

Number and Operations

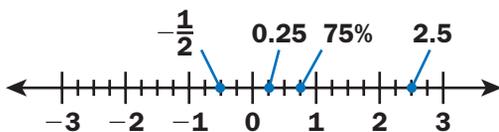
► **Understand numbers, ways of representing numbers, relationships among numbers, and number systems**

Fraction, Decimal, and Percent

A percent is a ratio that compares a number to 100. To write a percent as a fraction, drop the percent sign, and use the number as the numerator in a fraction with a denominator of 100. Simplify, if possible. For example, $76\% = \frac{76}{100}$, or $\frac{19}{25}$. To write a fraction as a percent, convert it to an equivalent fraction with a denominator of 100. For example, $\frac{3}{4} = \frac{75}{100}$, or 75%. A fraction can be expressed as a percent by first converting the fraction to a decimal (divide the numerator by the denominator) and then converting the decimal to a percent by moving the decimal point two places to the right.

Comparing Numbers on a Number Line

In order to compare and understand the relationship between real numbers in various forms, it is helpful to use a number line. The zero point on a number line is called the origin; the points to the left of the origin are negative, and those to the right are positive. The number line below shows how numbers in percent, decimal, fraction, and integer form can be compared.



Percents Greater Than 100 and Less Than 1

Percents greater than 100% represent values greater than 1. For example, if the weight of an object is 250% of another, it is 2.5, or $2\frac{1}{2}$, times the weight.

Percents less than 1 represent values less than $\frac{1}{100}$. In other words, 0.1% is one tenth of one percent, which can also be represented in decimal form as 0.001, or in fraction form as $\frac{1}{1,000}$. Similarly, 0.01% is one hundredth of one percent or 0.0001 or $\frac{1}{10,000}$.

Ratio, Rate, and Proportion

A ratio is a comparison of two numbers using division. If a basketball player makes 8 out of 10 free throws, the ratio is written as 8 to 10, 8:10, or $\frac{8}{10}$. Ratios are usually written in simplest form. In simplest form, the ratio “8 out of 10” is 4 to 5, 4:5, or $\frac{4}{5}$. A rate is a ratio of two measurements having different kinds of units—cups per gallon, or miles per hour, for example. When a rate is simplified so that it has a denominator of 1, it is called a unit rate. An example of a unit rate is 9 miles per hour. A proportion is an equation stating that two ratios are equal. $\frac{3}{18} = \frac{13}{78}$ is an example of a proportion. The cross products of a proportion are also equal. $\frac{3}{18} = \frac{13}{78}$ and $3 \times 78 = 18 \times 13$.

Representing Large and Small Numbers

In order to represent large and small numbers, it is important to understand the number system. Our number system is based on 10, and the value of each place is 10 times the value of the place to its right.



The value of a digit is the product of a digit and its place value. For instance, in the number 6,400, the 6 has a value of six thousands and the 4 has a value of four hundreds. A place value chart can help you read numbers. In the chart, each group of three digits is called a period. Commas separate the periods: the ones period, the thousands period, the millions period, and so on. Values to the right of the ones period are decimals. By understanding place value you can write very large numbers like 5 billion and more, and very small numbers that are less than 1.

Scientific Notation

When dealing with very large numbers like 1,500,000, or very small numbers like 0.000015, it is helpful to keep track of their value by writing the numbers in scientific notation. Powers of 10 with positive exponents are used with a decimal between 1 and 10 to express large numbers. The exponent represents the number of places the decimal point is moved to the right. So, 528,000 is written in scientific notation as 5.28×10^5 . Powers of 10 with negative exponents are used with a decimal between 1 and 10 to express small numbers. The exponent represents the number of places the decimal point is moved to the left. The number 0.00047 is expressed as 4.7×10^{-4} .

Factor, Multiple, and Prime Factorization

Two or more numbers that are multiplied to form a product are called factors. Divisibility rules can be used to determine whether 2, 3, 4, 5, 6, 8, 9, or 10 are factors of a given number. Multiples are the products of a given number and various integers.

For example, 8 is a multiple of 4 because $4 \times 2 = 8$. A prime number is a whole number that has exactly two factors: 1 and itself. A composite number is a whole number that has more than two factors. Zero and 1 are neither prime nor composite. A composite number can be expressed as the product of its prime factors. The prime factorization of 40 is $2 \times 2 \times 2 \times 5$, or $2^3 \times 5$. The numbers 2 and 5 are prime numbers.

Integers

A negative number is a number less than zero. Negative numbers like -8 , positive numbers like $+6$, and zero are members of the set of integers. Integers can be represented as points on a number line. A set of integers can be written $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ where \dots means “continues indefinitely.”

Real, Rational, and Irrational Numbers

The real number system is made up of the sets of rational and irrational numbers. Rational numbers are numbers that can be written in the form a/b where a and b are integers and $b \neq 0$. Examples are 0.45, $\frac{1}{2}$, and $\sqrt{36}$. Irrational numbers are non-repeating, non-terminating decimals. Examples are $\sqrt{71}$, π , and 0.020020002...

Complex and Imaginary Numbers

A complex number is a mathematical expression with a real number element and an imaginary number element. Imaginary numbers are multiples of i , the “imaginary” square root of -1 . Complex numbers are represented by $a + bi$, where a and b are real numbers and i represents the imaginary element. When a quadratic equation

does not have a real number solution, the solution can be represented by a complex number. Like real numbers, complex numbers can be added, subtracted, multiplied, and divided.

Vectors and Matrices

A matrix is a set of numbers or elements arranged in rows and columns to form a rectangle. The number of rows is represented by m and the number of columns is represented by n . To describe the number of rows and columns in a matrix, list the number of rows first using the format $m \times n$. Matrix A below is a 3×3 matrix because it has 3 rows and 3 columns. To name an element of a matrix, the letter i is used to denote the row and j is used to denote the column, and the element is labeled in the form $a_{i,j}$. In matrix A below, $a_{3,2}$ is 4.

$$\text{Matrix A} = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 6 & 8 \\ 3 & 4 & 5 \end{pmatrix}$$

A vector is a matrix with only one column or row of elements. A transposed column vector, or a column vector turned on its side, is a row vector. In the example below, row vector b' is the transpose of column vector b .

$$b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$b' = (1 \ 2 \ 3 \ 4)$$

► Understand meanings of operations and how they relate to one another

Properties of Addition and Multiplication

Properties are statements that are true for any numbers. For example, $3 + 8$ is the same as $8 + 3$ because each expression equals 11. This illustrates the Commutative Property of Addition. Likewise, $3 \times 8 = 8 \times 3$ illustrates the Commutative Property of Multiplication.

