CHAPTER 4

Introduction to Probability

CONTENTS

STATISTICS *in* PRACTICE

OCEANWIDE SEAFOOD*
SPRINGBORO, OHIO

Oceanwide Seafood is the leading provider of quality seafood in southwestern Ohio. The company stocks over 90 varieties of fresh and frozen seafood products from around the world and prepares specialty cuts according to customer specifications. Customers include major restaurants and retail food stores in Ohio, Kentucky, and Indiana. Established in 2005, the company has become successful by providing superior customer service and exceptional quality seafood.

Probability and statistical information are used for both operational and marketing decisions. For instance, a time series showing monthly sales is used to track the company's growth and to set future target sales levels. Statistics such as the mean customer order size and the mean number of days a customer takes to make payments help identify the firm's best customers as well as provide benchmarks for handling accounts receivable issues. In addition, data on monthly inventory levels are used in the analysis of operating profits and trends in product sales.

Probability analysis has helped Oceanwide determine reasonable and profitable prices for its products. For example, when Oceanwide receives a whole fresh fish from one of its suppliers, the fish must be processed and cut to fill individual customer orders. A fresh 100-pound whole tuna packed in ice might cost Oceanwide \$500. At first glance, the company's cost for tuna appears to be \$500/100 = \$5 per pound. However, due to the loss in the processing and cutting operation, a 100-pound whole tuna will not provide 100 pounds of finished product. If the processing and cutting operation yields 75% of the whole tuna, the number of pounds of finished product available for sale to customers would be .75(100) = 75 pounds, not 100 pounds. In this case, the company's actual cost of tuna would be \$500/75 = \$6.67 per pound. Thus, Oceanwide would need to use a cost of \$6.67 per pound to determine a profitable price to charge its customers.

*The authors are indebted to Dale Hartlage, president of Oceanwide Seafood Company, for providing this Statistics in Practice.

Fresh bluefin tuna are shipped to Oceanwide Seafood almost everyday © Gregor Kervina, 2009/ Used under license from Shutterstock.com.

To help determine the yield percentage that is likely for processing and cutting whole tuna, data were collected on the yields from a sample of whole tunas. Let Y denote the yield percentage for whole tuna. Using the data, Oceanwide was able to determine that 5% of the time the yield for whole tuna was at least 90%. In conditional probability notation, this probability is written $P(Y \geq 90\% \mid \text{Tuna}) = .05$; in other words, the probability that the yield will be at least 90% given that the fish is a tuna is .05. If Oceanwide established the selling price for tuna based on a 90% yield, 95% of the time the company would realize a yield less than expected. As a result, the company would be understating its cost per pound and also understating the price of tuna for its customers. Additional conditional probability information for other yield percentages helped management select an 70% yield as the basis for determining the cost of tuna and the price to charge its customers. Similar conditional probabilities for other seafood products helped management establish pricing yield percentages for each type of seafood product. In this chapter, you will learn how to compute and interpret conditional probabilities and other probabilities that are helpful in the decision-making process.

Managers often base their decisions on an analysis of uncertainties such as the following:

1. What are the chances that sales will decrease if we increase prices?
2. What is the likelihood a new assembly method will increase productivity?
3. How likely is it that the project will be finished on time?
4. What is the chance that a new investment will be profitable?

Some of the earliest work on probability originated in a series of letters between Pierre de Fermat and Blaise Pascal in the 1650s.

Probability is a numerical measure of the likelihood that an event will occur. Thus, probabilities can be used as measures of the degree of uncertainty associated with the four events previously listed. If probabilities are available, we can determine the likelihood of each event occurring.

Probability values are always assigned on a scale from 0 to 1. A probability near zero indicates an event is unlikely to occur; a probability near 1 indicates an event is almost certain to occur. Other probabilities between 0 and 1 represent degrees of likelihood that an event will occur. For example, if we consider the event "rain tomorrow," we understand that when the weather report indicates "a near-zero probability of rain," it means almost no chance of rain. However, if a .90 probability of rain is reported, we know that rain is likely to occur. A .50 probability indicates that rain is just as likely to occur as not. Figure 4.1 depicts the view of probability as a numerical measure of the likelihood of an event occurring.

4.1 Experiments, Counting Rules, and Assigning Probabilities

In discussing probability, we define an **experiment** as a process that generates well-defined outcomes. On any single repetition of an experiment, one and only one of the possible experimental outcomes will occur. Several examples of experiments and their associated outcomes follow.

Experiment	Experimental Outcomes
Toss a coin	Head, tail
Select a part for inspection	Defective, nondefective
Conduct a sales call	Purchase, no purchase
Roll a die	1, 2, 3, 4, 5, 6
Play a football game	Win, lose, tie

By specifying all possible experimental outcomes, we identify the **sample space** for an experiment.

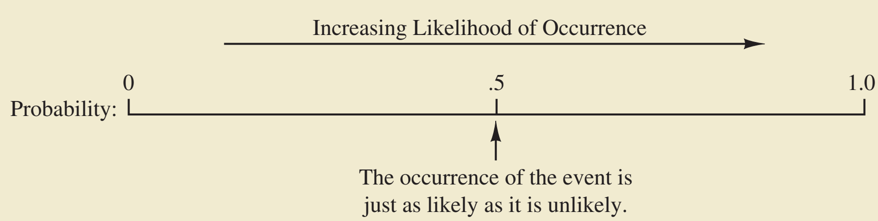
SAMPLE SPACE

The sample space for an experiment is the set of all experimental outcomes.

Experimental outcomes are also called sample points.

An experimental outcome is also called a **sample point** to identify it as an element of the sample space.

FIGURE 4.1 PROBABILITY AS A NUMERICAL MEASURE OF THE LIKELIHOOD OF AN EVENT OCCURRING



Consider the first experiment in the preceding table—tossing a coin. The upward face of the coin—a head or a tail—determines the experimental outcomes (sample points). If we let S denote the sample space, we can use the following notation to describe the sample space.

$$S = \{\text{Head, Tail}\}$$

The sample space for the second experiment in the table—selecting a part for inspection—can be described as follows:

$$S = \{\text{Defective, Nondefective}\}$$

Both of the experiments just described have two experimental outcomes (sample points). However, suppose we consider the fourth experiment listed in the table—rolling a die. The possible experimental outcomes, defined as the number of dots appearing on the upward face of the die, are the six points in the sample space for this experiment.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Counting Rules, Combinations, and Permutations

Being able to identify and count the experimental outcomes is a necessary step in assigning probabilities. We now discuss three useful counting rules.

Multiple-step experiments The first counting rule applies to multiple-step experiments. Consider the experiment of tossing two coins. Let the experimental outcomes be defined in terms of the pattern of heads and tails appearing on the upward faces of the two coins. How many experimental outcomes are possible for this experiment? The experiment of tossing two coins can be thought of as a two-step experiment in which step 1 is the tossing of the first coin and step 2 is the tossing of the second coin. If we use H to denote a head and T to denote a tail, (H, H) indicates the experimental outcome with a head on the first coin and a head on the second coin. Continuing this notation, we can describe the sample space (S) for this coin-tossing experiment as follows:

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

Thus, we see that four experimental outcomes are possible. In this case, we can easily list all of the experimental outcomes.

