x + w + \ldots) =$$

$$az + ay + ax + aw + \ldots + bz + by + bx + bw + \ldots + cz + cy + cx + cw + \ldots + dz + dy + dx + dw + \ldots$$