When evaluating expressions, it is often helpful to group or associate the numbers. The Associative Property says that the way in which numbers are grouped when added or multiplied does not change the sum or product. The following properties are also true:

- **Additive Identity Property:** When 0 is added to any number, the sum is the number.
- **Multiplicative Identity Property:** When any number is multiplied by 1, the product is the number.
- **Multiplicative Property of Zero:** When any number is multiplied by 0, the product is 0.

Rational Numbers

A number that can be written as a fraction is called a rational number. Terminating and repeating decimals are rational numbers because both can be written as fractions.



Decimals that are neither terminating nor repeating are called irrational numbers because they cannot be written as fractions. Terminating decimals can be converted to fractions by placing the number (without the decimal point) in the numerator. Count the number of places to the right of the decimal point, and in the denominator, place a 1 followed by a number of zeros equal to the number of places that you counted. The fraction can then be reduced to simplest form.

Writing a Fraction as a Decimal

Any fraction $\frac{a}{b}$, where $b \neq 0$, can be written as a decimal by dividing the numerator by the denominator. So, $\frac{a}{b} = a \div b$. If the division ends, or terminates, when the remainder is zero, the decimal is a terminating decimal. Not all fractions can be written as terminating decimals. Some have a repeating decimal. A bar indicates that the decimal repeats forever. For example, the fraction $\frac{4}{9}$ can be converted to a repeating decimal, $0.\overline{4}$

Adding and Subtracting Like Fractions

Fractions with the same denominator are called like fractions. To add like fractions, add the numerators and write the sum over the denominator. To add mixed numbers with like fractions, add the whole numbers and fractions separately, adding the numerators of the fractions, then simplifying if necessary. The rule for subtracting fractions with like denominators

is similar to the rule for adding. The numerators can be subtracted and the difference written over the denominator. Mixed numbers are written as improper fractions before subtracting. These same rules apply to adding or subtracting like algebraic fractions.

Adding and Subtracting Unlike Fractions

Fractions with different denominators are called unlike fractions. The least common multiple of the denominators is used to rename the fractions with a common denominator. After a common denominator is found, the numerators can then be added or subtracted. To add mixed numbers with unlike fractions, rename the mixed numbers as improper fractions. Then find a common denominator, add the numerators, and simplify the answer.

Multiplying Rational Numbers

To multiply fractions, multiply the numerators and multiply the denominators. If the numerators and denominators have common factors, they can be simplified before multiplication. If the fractions have different signs, then the product will be negative. Mixed numbers can be multiplied in the same manner, after first renaming them as improper fractions. A fraction that contains one or more variables in the numerator or denominator is called an algebraic fraction. Algebraic fractions may be multiplied using the same method described above.

Dividing Rational Numbers

To divide a number by a rational number (a fraction, for example), multiply the first number by the multiplicative inverse of the second. Two numbers whose product is 1 are called multiplicative inverses, or reciprocals. $\frac{7}{4} \times \frac{4}{7} = 1$. When dividing by a mixed number, first rename it as an improper fraction, and then multiply by its multiplicative inverse. This process of multiplying by a number's reciprocal can also be used when dividing algebraic fractions.

Adding Integers

To add integers with the same sign, add their absolute values. The sum then takes the same sign as the addends. The equation $-5 + (-2) = -7$ is an example of adding two integers with the same sign. To add integers with different signs, subtract their absolute values. The sum takes the same sign as the addend with the greater absolute value.

Subtracting Integers

The rules for adding integers are extended to the subtraction of integers. To subtract an integer, add its additive inverse. For example, to find the difference $2 - 5$, add the additive inverse of 5 to 2: $2 + (-5) = -3$. The rule for subtracting integers can be used to solve real-world problems and to evaluate algebraic expressions.

Additive Inverse Property

Two numbers with the same absolute value but different signs are called opposites. For example, -4 and 4 are opposites. An integer and its opposite are also called additive inverses. The Additive Inverse Property says that the sum of any number and its

additive inverse is zero. The Commutative, Associative, and Identity Properties also apply to integers. These properties help when adding more than two integers.

Absolute Value

In mathematics, when two integers on a number line are on opposite sides of zero, and they are the same distance from zero, they have the same absolute value. The symbol for absolute value is two vertical bars on either side of the number. For example, $|-5| = 5$.

Multiplying Integers

Since multiplication is repeated addition, $3(-7)$ means that -7 is used as an addend 3 times. By the Commutative Property of Multiplication, $3(-7) = -7(3)$. The product of two integers with different signs is always negative. The product of two integers with the same sign is always positive.

Dividing Integers

The quotient of two integers can be found by dividing the numbers using their absolute values. The quotient of two integers with the same sign is positive, and the quotient of two integers with a different sign is negative. $-12 \div (-4) = 3$ and $12 \div (-4) = -3$. The division of integers is used in statistics to find the average, or mean, of a set of data. When finding the mean of a set of numbers, find the sum of the numbers, and then divide by the number in the set.

Adding and Multiplying Vectors and Matrices

In order to add two matrices together, they must have the same number of rows and columns. In matrix addition, the



corresponding elements are added to each other. In other words $(a + b)_{ij} = a_{ij} + b_{ij}$. For example,

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+3 & 2+6 \\ 2+0 & 1+1 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 2 & 2 \end{pmatrix}$$

Matrix multiplication requires that the number of elements in each row in the first matrix is equal to the number of elements in each column in the second. The elements of the first row of the first matrix are multiplied by the corresponding elements of the first column of the second matrix and then added together to get the first element of the product matrix. To get the second element, the elements in the first row of the first matrix are multiplied by the corresponding elements in the second column of the second matrix then added, and so on, until every row of the first matrix is multiplied by every column of the second. See the example below.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (1 \times 3) + (2 \times 0) & (1 \times 6) + (2 \times 1) \\ (3 \times 3) + (4 \times 0) & (3 \times 6) + (4 \times 1) \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 9 & 22 \end{pmatrix}$$

Vector addition and multiplication are performed in the same way, but there is only one column and one row.

Permutations and Combinations

Permutations and combinations are used to determine the number of possible outcomes in different situations. An arrangement, listing, or pattern in which order is important is called a permutation. The symbol $P(6, 3)$ represents the number of permutations of 6 things taken 3 at a time. For $P(6, 3)$, there are $6 \times 5 \times 4$ or 120 possible outcomes. An arrangement or listing where order is not important is called a combination. The symbol $C(10, 5)$

represents the number of combinations of 10 things taken 5 at a time. For $C(10, 5)$, there are $(10 \times 9 \times 8 \times 7 \times 6) \div (5 \times 4 \times 3 \times 2 \times 1)$ or 252 possible outcomes.