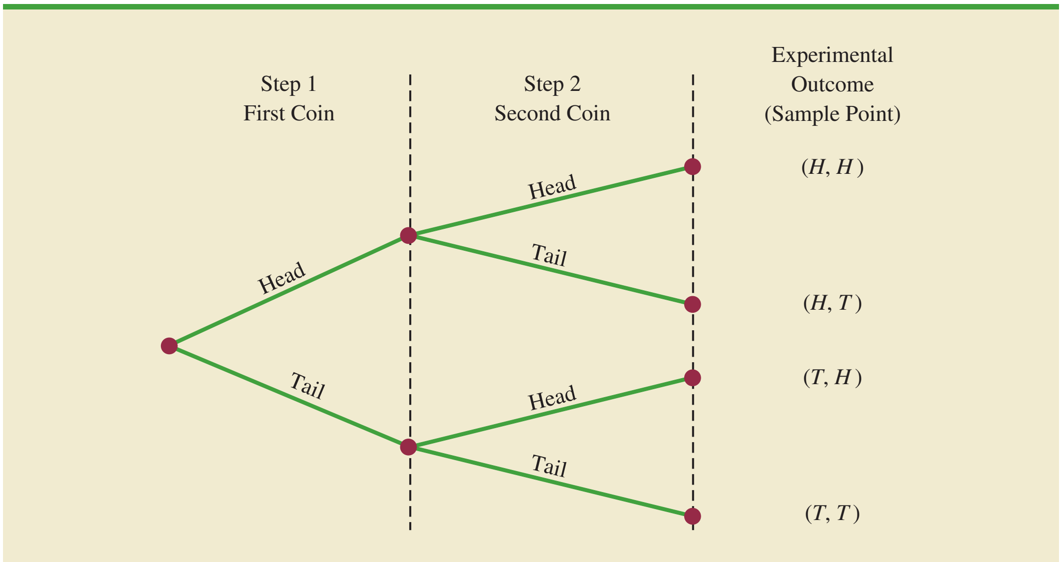
The counting rule for multiple-step experiments makes it possible to determine the number of experimental outcomes without listing them.

COUNTING RULE FOR MULTIPLE-STEP EXPERIMENTS

If an experiment can be described as a sequence of k steps with n_1 possible outcomes on the first step, n_2 possible outcomes on the second step, and so on, then the total number of experimental outcomes is given by $(n_1)(n_2) \ldots (n_k)$.

Viewing the experiment of tossing two coins as a sequence of first tossing one coin ($n_1 = 2$) and then tossing the other coin ($n_2 = 2$), we can see from the counting rule that $(2)(2) = 4$ distinct experimental outcomes are possible. As shown, they are $S = \{(H, H), (H, T), (T, H), (T, T)\}$. The number of experimental outcomes in an experiment involving tossing six coins is $(2)(2)(2)(2)(2)(2) = 64$.

FIGURE 4.2 TREE DIAGRAM FOR THE EXPERIMENT OF TOSSING TWO COINS



Without the tree diagram, one might think only three experimental outcomes are possible for two tosses of a coin: 0 heads, 1 head, and 2 heads.

A **tree diagram** is a graphical representation that helps in visualizing a multiple-step experiment. Figure 4.2 shows a tree diagram for the experiment of tossing two coins. The sequence of steps moves from left to right through the tree. Step 1 corresponds to tossing the first coin, and step 2 corresponds to tossing the second coin. For each step, the two possible outcomes are head or tail. Note that for each possible outcome at step 1 two branches correspond to the two possible outcomes at step 2. Each of the points on the right end of the tree corresponds to an experimental outcome. Each path through the tree from the leftmost node to one of the nodes at the right side of the tree corresponds to a unique sequence of outcomes.

Let us now see how the counting rule for multiple-step experiments can be used in the analysis of a capacity expansion project for the Kentucky Power & Light Company (KP&L). KP&L is starting a project designed to increase the generating capacity of one of its plants in northern Kentucky. The project is divided into two sequential stages or steps: stage 1 (design) and stage 2 (construction). Even though each stage will be scheduled and controlled as closely as possible, management cannot predict beforehand the exact time required to complete each stage of the project. An analysis of similar construction projects revealed possible completion times for the design stage of 2, 3, or 4 months and possible completion times for the construction stage of 6, 7, or 8 months. In addition, because of the critical need for additional electrical power, management set a goal of 10 months for the completion of the entire project.

Because this project has three possible completion times for the design stage (step 1) and three possible completion times for the construction stage (step 2), the counting rule for multiple-step experiments can be applied here to determine a total of $(3)(3) = 9$ experimental outcomes. To describe the experimental outcomes, we use a two-number notation; for instance, (2, 6) indicates that the design stage is completed in 2 months and the construction stage is completed in 6 months. This experimental outcome results in a total of $2 + 6 = 8$ months to complete the entire project. Table 4.1 summarizes the nine experimental outcomes for the KP&L problem. The tree diagram in Figure 4.3 shows how the nine outcomes (sample points) occur.

The counting rule and tree diagram help the project manager identify the experimental outcomes and determine the possible project completion times. From the information in

TABLE 4.1 EXPERIMENTAL OUTCOMES (SAMPLE POINTS) FOR THE KP&L PROJECT

Completion Time (months)			
Stage 1 Design	**Stage 2 Construction**	**Notation for Experimental Outcome**	**Total Project Completion Time (months)**
2	6	(2, 6)	8
2	7	(2, 7)	9
2	8	(2, 8)	10
3	6	(3, 6)	9
3	7	(3, 7)	10
3	8	(3, 8)	11
4	6	(4, 6)	10
4	7	(4, 7)	11
4	8	(4, 8)	12

FIGURE 4.3 TREE DIAGRAM FOR THE KP&L PROJECT

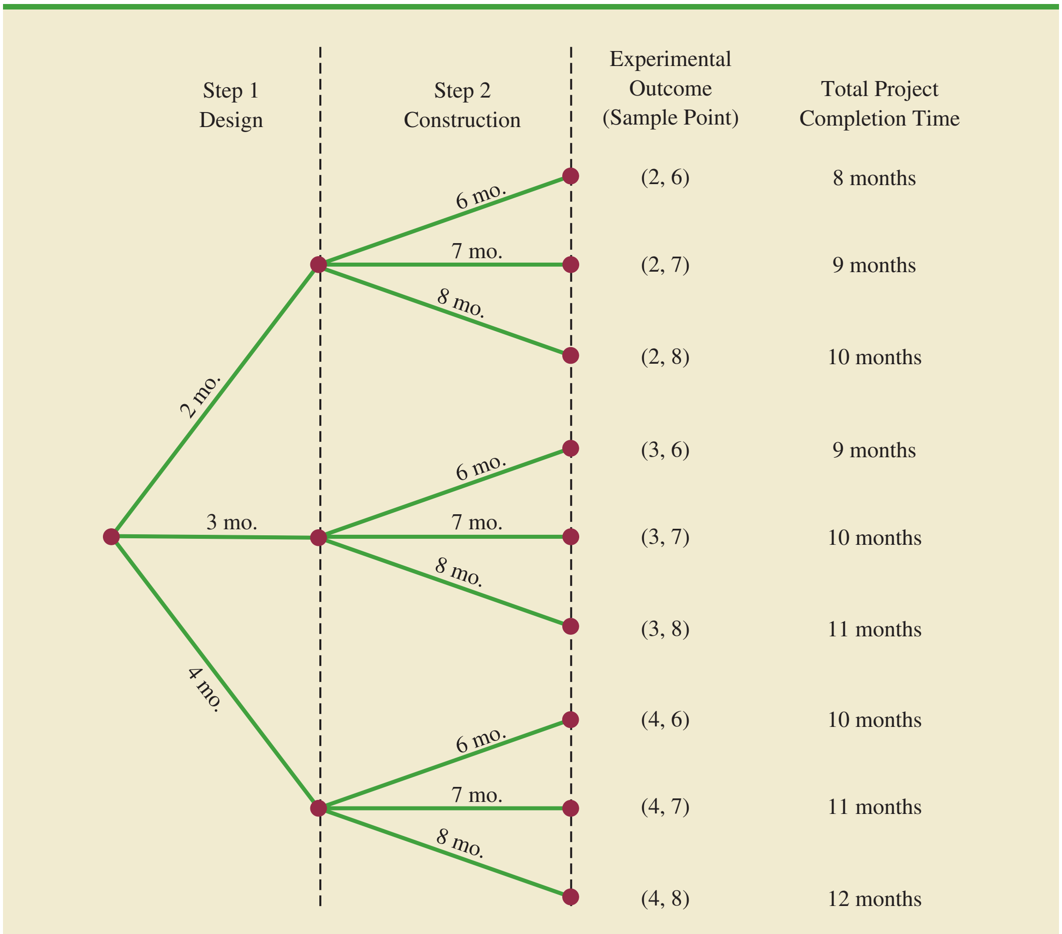


Figure 4.3, we see that the project will be completed in 8 to 12 months, with six of the nine experimental outcomes providing the desired completion time of 10 months or less. Even though identifying the experimental outcomes may be helpful, we need to consider how probability values can be assigned to the experimental outcomes before making an assessment of the probability that the project will be completed within the desired 10 months.

Combinations A second useful counting rule allows one to count the number of experimental outcomes when the experiment involves selecting n objects from a (usually larger) set of N objects. It is called the counting rule for combinations.

COUNTING RULE FOR COMBINATIONS

The number of combinations of N objects taken n at a time is

$$C_n^N = \binom{N}{n} = \frac{N!}{n!(N-n)!} \tag{4.1}$$

where

$$N! = N(N-1)(N-2)\cdots(2)(1)$$
$$n! = n(n-1)(n-2)\cdots(2)(1)$$

and, by definition,

$$0! = 1$$

In sampling from a finite population of size N, the counting rule for combinations is used to find the number of different samples of size n that can be selected.

The notation ! means *factorial;* for example, 5 factorial is $5! = (5)(4)(3)(2)(1) = 120$.

As an illustration of the counting rule for combinations, consider a quality control procedure in which an inspector randomly selects two of five parts to test for defects. In a group of five parts, how many combinations of two parts can be selected? The counting rule in equation (4.1) shows that with $N = 5$ and $n = 2$, we have

$$C_2^5 = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{(5)(4)(3)(2)(1)}{(2)(1)(3)(2)(1)} = \frac{120}{12} = 10$$

Thus, 10 outcomes are possible for the experiment of randomly selecting two parts from a group of five. If we label the five parts as A, B, C, D, and E, the 10 combinations or experimental outcomes can be identified as AB, AC, AD, AE, BC, BD, BE, CD, CE, and DE.

As another example, consider that the Florida lottery system uses the random selection of six integers from a group of 53 to determine the weekly winner. The counting rule for combinations, equation (4.1), can be used to determine the number of ways six different integers can be selected from a group of 53.

$$\binom{53}{6} = \frac{53!}{6!(53-6)!} = \frac{53!}{6!47!} = \frac{(53)(52)(51)(50)(49)(48)}{(6)(5)(4)(3)(2)(1)} = 22{,}957{,}480$$

The counting rule for combinations shows that the chance of winning the lottery is very unlikely.

The counting rule for combinations tells us that almost 23 million experimental outcomes are possible in the lottery drawing. An individual who buys a lottery ticket has 1 chance in 22,957,480 of winning.

Permutations A third counting rule that is sometimes useful is the counting rule for permutations. It allows one to compute the number of experimental outcomes when n objects are to be selected from a set of N objects where the order of selection is important. The same n objects selected in a different order are considered a different experimental outcome.

COUNTING RULE FOR PERMUTATIONS

The number of permutations of N objects taken n at a time is given by

$$P_n^N = n!\binom{N}{n} = \frac{N!}{(N-n)!} \qquad \textbf{(4.2)}$$

The counting rule for permutations closely relates to the one for combinations; however, an experiment results in more permutations than combinations for the same number of objects because every selection of n objects can be ordered in $n!$ different ways.

As an example, consider again the quality control process in which an inspector selects two of five parts to inspect for defects. How many permutations may be selected? The counting rule in equation (4.2) shows that with $N = 5$ and $n = 2$, we have

$$P_2^5 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{(5)(4)(3)(2)(1)}{(3)(2)(1)} = \frac{120}{6} = 20$$

Thus, 20 outcomes are possible for the experiment of randomly selecting two parts from a group of five when the order of selection must be taken into account. If we label the parts A, B, C, D, and E, the 20 permutations are AB, BA, AC, CA, AD, DA, AE, EA, BC, CB, BD, DB, BE, EB, CD, DC, CE, EC, DE, and ED.

Assigning Probabilities

Now let us see how probabilities can be assigned to experimental outcomes. The three approaches most frequently used are the classical, relative frequency, and subjective methods. Regardless of the method used, two **basic requirements for assigning probabilities** must be met.

BASIC REQUIREMENTS FOR ASSIGNING PROBABILITIES

1. The probability assigned to each experimental outcome must be between 0 and 1, inclusively. If we let E_i denote the ith experimental outcome and $P(E_i)$ its probability, then this requirement can be written as

$$0 \le P(E_i) \le 1 \text{ for all } i \qquad \textbf{(4.3)}$$

2. The sum of the probabilities for all the experimental outcomes must equal 1.0. For n experimental outcomes, this requirement can be written as

$$P(E_1) + P(E_2) + \cdots + P(E_n) = 1 \qquad \textbf{(4.4)}$$

The **classical method** of assigning probabilities is appropriate when all the experimental outcomes are equally likely. If n experimental outcomes are possible, a probability of $1/n$ is assigned to each experimental outcome. When using this approach, the two basic requirements for assigning probabilities are automatically satisfied.

For an example, consider the experiment of tossing a fair coin; the two experimental outcomes—head and tail—are equally likely. Because one of the two equally likely outcomes is a head, the probability of observing a head is 1/2, or .50. Similarly, the probability of observing a tail is also 1/2, or .50.

As another example, consider the experiment of rolling a die. It would seem reasonable to conclude that the six possible outcomes are equally likely, and hence each outcome is assigned a probability of 1/6. If $P(1)$ denotes the probability that one dot appears on the upward face of the die, then $P(1) = 1/6$. Similarly, $P(2) = 1/6$, $P(3) = 1/6$, $P(4) = 1/6$, $P(5) = 1/6$, and $P(6) = 1/6$. Note that these probabilities satisfy the two basic requirements of equations (4.3) and (4.4) because each of the probabilities is greater than or equal to zero and they sum to 1.0.