Powers and Exponents

An expression such as $3 \times 3 \times 3 \times 3$ can be written as a power. A power has two parts, a base and an exponent. $3 \times 3 \times 3 \times 3 = 3^4$. The base is the number that is multiplied (3). The exponent tells how many times the base is used as a factor (4 times). Numbers and variables can be written using exponents. For example, $8 \times 8 \times 8 \times m \times m \times m \times m \times m$ can be expressed $8^3 m^5$. Exponents also can be used with place value to express numbers in expanded form. Using this method, 1,462 can be written as $(1 \times 10^3) + (4 \times 10^2) + (6 \times 10^1) + (2 \times 10^0)$.

Squares and Square Roots

The square root of a number is one of two equal factors of a number. Every positive number has both a positive and a negative square root. For example, since $8 \times 8 = 64$, 8 is a square root of 64. Since $(-8) \times (-8) = 64$, -8 is also a square root of 64. The notation $\sqrt{\quad}$ indicates the positive square root, $-\sqrt{\quad}$ indicates the negative square root, and $\pm\sqrt{\quad}$ indicates both square roots. For example, $\sqrt{81} = 9$, $-\sqrt{49} = -7$, and $\pm\sqrt{4} = \pm 2$. The square root of a negative number is an imaginary number because any two factors of a negative number must have different signs, and are therefore not equivalent.

Logarithm

A logarithm is the inverse of exponentiation. The logarithm of a number x in base b is equal to the number n . Therefore, $b^n = x$ and $\log_b x = n$. For example, $\log_4(64) = 3$ because $4^3 = 64$.

The most commonly used bases for logarithms are 10, the common logarithm; 2, the binary logarithm; and the constant e , the natural logarithm (also called $\ln(x)$ instead of $\log_e(x)$). Below is a list of some of the rules of logarithms that are important to understand if you are going to use them.

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

$$\log_b(x/y) = \log_b(x) - \log_b(y)$$

$$\log_b(1/x) = -\log_b(x)$$

$$\log_b(x)y = y\log_b(x)$$

Compute fluently and make reasonable estimates

Estimation by Rounding

When rounding numbers, look at the digit to the right of the place to which you are rounding. If the digit is 5 or greater, round up. If it is less than 5, round down. For example, to round 65,137 to the nearest hundred, look at the number in the tens place. Since 3 is less than 5, round down to 65,100. To round the same number to the nearest ten thousandth, look at the number in the thousandths place. Since it is 5, round up to 70,000.

Finding Equivalent Ratios

Equivalent ratios have the same meaning. Just like finding equivalent fractions, to find an equivalent ratio, multiply or divide both sides by the same number. For example, you can multiply 7 by both sides of the ratio 6:8 to get 42:56. Instead, you can also divide

both sides of the same ratio by 2 to get 3:4. Find the simplest form of a ratio by dividing to find equivalent ratios until you can't go any further without going into decimals. So, 160:240 in simplest form is 2:3. To write a ratio in the form 1:n, divide both sides by the left-hand number. In other words, to change 8:20 to 1:n, divide both sides by 8 to get 1:2.5.

Front-End Estimation

Front-end estimation can be used to quickly estimate sums and differences before adding or subtracting. To use this technique, add or subtract just the digits of the two highest place values, and replace the other place values with zero. This will give you an estimation of the solution of a problem. For example, $93,471 - 22,825$ can be changed to $93,000 - 22,000$ or 71,000. This estimate can be compared to your final answer to judge its correctness.

Judging Reasonableness

When solving an equation, it is important to check your work by considering how reasonable your answer is. For example, consider the equation $9\frac{3}{4} \times 4\frac{1}{3}$. Since $9\frac{3}{4}$ is between 9 and 10 and $4\frac{1}{3}$ is between 4 and 5, only values that are between 9×4 or 36 and 10×5 or 50 will be reasonable. You can also use front-end estimation, or you can round and estimate a reasonable answer. In the equation 73×25 , you can round and solve to estimate a reasonable answer to be near 70×30 or 2,100.

Algebra

► Understand patterns, relations, and functions

Relation

A relation is a generalization comparing sets of ordered pairs for an equation or inequality such as $x = y + 1$ or $x > y$. The first element in each pair, the x values, form the domain. The second values in each pair, the y values, form the range.

Function

A function is a special relation in which each member of the domain is paired with exactly one member in the range. Functions may be represented using ordered pairs, tables, or graphs. One way to determine whether a relation is a function is to use the vertical line test. Using an object to represent a vertical line, move the object from left to right across the graph. If, for each value of x in the domain, the object passes through no more than one point on the graph, then the graph represents a function.

Linear and Nonlinear Functions

Linear functions have graphs that are straight lines. These graphs represent constant rates of change. In other words, the slope between any two pairs of points on the graph is the same. Nonlinear functions do not have constant rates of change. The slope changes along these graphs. Therefore, the graphs of nonlinear functions are *not* straight lines. Graphs of curves represent nonlinear functions. The equation for a linear function can be written in the form $y = mx + b$, where m represents the

constant rate of change, or the slope. Therefore, you can determine whether a function is linear by looking at the equation. For example, the equation $y = \frac{3}{x}$ is nonlinear because x is in the denominator and the equation cannot be written in the form $y = mx + b$. A nonlinear function does not increase or decrease at a constant rate. You can check this by using a table and finding the increase or decrease in y for each regular increase in x . For example, if for each increase in x by 2, y does not increase or decrease the same amount each time, the function is nonlinear.

Linear Equations in Two Variables

In a linear equation with two variables, such as $y = x - 3$, the variables appear in separate terms and neither variable contains an exponent other than 1. The graphs of all linear equations are straight lines. All points on a line are solutions of the equation that is graphed.

Quadratic and Cubic Functions

A quadratic function is a polynomial equation of the second degree, generally expressed as $ax^2 + bx + c = 0$, where a , b , and c are real numbers and a is not equal to zero. Similarly, a cubic function is a polynomial equation of the third degree, usually expressed as $ax^3 + bx^2 + cx + d = 0$. Quadratic functions can be graphed using an equation or a table of values. For example, to graph $y = 3x^2 + 1$, substitute the values -1 , -0.5 , 0 , 0.5 , and 1 for x to yield the point coordinates $(-1, 4)$, $(-0.5, 1.75)$, $(0, 1)$, $(0.5, 1.75)$, and $(1, 4)$. Plot these points on a coordinate grid and connect

the points in the form of a parabola. Cubic functions also can be graphed by making a table of values. The points of a cubic function form a curve. There is one point at which the curve changes from opening upward to opening downward, or vice versa, called the point of inflection.