The **relative frequency method** of assigning probabilities is appropriate when data are available to estimate the proportion of the time the experimental outcome will occur if the experiment is repeated a large number of times. As an example, consider a study of waiting times in the X-ray department for a local hospital. A clerk recorded the number of patients waiting for service at 9:00 A.M. on 20 successive days and obtained the following results.

Number Waiting		Number of Days Outcome Occurred
0		2
1		5
2		6
3		4
4		3
	Total	20

These data show that on 2 of the 20 days, zero patients were waiting for service; on 5 of the days, one patient was waiting for service; and so on. Using the relative frequency method, we would assign a probability of $2/20 = .10$ to the experimental outcome of zero patients waiting for service, $5/20 = .25$ to the experimental outcome of one patient waiting, $6/20 = .30$ to two patients waiting, $4/20 = .20$ to three patients waiting, and $3/20 = .15$ to four patients waiting. As with the classical method, using the relative frequency method automatically satisfies the two basic requirements of equations (4.3) and (4.4).

The **subjective method** of assigning probabilities is most appropriate when one cannot realistically assume that the experimental outcomes are equally likely and when little relevant data are available. When the subjective method is used to assign probabilities to the experimental outcomes, we may use any information available, such as our experience or intuition. After considering all available information, a probability value that expresses our *degree of belief* (on a scale from 0 to 1) that the experimental outcome will occur is specified. Because subjective probability expresses a person's degree of belief, it is personal. Using the subjective method, different people can be expected to assign different probabilities to the same experimental outcome.

The subjective method requires extra care to ensure that the two basic requirements of equations (4.3) and (4.4) are satisfied. Regardless of a person's degree of belief, the probability value assigned to each experimental outcome must be between 0 and 1, inclusive, and the sum of all the probabilities for the experimental outcomes must equal 1.0.

Consider the case in which Tom and Judy Elsbernd make an offer to purchase a house. Two outcomes are possible:

$$E_1 = \text{their offer is accepted}$$
$$E_2 = \text{their offer is rejected}$$

Judy believes that the probability their offer will be accepted is .8; thus, Judy would set $P(E_1) = .8$ and $P(E_2) = .2$. Tom, however, believes that the probability that their offer will be accepted is .6; hence, Tom would set $P(E_1) = .6$ and $P(E_2) = .4$. Note that Tom's probability estimate for E_1 reflects a greater pessimism that their offer will be accepted.

Bayes' theorem (see Section 4.5) provides a means for combining subjectively determined prior probabilities with probabilities obtained by other means to obtain revised, or posterior, probabilities.

Both Judy and Tom assigned probabilities that satisfy the two basic requirements. The fact that their probability estimates are different emphasizes the personal nature of the subjective method.

Even in business situations where either the classical or the relative frequency approach can be applied, managers may want to provide subjective probability estimates. In such cases, the best probability estimates often are obtained by combining the estimates from the classical or relative frequency approach with subjective probability estimates.

Probabilities for the KP&L Project

To perform further analysis on the KP&L project, we must develop probabilities for each of the nine experimental outcomes listed in Table 4.1. On the basis of experience and judgment, management concluded that the experimental outcomes were not equally likely. Hence, the classical method of assigning probabilities could not be used. Management then decided to conduct a study of the completion times for similar projects undertaken by KP&L over the past three years. The results of a study of 40 similar projects are summarized in Table 4.2.

After reviewing the results of the study, management decided to employ the relative frequency method of assigning probabilities. Management could have provided subjective probability estimates, but felt that the current project was quite similar to the 40 previous projects. Thus, the relative frequency method was judged best.

In using the data in Table 4.2 to compute probabilities, we note that outcome (2, 6)—stage 1 completed in 2 months and stage 2 completed in 6 months—occurred six times in the 40 projects. We can use the relative frequency method to assign a probability of 6/40 = .15 to this outcome. Similarly, outcome (2, 7) also occurred in six of the 40 projects, providing a 6/40 = .15 probability. Continuing in this manner, we obtain the probability assignments for the sample points of the KP&L project shown in Table 4.3. Note that $P(2, 6)$ represents the probability of the sample point (2, 6), $P(2, 7)$ represents the probability of the sample point (2, 7), and so on.

TABLE 4.2 COMPLETION RESULTS FOR 40 KP&L PROJECTS

Completion Time (months)			
Stage 1 Design	Stage 2 Construction	Sample Point	Number of Past Projects Having These Completion Times
2	6	(2, 6)	6
2	7	(2, 7)	6
2	8	(2, 8)	2
3	6	(3, 6)	4
3	7	(3, 7)	8
3	8	(3, 8)	2
4	6	(4, 6)	2
4	7	(4, 7)	4
4	8	(4, 8)	6
		Total	40

TABLE 4.3 PROBABILITY ASSIGNMENTS FOR THE KP&L PROJECT BASED ON THE RELATIVE FREQUENCY METHOD

Sample Point	Project Completion Time	Probability of Sample Point
(2, 6)	8 months	$P(2, 6) = 6/40 = .15$
(2, 7)	9 months	$P(2, 7) = 6/40 = .15$
(2, 8)	10 months	$P(2, 8) = 2/40 = .05$
(3, 6)	9 months	$P(3, 6) = 4/40 = .10$
(3, 7)	10 months	$P(3, 7) = 8/40 = .20$
(3, 8)	11 months	$P(3, 8) = 2/40 = .05$
(4, 6)	10 months	$P(4, 6) = 2/40 = .05$
(4, 7)	11 months	$P(4, 7) = 4/40 = .10$
(4, 8)	12 months	$P(4, 8) = 6/40 = .15$
		Total 1.00

NOTES AND COMMENTS

1. In statistics, the notion of an experiment differs somewhat from the notion of an experiment in the physical sciences. In the physical sciences, researchers usually conduct an experiment in a laboratory or a controlled environment in order to learn about cause and effect. In statistical experiments, probability determines outcomes. Even though the experiment is repeated in exactly the same way, an entirely different outcome may occur. Because of this influence of probability on the outcome, the experiments of statistics are sometimes called *random experiments.*

2. When drawing a random sample without replacement from a population of size N, the counting rule for combinations is used to find the number of different samples of size n that can be selected.

Exercises

Methods

1. An experiment has three steps with three outcomes possible for the first step, two outcomes possible for the second step, and four outcomes possible for the third step. How many experimental outcomes exist for the entire experiment?

SELF test

2. How many ways can three items be selected from a group of six items? Use the letters A, B, C, D, E, and F to identify the items, and list each of the different combinations of three items.

3. How many permutations of three items can be selected from a group of six? Use the letters A, B, C, D, E, and F to identify the items, and list each of the permutations of items B, D, and F.

4. Consider the experiment of tossing a coin three times.
   a. Develop a tree diagram for the experiment.
   b. List the experimental outcomes.
   c. What is the probability for each experimental outcome?

5. Suppose an experiment has five equally likely outcomes: E_1, E_2, E_3, E_4, E_5. Assign probabilities to each outcome and show that the requirements in equations (4.3) and (4.4) are satisfied. What method did you use?

SELF test

6. An experiment with three outcomes has been repeated 50 times, and it was learned that E_1 occurred 20 times, E_2 occurred 13 times, and E_3 occurred 17 times. Assign probabilities to the outcomes. What method did you use?

7. A decision maker subjectively assigned the following probabilities to the four outcomes of an experiment: $P(E_1) = .10$, $P(E_2) = .15$, $P(E_3) = .40$, and $P(E_4) = .20$. Are these probability assignments valid? Explain.

Applications

8. In the city of Milford, applications for zoning changes go through a two-step process: a review by the planning commission and a final decision by the city council. At step 1 the planning commission reviews the zoning change request and makes a positive or negative recommendation concerning the change. At step 2 the city council reviews the planning commission's recommendation and then votes to approve or to disapprove the zoning change. Suppose the developer of an apartment complex submits an application for a zoning change. Consider the application process as an experiment.
   a. How many sample points are there for this experiment? List the sample points.
   b. Construct a tree diagram for the experiment.

SELF test

9. Simple random sampling uses a sample of size n from a population of size N to obtain data that can be used to make inferences about the characteristics of a population. Suppose that, from a population of 50 bank accounts, we want to take a random sample of four accounts in order to learn about the population. How many different random samples of four accounts are possible?

SELF test

10. Many students accumulate debt by the time they graduate from college. Shown in the following table is the percentage of graduates with debt and the average amount of debt for these graduates at four universities and four liberal arts colleges (*U.S. News and World Report, America's Best Colleges,* 2008).

University	% with Debt	Amount($)	College	% with Debt	Amount($)
Pace	72	32,980	Wartburg	83	28,758
Iowa State	69	32,130	Morehouse	94	27,000
Massachusetts	55	11,227	Wellesley	55	10,206
SUNY—Albany	64	11,856	Wofford	49	11,012

   a. If you randomly choose a graduate of Morehouse College, what is the probability that this individual graduated with debt?
   b. If you randomly choose one of these eight institutions for a follow-up study on student loans, what is the probability that you will choose an institution with more than 60% of its graduates having debt?
   c. If you randomly choose one of these eight institutions for a follow-up study on student loans, what is the probability that you will choose an institution whose graduates with debts have an average debt of more than $30,000?
   d. What is the probability that a graduate of Pace University does not have debt?
   e. For graduates of Pace University with debt, the average amount of debt is $32,980. Considering all graduates from Pace University, what is the average debt per graduate?

11. The National Highway Traffic Safety Administration (NHTSA) conducted a survey to learn about how drivers throughout the United States are using seat belts (Associated Press, August 25, 2003). Sample data consistent with the NHTSA survey are as follows.

	Driver Using Seat Belt?	
Region	**Yes**	**No**
Northeast	148	52
Midwest	162	54
South	296	74
West	252	48
Total	858	228

a. For the United States, what is the probability that a driver is using a seat belt?
b. The seat belt usage probability for a U.S. driver a year earlier was .75. NHTSA chief Dr. Jeffrey Runge had hoped for a .78 probability in 2003. Would he have been pleased with the 2003 survey results?
c. What is the probability of seat belt usage by region of the country? What region has the highest seat belt usage?
d. What proportion of the drivers in the sample came from each region of the country? What region had the most drivers selected? What region had the second most drivers selected?
e. Assuming the total number of drivers in each region is the same, do you see any reason why the probability estimate in part (a) might be too high? Explain.

12. The Powerball lottery is played twice each week in 28 states, the Virgin Islands, and the District of Columbia. To play Powerball a participant must purchase a ticket and then select five numbers from the digits 1 through 55 and a Powerball number from the digits 1 through 42. To determine the winning numbers for each game, lottery officials draw five white balls out of a drum with 55 white balls, and one red ball out of a drum with 42 red balls. To win the jackpot, a participant's numbers must match the numbers on the five white balls in any order and the number on the red Powerball. Eight coworkers at the ConAgra Foods plant in Lincoln, Nebraska, claimed the record $365 million jackpot on February 18, 2006, by matching the numbers 15-17-43-44-49 and the Powerball number 29. A variety of other cash prizes are awarded each time the game is played. For instance, a prize of $200,000 is paid if the participant's five numbers match the numbers on the five white balls (Powerball website*,* March 19, 2006).
    a. Compute the number of ways the first five numbers can be selected.
    b. What is the probability of winning a prize of $200,000 by matching the numbers on the five white balls?
    c. What is the probability of winning the Powerball jackpot?

13. A company that manufactures toothpaste is studying five different package designs. Assuming that one design is just as likely to be selected by a consumer as any other design, what selection probability would you assign to each of the package designs? In an actual experiment, 100 consumers were asked to pick the design they preferred. The following data were obtained. Do the data confirm the belief that one design is just as likely to be selected as another? Explain.

Design	Number of Times Preferred
1	5
2	15
3	30
4	40
5	10

4.2 Events and Their Probabilities

In the introduction to this chapter we used the term *event* much as it would be used in everyday language. Then, in Section 4.1 we introduced the concept of an experiment and its associated experimental outcomes or sample points. Sample points and events provide the foundation for the study of probability. As a result, we must now introduce the formal definition of an **event** as it relates to sample points. Doing so will provide the basis for determining the probability of an event.

EVENT

An event is a collection of sample points.

For an example, let us return to the KP&L project and assume that the project manager is interested in the event that the entire project can be completed in 10 months or less. Referring to Table 4.3, we see that six sample points—(2, 6), (2, 7), (2, 8), (3, 6), (3, 7), and (4, 6)—provide a project completion time of 10 months or less. Let C denote the event that the project is completed in 10 months or less; we write

$$C = \{(2, 6), (2, 7), (2, 8), (3, 6), (3, 7), (4, 6)\}$$

Event C is said to occur if *any one* of these six sample points appears as the experimental outcome.

Other events that might be of interest to KP&L management include the following.

L = The event that the project is completed in *less* than 10 months
M = The event that the project is completed in *more* than 10 months

Using the information in Table 4.3, we see that these events consist of the following sample points.

$$L = \{(2, 6), (2, 7), (3, 6)\}$$
$$M = \{(3, 8), (4, 7), (4, 8)\}$$

A variety of additional events can be defined for the KP&L project, but in each case the event must be identified as a collection of sample points for the experiment.

Given the probabilities of the sample points shown in Table 4.3, we can use the following definition to compute the probability of any event that KP&L management might want to consider.

PROBABILITY OF AN EVENT

The probability of any event is equal to the sum of the probabilities of the sample points in the event.