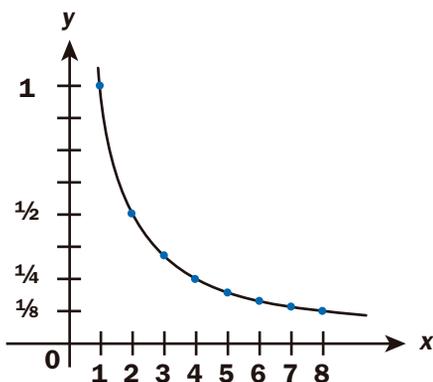
Slope

Slope is the ratio of the rise, or vertical change, to the run, or horizontal change of a line: slope = rise/run. Slope (m) is the same for any two points on a straight line and can be found by using the coordinates of any two points on the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \text{ where } x_2 \neq x_1.$$

Asymptotes

An asymptote is a straight line that a curve approaches but never actually meets or crosses. Theoretically, the asymptote meets the curve at infinity. For example, in the function $f(x) = \frac{1}{x}$, two asymptotes are being approached: the line $y = 0$ and $x = 0$. See the graph of the function below.



► Represent and analyze mathematical situations and structures using algebraic symbols

Variables and Expressions

Algebra is a language of symbols. A variable is a placeholder for a changing value. Any letter, such as x , can be used as a variable. Expressions such as $x + 2$ and $4x$ are algebraic expressions because they represent sums and/or products of variables and numbers. Usually, mathematicians avoid the use of i and e for variables because they have other mathematical meanings ($i = \sqrt{-1}$ and e is used with natural logarithms). To evaluate an algebraic expression, replace the variable or variables with known values, and then solve using order of operations. Translate verbal phrases into algebraic expressions by first defining a variable: choose a variable and a quantity for the variable to represent. In this way, algebraic expressions can be used to represent real-world situations.

Constant and Coefficient

A constant is a fixed value unlike a variable, which can change. Constants are usually represented by numbers, but they can also be represented by symbols. For example, π is a symbolic representation of the value 3.1415.... A coefficient is a constant by which a variable or other object is multiplied. For example, in the expression $7x^2 + 5x + 9$, the coefficient of x^2 is 7 and the coefficient of x is 5. The number 9 is a constant and not a coefficient.

Monomial and Polynomial

A monomial is a number, a variable, or a product of numbers and/or variables such as 3×4 . An algebraic expression that



contains one or more monomials is called a polynomial. In a polynomial, there are no terms with variables in the denominator and no terms with variables under a radical sign. Polynomials can be classified by the number of terms contained in the expression. Therefore, a polynomial with two terms is called a binomial ($z^2 - 1$), and a polynomial with three terms is called a trinomial ($2y^3 + 4y^2 - y$). Polynomials also can be classified by their degrees. The degree of a monomial is the sum of the exponents of its variables. The degree of a nonzero constant such as 6 or 10 is 0. The constant 0 has no degree. For example, the monomial $4b^5c^2$ had a degree of 7. The degree of a polynomial is the same as that of the term with the greatest degree. For example, the polynomial $3x^4 - 2y^3 + 4y^2 - y$ has a degree of 4.

Equation

An equation is a mathematical sentence that states that two expressions are equal. The two expressions in an equation are always separated by an equal sign. When solving for a variable in an equation, you must perform the same operations on both sides of the equation in order for the mathematical sentence to remain true.

Solving Equations with Variables

To solve equations with variables on both sides, use the Addition or Subtraction Property of Equality to write an equivalent equation with the variables on the same side. For example, to solve $5x - 8 = 3x$, subtract $3x$ from each side to get $2x - 8 = 0$. Then add 8 to each side to get $2x = 8$. Finally, divide each side by 2 to find that $x = 4$.

Solving Equations with Grouping Symbols

Equations often contain grouping symbols such as parentheses or brackets. The first step in solving these equations is to use the Distributive Property to remove the grouping symbols. For example $5(x + 2) = 25$ can be changed to $5x + 10 = 25$, and then solved to find that $x = 3$.

Some equations have no solution. That is, there is no value of the variable that results in a true sentence. For such an equation, the solution set is called the null or empty set, and is represented by the symbol \emptyset or $\{\}$. Other equations may have every number as the solution. An equation that is true for every value of the variable is called the identity.

Inequality

A mathematical sentence that contains the symbols $<$ (less than), $>$ (greater than), \leq (less than or equal to), or \geq (greater than or equal to) is called an inequality. For example, the statement that it is legal to drive 55 miles per hour or slower on a stretch of the highway can be shown by the sentence $s \leq 55$. Inequalities with variables are called open sentences. When a variable is replaced with a number, the inequality may be true or false.

Solving Inequalities

Solving an inequality means finding values for the variable that make the inequality true. Just as with equations, when you add or subtract the same number from each side of an inequality, the inequality remains true. For example, if you add 5 to each side of the inequality $3x < 6$, the resulting inequality $3x + 5 < 11$ is also true. Adding or subtracting the same number from each side of an inequality does not



affect the inequality sign. When multiplying or dividing each side of an inequality by the same positive number, the inequality remains true. In such cases, the inequality symbol does not change. When multiplying or dividing each side of an inequality by a negative number, the inequality symbol must be reversed. For example, when dividing each side of the inequality $-4x \geq -8$ by -2 , the inequality sign must be changed to \leq for the resulting inequality, $2x \leq 4$, to be true. Since the solutions to an inequality include all rational numbers satisfying it, inequalities have an infinite number of solutions.

Representing Inequalities on a Number Line

The solutions of inequalities can be graphed on a number line. For example, if the solution of an inequality is $x < 5$, start an arrow at 5 on the number line, and continue the arrow to the left to show all values less than 5 as the solution. Put an open circle at 5 to show that the point 5 is *not* included in the graph. Use a closed circle when graphing solutions that are greater than or equal to, or less than or equal to, a number.

Order of Operations

Solving a problem may involve using more than one operation. The answer can depend on the order in which you do the operations. To make sure that there is just one answer to a series of computations, mathematicians have agreed upon an order in which to do the operations. First simplify within the parentheses, and then evaluate any exponents. Then multiply and divide from left to right, and finally add and subtract from left to right.

Parametric Equations

Given an equation with more than one unknown, a statistician can draw conclusions about those unknown quantities through the use of parameters, independent variables that the statistician already knows something about. For example, you can find the velocity of an object if you make some assumptions about distance and time parameters.

Recursive Equations

In recursive equations, every value is determined by the previous value. You must first plug an initial value into the equation to get the first value, and then you can use the first value to determine the next one, and so on. For example, in order to determine what the population of pigeons will be in New York City in three years, you can use an equation with the birth, death, immigration, and emigration rates of the birds. Input the current population size into the equation to determine next year's population size, then repeat until you have calculated the value for which you are looking.

 **Use mathematical models to represent and understand quantitative relationships**

Solving Systems of Equations

Two or more equations together are called a system of equations. A system of equations can have one solution, no solution, or infinitely many solutions. One method for solving a system of equations is to graph the equations on the same coordinate plane. The coordinates of the point where the

graphs intersect is the solution. In other words, the solution of a system is the ordered pair that is a solution of all equations. A more accurate way to solve a system of two equations is by using a method called substitution. Write both equations in terms of y . Replace y in the first equation with the right side of the second equation. Check the solution by graphing. You can solve a system of three equations using matrix algebra.