Using this definition, we calculate the probability of a particular event by adding the probabilities of the sample points (experimental outcomes) that make up the event. We can now compute the probability that the project will take 10 months or less to complete. Because this event is given by $C = \{(2, 6), (2, 7), (2, 8), (3, 6), (3, 7), (4, 6)\}$, the probability of event C, denoted $P(C)$, is given by

$$P(C) = P(2, 6) + P(2, 7) + P(2, 8) + P(3, 6) + P(3, 7) + P(4, 6)$$

Refer to the sample point probabilities in Table 4.3; we have

$$P(C) = .15 + .15 + .05 + .10 + .20 + .05 = .70$$

Similarly, because the event that the project is completed in less than 10 months is given by $L = \{(2, 6), (2, 7), (3, 6)\}$, the probability of this event is given by

$$\begin{aligned} P(L) &= P(2, 6) + P(2, 7) + P(3, 6) \\ &= .15 + .15 + .10 = .40 \end{aligned}$$

Finally, for the event that the project is completed in more than 10 months, we have $M = \{(3, 8), (4, 7), (4, 8)\}$ and thus

$$\begin{aligned} P(M) &= P(3, 8) + P(4, 7) + P(4, 8) \\ &= .05 + .10 + .15 = .30 \end{aligned}$$

Using these probability results, we can now tell KP&L management that there is a .70 probability that the project will be completed in 10 months or less, a .40 probability that the project will be completed in less than 10 months, and a .30 probability that the project will be completed in more than 10 months. This procedure of computing event probabilities can be repeated for any event of interest to the KP&L management.

Any time that we can identify all the sample points of an experiment and assign probabilities to each, we can compute the probability of an event using the definition. However, in many experiments the large number of sample points makes the identification of the sample points, as well as the determination of their associated probabilities, extremely cumbersome, if not impossible. In the remaining sections of this chapter, we present some basic probability relationships that can be used to compute the probability of an event without knowledge of all the sample point probabilities.

NOTES AND COMMENTS

1. The sample space, S, is an event. Because it contains all the experimental outcomes, it has a probability of 1; that is, $P(S) = 1$.
2. When the classical method is used to assign probabilities, the assumption is that the experimental outcomes are equally likely. In such cases, the probability of an event can be computed by counting the number of experimental outcomes in the event and dividing the result by the total number of experimental outcomes.

Exercises

Methods

14. An experiment has four equally likely outcomes: E_1, E_2, E_3, and E_4.
    a. What is the probability that E_2 occurs?
    b. What is the probability that any two of the outcomes occur (e.g., E_1 or E_3)?
    c. What is the probability that any three of the outcomes occur (e.g., E_1 or E_2 or E_4)?

SELF test

15. Consider the experiment of selecting a playing card from a deck of 52 playing cards. Each card corresponds to a sample point with a 1/52 probability.
    a. List the sample points in the event an ace is selected.
    b. List the sample points in the event a club is selected.
    c. List the sample points in the event a face card (jack, queen, or king) is selected.
    d. Find the probabilities associated with each of the events in parts (a), (b), and (c).

16. Consider the experiment of rolling a pair of dice. Suppose that we are interested in the sum of the face values showing on the dice.
    a. How many sample points are possible? (*Hint:* Use the counting rule for multiple-step experiments.)
    b. List the sample points.
    c. What is the probability of obtaining a value of 7?
    d. What is the probability of obtaining a value of 9 or greater?
    e. Because each roll has six possible even values (2, 4, 6, 8, 10, and 12) and only five possible odd values (3, 5, 7, 9, and 11), the dice should show even values more often than odd values. Do you agree with this statement? Explain.
    f. What method did you use to assign the probabilities requested?

Applications

SELF test

17. Refer to the KP&L sample points and sample point probabilities in Tables 4.2 and 4.3.
    a. The design stage (stage 1) will run over budget if it takes 4 months to complete. List the sample points in the event the design stage is over budget.
    b. What is the probability that the design stage is over budget?
    c. The construction stage (stage 2) will run over budget if it takes 8 months to complete. List the sample points in the event the construction stage is over budget.
    d. What is the probability that the construction stage is over budget?
    e. What is the probability that both stages are over budget?
18. To investigate how often families eat at home, Harris Interactive surveyed 496 adults living with children under the age of 18 (*USA Today,* January 3, 2007). The survey results are shown in the following table.

Number of Family Meals per Week	Number of Survey Responses
0	11
1	11
2	30
3	36
4	36
5	119
6	114
7 or more	139

For a randomly selected family with children under the age of 18, compute the following.
    a. The probability the family eats no meals at home during the week.
    b. The probability the family eats at least four meals at home during the week.
    c. The probability the family eats two or fewer meals at home during the week.
19. The National Sporting Goods Association conducted a survey of persons 7 years of age or older about participation in sports activities (*Statistical Abstract of the United States*, 2002). The total population in this age group was reported at 248.5 million, with 120.9 million male and 127.6 million female. The number of participants for the top five sports activities appears here.

	Participants (millions)	
Activity	**Male**	**Female**
Bicycle riding	22.2	21.0
Camping	25.6	24.3
Exercise walking	28.7	57.7
Exercising with equipment	20.4	24.4
Swimming	26.4	34.4

    a. For a randomly selected female, estimate the probability of participation in each of the sports activities.
    b. For a randomly selected male, estimate the probability of participation in each of the sports activities.
    c. For a randomly selected person, what is the probability the person participates in exercise walking?
    d. Suppose you just happen to see an exercise walker going by. What is the probability the walker is a woman? What is the probability the walker is a man?

20. *Fortune* magazine publishes an annual list of the 500 largest companies in the United States. The following data show the five states with the largest number of *Fortune* 500 companies (*The New York Times Almanac,* 2006).

State	Number of Companies
New York	54
California	52
Texas	48
Illinois	33
Ohio	30

Suppose a *Fortune* 500 company is chosen for a follow-up questionnaire. What are the probabilities of the following events?

a. Let N be the event the company is headquartered in New York. Find $P(N)$.
b. Let T be the event the company is headquartered in Texas. Find $P(T)$.
c. Let B be the event the company is headquartered in one of these five states. Find $P(B)$.

21. The U.S. adult population by age is as follows (*The World Almanac,* 2009). The data are in millions of people.

Age	Number
18 to 24	29.8
25 to 34	40.0
35 to 44	43.4
45 to 54	43.9
55 to 64	32.7
65 and over	37.8

Assume that a person will be randomly chosen from this population.

a. What is the probability the person is 18 to 24 years old?
b. What is the probability the person is 18 to 34 years old?
c. What is the probability the person is 45 or older?

4.3 Some Basic Relationships of Probability

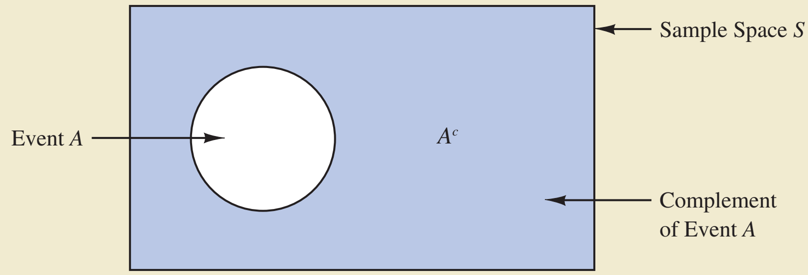
Complement of an Event

Given an event A, the **complement of A** is defined to be the event consisting of all sample points that are *not* in A. The complement of A is denoted by A^c. Figure 4.4 is a diagram, known as a **Venn diagram**, which illustrates the concept of a complement. The rectangular area represents the sample space for the experiment and as such contains all possible sample points. The circle represents event A and contains only the sample points that belong to A. The shaded region of the rectangle contains all sample points not in event A and is by definition the complement of A.