Graphing Inequalities

To graph an inequality, first graph the related equation, which is the boundary. All points in the shaded region are solutions of the inequality. If an inequality contains the symbol \leq or \geq , then use a solid line to indicate that the boundary is included in the graph. If an inequality contains the symbol $<$ or $>$, then use a dashed line to indicate that the boundary is not included in the graph.

► Analyze change in various contexts

Rate of Change

A change in one quantity with respect to another quantity is called the rate of change. Rates of change can be described using slope:

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}$$

You can find rates of change from an equation, a table, or a graph. A special type

of linear equation that describes rate of change is called a direct variation. The graph of a direct variation always passes through the origin and represents a proportional situation. In the equation $y = kx$, k is called the constant of variation. It is the slope, or rate of change. As x increases in value, y increases or decreases at a constant rate k , or y varies directly with x . Another way to say this is that y is directly proportional to x . The direct variation $y = kx$ also can be written as $k = \frac{y}{x}$. In this form, you can see that the ratio of y to x is the same for any corresponding values of y and x .

Slope-Intercept Form

Equations written as $y = mx + b$, where m is the slope and b is the y -intercept, are linear equations in slope-intercept form. For example, the graph of $y = 5x - 6$ is a line that has a slope of 5 and crosses the y -axis at $(0, -6)$. Sometimes you must first write an equation in slope-intercept form before finding the slope and y -intercept. For example, the equation $2x + 3y = 15$ can be expressed in slope-intercept form by subtracting $2x$ from each side and then dividing by 3: $y = -\frac{2}{3}x + 5$, revealing a slope of $-\frac{2}{3}$ and a y -intercept of 5. You can use the slope-intercept form of an equation to graph a line easily. Graph the y -intercept and use the slope to find another point on the line, then connect the two points with a line.

Geometry

► **Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships**

Angles

Two rays that have the same endpoint form an angle. The common endpoint is called the vertex, and the two rays that make up the angle are called the sides of the angle. The most common unit of measure for angles is the degree. Protractors can be used to measure angles or to draw an angle of a given measure. Angles can be classified by their degree measure. Acute angles have measures less than 90° but greater than 0° . Obtuse angles have measures greater than 90° but less than 180° . Right angles have measures of 90° .

Triangles

A triangle is a figure formed by three line segments that intersect only at their endpoints. The sum of the measures of the angles of a triangle is 180° . Triangles can be classified by their angles. An acute triangle contains all acute angles. An obtuse triangle has one obtuse angle. A right triangle has one right angle. Triangles can also be classified by their sides. A scalene triangle has no congruent sides. An isosceles triangle has at least two congruent sides. In an equilateral triangle all sides are congruent.

Quadrilaterals

A quadrilateral is a closed figure with four sides and four vertices. The segments of a quadrilateral intersect only at their endpoints. Quadrilaterals can be separated into two triangles. Since the sum of the interior angles of all triangles totals 180° , the measures of the interior angles of a quadrilateral equal 360° . Quadrilaterals are classified according to their characteristics, and include trapezoids, parallelograms, rectangles, squares, and rhombuses.

Two-Dimensional Figures

A two-dimensional figure exists within a plane and has only the dimensions of length and width. Examples of two-dimensional figures include circles and polygons. Polygons are figures that have three or more angles, including triangles, quadrilaterals, pentagons, hexagons, and many more. The sum of the angles of any polygon totals at least 180° (triangle), and each additional side adds 180° to the measure of the first three angles. The sum of the angles of a quadrilateral, for example, is 360° . The sum of the angles of a pentagon is 540° .

Three-Dimensional Figures

A plane is a two-dimensional flat surface that extends in all directions. Intersecting planes can form the edges and vertices of three-dimensional figures or solids. A polyhedron is a solid with flat surfaces that

are polygons. Polyhedrons are composed of faces, edges, and vertices and are differentiated by their shape and by their number of bases. Skew lines are lines that lie in different planes. They are neither intersecting nor parallel.

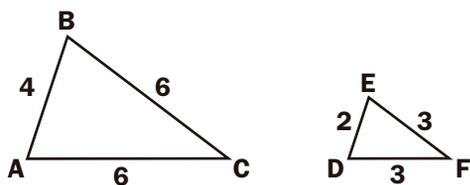
Congruence

Figures that have the same size and shape are congruent. The parts of congruent triangles that match are called corresponding parts. Congruence statements are used to identify corresponding parts of congruent triangles. When writing a congruence statement, the letters must be written so that corresponding vertices appear in the same order. Corresponding parts can be used to find the measures of angles and sides in a figure that is congruent to a figure with known measures.

Similarity

If two figures have the same shape but not the same size they are called similar figures. For example, the triangles below are similar, so angles A , B , and C have the same measurements as angles D , E , and F , respectively. However, segments AB , BC , and CA do not have the same measurements as segments DE , EF , and FD , but the measures of the sides are proportional.

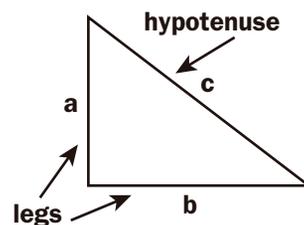
For example, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.



Solid figures are considered to be similar if they have the same shape and their corresponding linear measures are proportional. As with two-dimensional figures, they can be tested for similarity by comparing corresponding measures. If the compared ratios are proportional, then the figures are similar solids. Missing measures of similar solids can also be determined by using proportions.

The Pythagorean Theorem

In a right triangle, the sides that are adjacent to the right angle are called legs. The side opposite the right angle is the hypotenuse.

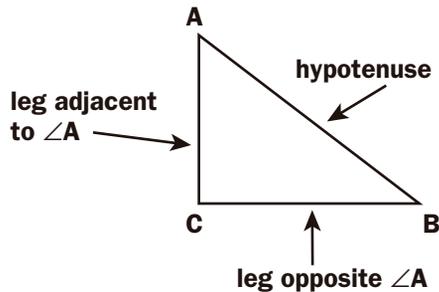


The Pythagorean Theorem describes the relationship between the lengths of the legs a and b and the hypotenuse c . It states that if a triangle is a right triangle, then the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. In symbols, $c^2 = a^2 + b^2$.

Sine, Cosine, and Tangent Ratios

Trigonometry is the study of the properties of triangles. A trigonometric ratio is a ratio of the lengths of two sides of a right triangle. The most common trigonometric

ratios are the sine, cosine, and tangent ratios. These ratios are abbreviated as *sin*, *cos*, and *tan*, respectively.



If $\angle A$ is an acute angle of a right triangle, then

$$\sin \angle A = \frac{\text{measure of leg opposite } \angle A}{\text{measure of hypotenuse}}$$

$$\cos \angle A = \frac{\text{measure of leg adjacent to } \angle A}{\text{measure of leg opposite } \angle A}, \text{ and}$$

$$\tan \angle A = \frac{\text{measure of leg opposite } \angle A}{\text{measure of leg adjacent to } \angle A}$$

► Specify locations and describe spatial relationships using coordinate geometry and other representational systems

Polygons

A polygon is a simple, closed figure formed by three or more line segments. The line segments meet only at their endpoints. The points of intersection are called vertices, and the line segments are called sides. Polygons are classified by the number of sides they have. The diagonals of a polygon divide the polygon into triangles. The number of triangles formed is two less than the number of sides. To find the sum of the measures of the interior angles of any polygon, multiply the number of triangles within the polygon

by 180. That is, if n equals the number of sides, then $(n - 2)180$ gives the sum of the measures of the polygon's interior angles.

Cartesian Coordinates

In the Cartesian coordinate system, the y -axis extends above and below the origin and the x -axis extends to the right and left of the origin, which is the point at which the x - and y -axes intersect. Numbers below and to the left of the origin are negative. A point graphed on the coordinate grid is said to have an x -coordinate and a y -coordinate. For example, the point $(1, -2)$ has as its x -coordinate the number 1, and has as its y -coordinate the number -2 . This point is graphed by locating the position on the grid that is 1 unit to the right of the origin and 2 units below the origin.

The x -axis and the y -axis separate the coordinate plane into four regions, called quadrants. The axes and points located on the axes themselves are not located in any of the quadrants. The quadrants are labeled I to IV, starting in the upper right and proceeding counterclockwise. In quadrant I, both coordinates are positive. In quadrant II, the x -coordinate is negative and the y -coordinate is positive. In quadrant III, both coordinates are negative. In quadrant IV, the x -coordinate is positive and the y -coordinate is negative. A coordinate graph can be used to show algebraic relationships among numbers.

► Apply transformations and use symmetry to analyze mathematical situations

Similar Triangles and Indirect Measurement

Triangles that have the same shape but not necessarily the same dimensions are called similar triangles. Similar triangles have corresponding angles and corresponding

sides. Arcs are used to show congruent angles. If two triangles are similar, then the corresponding angles have the same measure, and the corresponding sides are proportional. Therefore, to determine the measures of the sides of similar triangles when some measures are known, proportions can be used.

Transformations

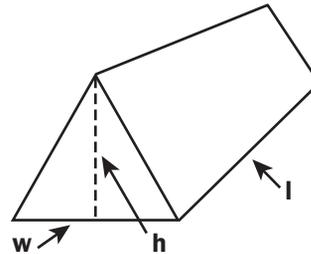
A transformation is a movement of a geometric figure. There are several types of transformations. In a translation, also called a slide, a figure is slid from one position to another without turning it. Every point of the original figure is moved the same distance and in the same direction. In a reflection, also called a flip, a figure is flipped over a line to form a mirror image. Every point of the original figure has a corresponding point on the other side of the line of symmetry. In a rotation, also called a turn, a figure is turned around a fixed point. A figure may be rotated 90° clockwise, 90° counterclockwise, or 180° . A dilation transforms each line to a parallel line whose length is a fixed multiple of the length of the original line to create a similar figure that will be either larger or smaller.

► **Use visualizations, spatial reasoning, and geometric modeling to solve problems**

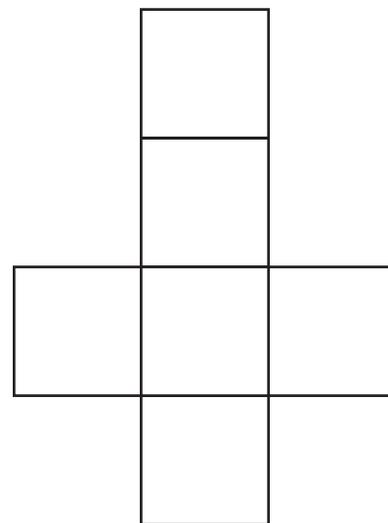
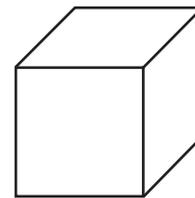
Two-Dimensional Representations of Three-Dimensional Objects

Three-dimensional objects can be represented in a two-dimensional drawing in order to more easily determine properties such as surface area and volume. When you look at the rectangular prism below, you can see the orientation of its three dimensions, length, width, and height. Using the

drawing and the formulas for surface area and volume, you can easily calculate these properties.



Another way to represent a three-dimensional object in a two-dimensional plane is by using a net, which is the unfolded representation. Imagine cutting the vertices of a box until it is flat then drawing an outline of it. That's a net. Most objects have more than one net, but any one can be measured to determine surface area. Below is a cube and one of its nets.



Measurement

► *Understand measurable attributes of objects and the units, systems, and processes of measurement*

Customary System

The customary system is the system of weights and measures used in the United States. The main units of weight are ounces, pounds (1 equal to 16 ounces), and tons (1 equal to 2,000 pounds). Length is typically measured in inches, feet (1 equal to 12 inches), yards (1 equal to 3 feet), and miles (1 equal to 5,280 feet), while area is measured in square feet and acres (1 equal to 43,560 square feet). Liquid is measured in cups, pints (1 equal to 2 cups), quarts (1 equal to 2 pints), and gallons (1 equal to 4 quarts). Finally, temperature is measured in degrees Fahrenheit.

Metric System

The metric system is a decimal system of weights and measurements in which the prefixes of the words for the units of measure indicate the relationships between the different measurements. In this system, the main units of weight, or mass, are grams and kilograms. Length is measured in millimeters, centimeters, meters, and kilometers, and the units of area are square millimeters, centimeters, meters, and kilometers. Liquid is typically measured in milliliters and liters, while temperature is in degrees Celsius.

Selecting Units of Measure

When measuring something, it is important to select the appropriate type and size of unit. For example, in the United States it would be appropriate when describing

someone's height to use feet and inches. These units of height or length are good to use because they are in the customary system, and they are of appropriate size. In the customary system, use inches, feet, and miles for lengths and perimeters; square inches, feet, and miles for area and surface area; and cups, pints, quarts, gallons or cubic inches and feet (and less commonly miles) for volume. In the metric system use millimeters, centimeters, meters, and kilometers for lengths and perimeters; square units millimeters, centimeters, meters, and kilometers for area and surface area; and milliliters and liters for volume. Finally, always use degrees to measure angles.