In any probability application, either event A or its complement A^c must occur. Therefore, we have

$$P(A) + P(A^c) = 1$$

FIGURE 4.4 COMPLEMENT OF EVENT A IS SHADED



Solving for $P(A)$, we obtain the following result.

COMPUTING PROBABILITY USING THE COMPLEMENT

$$P(A) = 1 - P(A^c) \tag{4.5}$$

Equation (4.5) shows that the probability of an event A can be computed easily if the probability of its complement, $P(A^c)$, is known.

As an example, consider the case of a sales manager who, after reviewing sales reports, states that 80% of new customer contacts result in no sale. By allowing A to denote the event of a sale and A^c to denote the event of no sale, the manager is stating that $P(A^c) = .80$. Using equation (4.5), we see that

$$P(A) = 1 - P(A^c) = 1 - .80 = .20$$

We can conclude that a new customer contact has a .20 probability of resulting in a sale.

In another example, a purchasing agent states a .90 probability that a supplier will send a shipment that is free of defective parts. Using the complement, we can conclude that there is a $1 - .90 = .10$ probability that the shipment will contain defective parts.

Addition Law

The addition law is helpful when we are interested in knowing the probability that at least one of two events occurs. That is, with events A and B we are interested in knowing the probability that event A or event B or both occur.

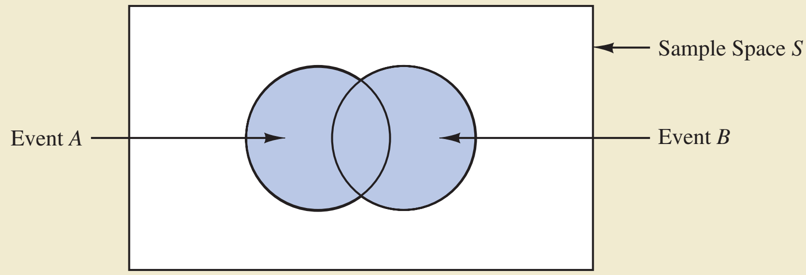
Before we present the addition law, we need to discuss two concepts related to the combination of events: the *union* of events and the *intersection* of events. Given two events A and B, the **union of A and B** is defined as follows.

UNION OF TWO EVENTS

The *union* of A and B is the event containing *all* sample points belonging to A *or* B *or both.* The union is denoted by $A \cup B$.

The Venn diagram in Figure 4.5 depicts the union of events A and B. Note that the two circles contain all the sample points in event A as well as all the sample points in event B.

FIGURE 4.5 UNION OF EVENTS *A* AND *B* IS SHADED



The fact that the circles overlap indicates that some sample points are contained in both *A* and *B*.

The definition of the **intersection of *A* and *B*** follows.

INTERSECTION OF TWO EVENTS

Given two events *A* and *B*, the *intersection* of *A* and *B* is the event containing the sample points belonging to *both A and B*. The intersection is denoted by $A \cap B$.

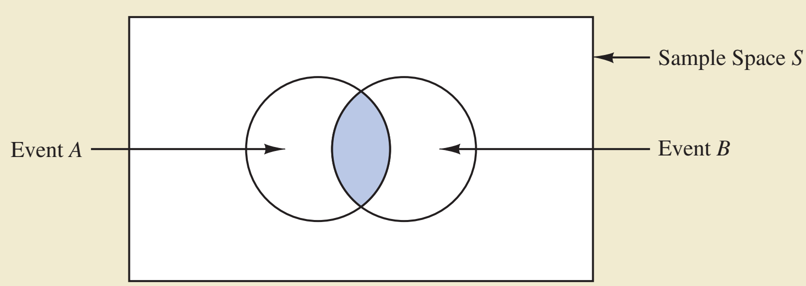
The Venn diagram depicting the intersection of events *A* and *B* is shown in Figure 4.6. The area where the two circles overlap is the intersection; it contains the sample points that are in both *A* and *B*.

Let us now continue with a discussion of the addition law. The **addition law** provides a way to compute the probability that event *A* or event *B* or both occur. In other words, the addition law is used to compute the probability of the union of two events. The addition law is written as follows.

ADDITION LAW

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{4.6}$$

FIGURE 4.6 INTERSECTION OF EVENTS *A* AND *B* IS SHADED



To understand the addition law intuitively, note that the first two terms in the addition law, $P(A) + P(B)$, account for all the sample points in $A \cup B$. However, because the sample points in the intersection $A \cap B$ are in both A and B, when we compute $P(A) + P(B)$, we are in effect counting each of the sample points in $A \cap B$ twice. We correct for this overcounting by subtracting $P(A \cap B)$.

As an example of an application of the addition law, let us consider the case of a small assembly plant with 50 employees. Each worker is expected to complete work assignments on time and in such a way that the assembled product will pass a final inspection. On occasion, some of the workers fail to meet the performance standards by completing work late or assembling a defective product. At the end of a performance evaluation period, the production manager found that 5 of the 50 workers completed work late, 6 of the 50 workers assembled a defective product, and 2 of the 50 workers both completed work late *and* assembled a defective product.

Let

$$L = \text{the event that the work is completed late}$$
$$D = \text{the event that the assembled product is defective}$$

The relative frequency information leads to the following probabilities.

$$P(L) = \frac{5}{50} = .10$$
$$P(D) = \frac{6}{50} = .12$$
$$P(L \cap D) = \frac{2}{50} = .04$$

After reviewing the performance data, the production manager decided to assign a poor performance rating to any employee whose work was either late or defective; thus the event of interest is $L \cup D$. What is the probability that the production manager assigned an employee a poor performance rating?

Note that the probability question is about the union of two events. Specifically, we want to know $P(L \cup D)$. Using equation (4.6), we have

$$P(L \cup D) = P(L) + P(D) - P(L \cap D)$$

Knowing values for the three probabilities on the right side of this expression, we can write

$$P(L \cup D) = .10 + .12 - .04 = .18$$

This calculation tells us that there is a .18 probability that a randomly selected employee received a poor performance rating.

As another example of the addition law, consider a recent study conducted by the personnel manager of a major computer software company. The study showed that 30% of the employees who left the firm within two years did so primarily because they were dissatisfied with their salary, 20% left because they were dissatisfied with their work assignments, and 12% of the former employees indicated dissatisfaction with *both* their salary and their work assignments. What is the probability that an employee who leaves within two years does so because of dissatisfaction with salary, dissatisfaction with the work assignment, or both?

Let

$$S = \text{the event that the employee leaves because of salary}$$
$$W = \text{the event that the employee leaves because of work assignment}$$

We have $P(S) = .30$, $P(W) = .20$, and $P(S \cap W) = .12$. Using equation (4.6), the addition law, we have

$$P(S \cup W) = P(S) + P(W) - P(S \cap W) = .30 + .20 - .12 = .38.$$

We find a .38 probability that an employee leaves for salary or work assignment reasons.

Before we conclude our discussion of the addition law, let us consider a special case that arises for **mutually exclusive events**.