► *Apply appropriate techniques, tools, and formulas to determine measurements*

Precision and Significant Digits

The precision of measurement is the exactness to which a measurement is made. Precision depends on the smallest unit of measure being used, or the precision unit. One way to record a measure is to estimate to the nearest precision unit. A more precise method is to include all of the digits that are actually measured, plus one estimated digit. The digits recorded, called significant digits, indicate the precision of the measurement. There are special rules for determining significant digits. If a number contains a decimal point, the number of significant digits is found by counting from left to right, starting with the first nonzero digit. If the number does not contain a decimal point, the number of significant digits is

found by counting the digits from left to right, starting with the first digit and ending with the last nonzero digit.

Surface Area

The amount of material needed to cover the surface of a figure is called the surface area. It can be calculated by finding the area of each face and adding them together. To find the surface area of a rectangular prism, for example, the formula $S = 2lw + 2lh + 2wh$ applies. A cylinder, on the other hand, may be unrolled to reveal two circles and a rectangle. Its surface area can be determined by finding the area of the two circles, $2\pi r^2$, and adding it to the area of the rectangle, $2\pi rh$ (the length of the rectangle is the circumference of one of the circles), or $S = 2\pi r^2 + 2\pi rh$. The surface area of a pyramid is measured in a slightly different way because the sides of a pyramid are triangles that intersect at the vertex. These sides are called lateral faces and the height of each is called the slant height. The sum of their areas is the lateral area of a pyramid. The surface area of a square pyramid is the lateral area $\frac{1}{2}bh$ (area of a lateral face) times 4 (number of lateral faces), plus the area of the base. The surface area of a cone is the area of its circular base (πr^2) plus its lateral area (πrl , where l is the slant height).

Volume

Volume is the measure of space occupied by a solid region. To find the volume of a prism, the area of the base is multiplied by

the measure of the height, $V = bh$. A solid containing several prisms can be broken down into its component prisms. Then the volume of each component can be found and the volumes added. The volume of a cylinder can be determined by finding the area of its circular base, πr^2 , and then multiplying by the height of the cylinder. A pyramid has one-third the volume of a prism with the same base and height. To find the volume of a pyramid, multiply the area of the base by the pyramid's height, and then divide by 3. Simply stated, the formula for the volume of a pyramid is $V = \frac{1}{3}bh$. A cone is a three-dimensional figure with one circular base and a curved surface connecting the base and the vertex. The volume of a cone is one-third the volume of a cylinder with the same base area and height. Like a pyramid, the formula for the volume of a cone is $V = \frac{1}{3}bh$. More specifically, the formula is $V = \frac{1}{3}\pi r^2h$.

Upper and Lower Bounds

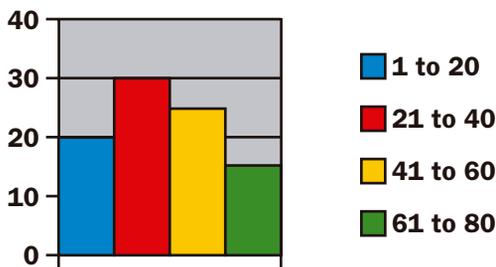
Upper and lower bounds have to do with the accuracy of a measurement. When a measurement is given, the degree of accuracy is also stated to tell you what the upper and lower bounds of the measurement are. The upper bound is the largest possible value that a measurement could have had before being rounded down, and the lower bound is the lowest possible value it could have had before being rounded up.

Data Analysis and Probability

► **Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them**

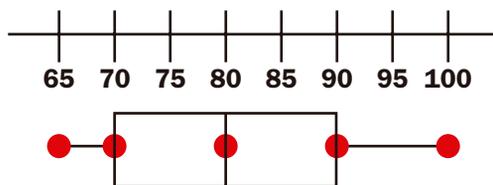
Histograms

A histogram displays numerical data that have been organized into equal intervals using bars that have the same width and no space between them. While a histogram does not give exact data points, its shape shows the distribution of the data. Histograms also can be used to compare data.



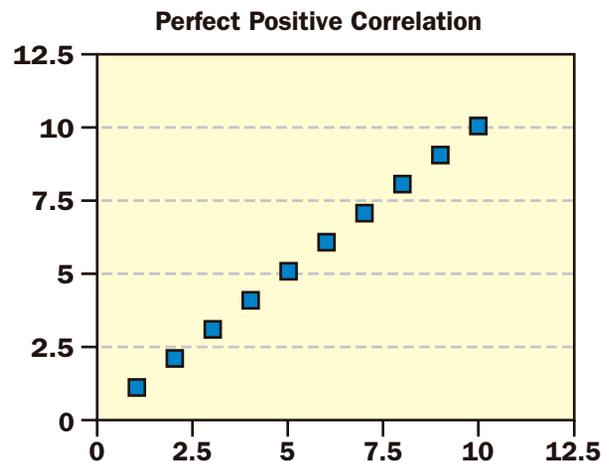
Box-and-Whisker Plot

A box-and-whisker plot displays the measures of central tendency and variation. A box is drawn around the quartile values, and whiskers extend from each quartile to the extreme data points. To make a box plot for a set of data, draw a number line that covers the range of data. Find the median, the extremes, and the upper and lower quartiles. Mark these points on the number line, then draw a box and the whiskers. The length of a whisker or box shows whether the values of the data in that part are concentrated or spread out.



Scatter Plots

A scatter plot is a graph that shows the relationship between two sets of data. In a scatter plot, two sets of data are graphed as ordered pairs on a coordinate system. Two sets of data can have a positive correlation (as x increases, y increases), a negative correlation (as x increases, y decreases), or no correlation (no obvious pattern is shown). Scatter plots can be used to spot trends, draw conclusions, and make predictions about data.



Randomization

The idea of randomization is a very important principle of statistics and the design of experiments. Data must be selected randomly to prevent bias from influencing the results. For example, you want to know the average income of people in your town but you can only use a sample of 100 individuals to make determinations about everyone. If you select 100 individuals who are all doctors, you will have a biased sample. However, if you chose a random sample of 100 people out of the phone book, you are much more likely to accurately represent average income in the town.

Statistics and Parameters

Statistics is a science that involves collecting, analyzing, and presenting data. The data can be collected in various ways—for example through a census or by making physical measurements. The data can then be analyzed by creating summary statistics, which have to do with the distribution of the data sample, including the mean, range, and standard error. They can also be illustrated in tables and graphs, like box-plots, scatter plots, and histograms. The presentation of the data typically involves describing the strength or validity of the data and what they show. For example, an analysis of ancestry of people in a city might tell you something about immigration patterns, unless the data set is very small or biased in some way, in which case it is not likely to be very accurate or useful.

Categorical and Measurement Data

When analyzing data, it is important to understand if the data is qualitative or quantitative. Categorical data is qualitative and measurement, or numerical, data is quantitative. Categorical data describes a quality of something and can be placed into different categories. For example, if you are analyzing the number of students in different grades in a school, each grade is a category. On the other hand, measurement data is continuous, like height, weight, or any other measurable variable. Measurement data can be converted into categorical data if you decide to group the data. Using height as an example, you can group the continuous data set into categories like under 5 feet, 5 feet to 5 feet 5 inches, over 5 feet 5 inches to 6 feet, and so on.

Univariate and Bivariate Data

In data analysis, a researcher can analyze one variable at a time or look at how multiple variables behave together. Univariate data involves only one variable, for example height in humans. You can measure the height in a population of people then plot the results in a histogram to look at how height is distributed in humans. To summarize univariate data, you can use statistics like the mean, mode, median, range, and standard deviation, which is a measure of variation. When looking at more than one variable at once, you use multivariate data. Bivariate data involves two variables. For example, you can look at height and age in humans together by gathering information on both variables from individuals in a population. You can then plot both variables in a scatter plot, look at how the variables behave in relation to each other, and create an equation that represents the relationship, also called a regression. These equations could help answer questions such as, for example, does height increase with age in humans?

► **Select and use appropriate statistical methods to analyze data**

Measures of Central Tendency

When you have a list of numerical data, it is often helpful to use one or more numbers to represent the whole set. These numbers are called measures of central tendency. Three measures of central tendency are mean, median, and mode. The mean is the sum of the data divided by the number of items in the data set. The median is the middle number of the ordered data (or the mean of the two middle numbers). The mode is the number or numbers that occur most often.

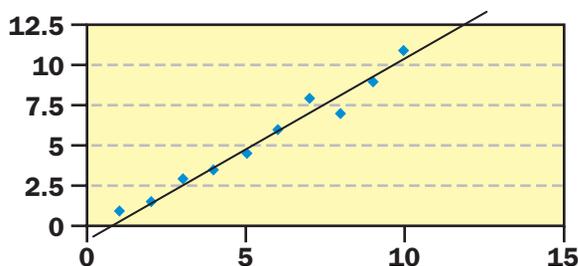
These measures of central tendency allow data to be analyzed and better understood.

Measures of Spread

In statistics, measures of spread or variation are used to describe how data are distributed. The range of a set of data is the difference between the greatest and the least values of the data set. The quartiles are the values that divide the data into four equal parts. The median of data separates the set in half. Similarly, the median of the lower half of a set of data is the lower quartile. The median of the upper half of a set of data is the upper quartile. The interquartile range is the difference between the upper quartile and the lower quartile.

Line of Best Fit

When real-life data are collected, the points graphed usually do not form a straight line, but they may approximate a linear relationship. A line of best fit is a line that lies very close to most of the data points. It can be used to predict data. You also can use the equation of the best-fit line to make predictions.



Stem and Leaf Plots

In a stem and leaf plot, numerical data are listed in ascending or descending order. The greatest place value of the data is used for the stems. The next greatest place value forms the leaves. For example, if the least

number in a set of data is 8 and the greatest number is 95, draw a vertical line and write the stems from 0 to 9 to the left of the line. Write the leaves from 0 to the right of the line, with the corresponding stem. Next, rearrange the leaves so they are ordered from least to greatest. Then include a key or explanation, such as $1|3 = 13$. Notice that the stem-and-leaf plot below is like a histogram turned on its side.

```

0|8
1|3 6
2|5 6 9
3|0 2 7 8
4|0 1 4 7 9
5|1 4 5 8
6|1 3 7
7|5 8
8|2 6
9|5
    
```

Key: $1|3 = 13$

Develop and evaluate inferences and predictions that are based on data

Sampling Distribution

The sampling distribution of a population is the distribution that would result if you could take an infinite number of samples from the population, average each, and then average the averages. The more normal the distribution of the population, that is, how closely the distribution follows a bell curve, the more likely the sampling distribution will also follow a normal distribution. Furthermore, the larger the sample, the more likely it will accurately represent the entire population. For instance, you are more likely to gain more representative results from a population of 1,000 with a sample of 100 than with a sample of 2.

**Validity**

In statistics, validity refers to acquiring results that accurately reflect that which is being measured. In other words, it is important when performing statistical analyses, to ensure that the data are valid in that the sample being analyzed represents the population to the best extent possible. Randomization of data and using appropriate sample sizes are two important aspects of making valid inferences about a population.

► **Understand and apply basic concepts of probability**

Complementary, Mutually Exclusive Events

To understand probability theory, it is important to know if two events are mutually exclusive, or complementary: the occurrence of one event automatically implies the non-occurrence of the other. That is, two complementary events cannot both occur. If you roll a pair of dice, the event of rolling 6 and rolling doubles have an outcome in common (3, 3), so they are not mutually exclusive. If you roll (3, 3), you also roll doubles. However, the events of rolling a 9 and rolling doubles are mutually exclusive because they have no outcomes in common. If you roll a 9, you will not also roll doubles.

Independent and Dependent Events

Determining the probability of a series of events requires that you know whether the events are independent or dependent. An independent event has no influence on the occurrence of subsequent events, whereas, a dependent event does influence subsequent events. The chances that a woman's first child will be a girl are $\frac{1}{2}$, and the chances that her second child will be a girl are also

$\frac{1}{2}$ because the two events are independent of each other. However, if there are 7 red marbles in a bag of 15 marbles, the chances that the first marble you pick will be red are $\frac{7}{15}$ and if you indeed pick a red marble and remove it, you have reduced the chances of picking another red marble to $\frac{6}{14}$.

Sample Space

The sample space is the group of all possible outcomes for an event. For example, if you are tossing a single six-sided die, the sample space is {1, 2, 3, 4, 5, 6}. Similarly, you can determine the sample space for the possible outcomes of two events. If you are going to toss a coin twice, the sample space is {(heads, heads), (heads, tails), (tails, heads), (tails, tails)}.

Computing the Probability of a Compound Event

If two events are independent, the outcome of one event does not influence the outcome of the second. For example, if a bag contains 2 blue and 3 red marbles, then the probability of selecting a blue marble, replacing it, and then selecting a red marble is $P(A) \times P(B) = \frac{2}{5} \times \frac{3}{5}$ or $\frac{6}{25}$.

If two events are dependent, the outcome of one event affects the outcome of the second. For example, if a bag contains 2 blue and 3 red marbles, then the probability of selecting a blue and then a red marble without replacing the first marble is $P(A) \times P(B \text{ following } A) = \frac{2}{5} \times \frac{3}{4}$ or $\frac{3}{10}$. Two events that cannot happen at the same time are mutually exclusive. For example, when you roll two number cubes, you cannot roll a sum that is both 5 and even. So, $P(A \text{ or } B) = \frac{4}{36} + \frac{18}{36}$ or $\frac{11}{18}$.