MUTUALLY EXCLUSIVE EVENTS

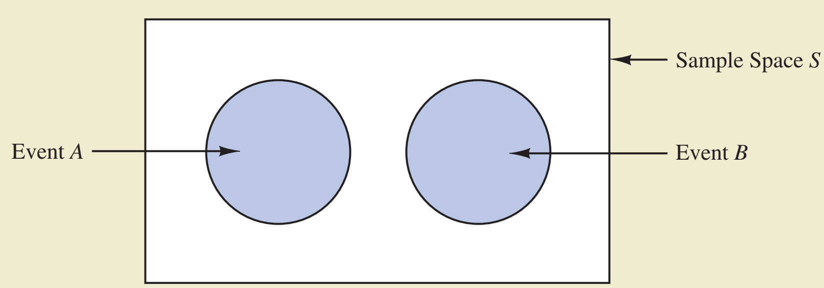
Two events are said to be mutually exclusive if the events have no sample points in common.

Events A and B are mutually exclusive if, when one event occurs, the other cannot occur. Thus, a requirement for A and B to be mutually exclusive is that their intersection must contain no sample points. The Venn diagram depicting two mutually exclusive events A and B is shown in Figure 4.7. In this case $P(A \cap B) = 0$ and the addition law can be written as follows.

ADDITION LAW FOR MUTUALLY EXCLUSIVE EVENTS

$$P(A \cup B) = P(A) + P(B)$$

FIGURE 4.7 MUTUALLY EXCLUSIVE EVENTS



Exercises

Methods

22. Suppose that we have a sample space with five equally likely experimental outcomes: E_1, E_2, E_3, E_4, E_5. Let

$$A = \{E_1, E_2\}$$
$$B = \{E_3, E_4\}$$
$$C = \{E_2, E_3, E_5\}$$

a. Find $P(A)$, $P(B)$, and $P(C)$.
b. Find $P(A \cup B)$. Are A and B mutually exclusive?
c. Find A^c, C^c, $P(A^c)$, and $P(C^c)$.
d. Find $A \cup B^c$ and $P(A \cup B^c)$.
e. Find $P(B \cup C)$.

SELF test

23. Suppose that we have a sample space $S = \{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$, where $E_1, E_2, \ldots, E_7$ denote the sample points. The following probability assignments apply: $P(E_1) = .05$, $P(E_2) = .20$, $P(E_3) = .20$, $P(E_4) = .25$, $P(E_5) = .15$, $P(E_6) = .10$, and $P(E_7) = .05$. Let

$$A = \{E_1, E_4, E_6\}$$
$$B = \{E_2, E_4, E_7\}$$
$$C = \{E_2, E_3, E_5, E_7\}$$

a. Find $P(A)$, $P(B)$, and $P(C)$.
b. Find $A \cup B$ and $P(A \cup B)$.
c. Find $A \cap B$ and $P(A \cap B)$.
d. Are events A and C mutually exclusive?
e. Find B^c and $P(B^c)$.

Applications

24. Clarkson University surveyed alumni to learn more about what they think of Clarkson. One part of the survey asked respondents to indicate whether their overall experience at Clarkson fell short of expectations, met expectations, or surpassed expectations. The results showed that 4% of the respondents did not provide a response, 26% said that their experience fell short of expectations, and 65% of the respondents said that their experience met expectations.
a. If we chose an alumnus at random, what is the probability that the alumnus would say their experience *surpassed* expectations?
b. If we chose an alumnus at random, what is the probability that the alumnus would say their experience met or surpassed expectations?

25. The U.S. Census Bureau provides data on the number of young adults, ages 18–24, who are living in their parents' home.[1] Let

M = the event a male young adult is living in his parents' home
F = the event a female young adult is living in her parents' home

If we randomly select a male young adult and a female young adult, the Census Bureau data enable us to conclude $P(M) = .56$ and $P(F) = .42$ (*The World Almanac,* 2006). The probability that both are living in their parents' home is .24.
a. What is the probability at least one of the two young adults selected is living in his or her parents' home?
b. What is the probability both young adults selected are living on their own (neither is living in their parents' home)?

[1]The data include single young adults who are living in college dormitories because it is assumed these young adults will return to their parents' home when school is not in session.

26. Information about mutual funds provided by Morningstar Investment Research includes the type of mutual fund (Domestic Equity, International Equity, or Fixed Income) and the Morningstar rating for the fund. The rating is expressed from 1-star (lowest rating) to 5-star (highest rating). A sample of 25 mutual funds was selected from *Morningstar Funds500* (2008). The following counts were obtained:
    - Sixteen mutual funds were Domestic Equity funds.
    - Thirteen mutual funds were rated 3-star or less.
    - Seven of the Domestic Equity funds were rated 4-star.
    - Two of the Domestic Equity funds were rated 5-star.

    Assume that one of these 25 mutual funds will be randomly selected in order to learn more about the mutual fund and its investment strategy.
    a. What is the probability of selecting a Domestic Equity fund?
    b. What is the probability of selecting a fund with a 4-star or 5-star rating?
    c. What is the probability of selecting a fund that is both a Domestic Equity fund *and* a fund with a 4-star or 5-star rating?
    d. What is the probability of selecting a fund that is a Domestic Equity fund *or* a fund with a 4-star or 5-star rating?

27. What NCAA college basketball conferences have the higher probability of having a team play in college basketball's national championship game? Over the last 20 years, the Atlantic Coast Conference (ACC) ranks first by having a team in the championship game 10 times. The Southeastern Conference (SEC) ranks second by having a team in the championship game 8 times. However, these two conferences have both had teams in the championship game only one time, when Arkansas (SEC) beat Duke (ACC) 76–70 in 1994 (NCAA website, April 2009). Use these data to estimate the following probabilities.
    a. What is the probability the ACC will have a team in the championship game?
    b. What is the probability the SEC will have team in the championship game?
    c. What is the probability the ACC and SEC will both have teams in the championship game?
    d. What is the probability at least one team from these two conferences will be in the championship game? That is, what is the probability a team from the ACC or SEC will play in the championship game?
    e. What is the probability that the championship game will not a have team from one of these two conferences?

SELF test

28. A survey of magazine subscribers showed that 45.8% rented a car during the past 12 months for business reasons, 54% rented a car during the past 12 months for personal reasons, and 30% rented a car during the past 12 months for both business and personal reasons.
    a. What is the probability that a subscriber rented a car during the past 12 months for business or personal reasons?
    b. What is the probability that a subscriber did not rent a car during the past 12 months for either business or personal reasons?

29. High school seniors with strong academic records apply to the nation's most selective colleges in greater numbers each year. Because the number of slots remains relatively stable, some colleges reject more early applicants. The University of Pennsylvania received 2851 applications for early admission. Of this group, it admitted 1033 students early, rejected 854 outright, and deferred 964 to the regular admission pool for further consideration. In the past, Penn has admitted 18% of the deferred early admission applicants during the regular admission process. Counting the students admitted early and the students admitted during the regular admission process, the total class size was 2375 (*USA Today,* January 24, 2001). Let E, R, and D represent the events that a student who applies for early admission is admitted early, rejected outright, or deferred to the regular admissions pool.
    a. Use the data to estimate $P(E)$, $P(R)$, and $P(D)$.
    b. Are events E and D mutually exclusive? Find $P(E \cap D)$.

c. For the 2375 students admitted to Penn, what is the probability that a randomly selected student was accepted during early admission?
d. Suppose a student applies to Penn for early admission. What is the probability the student will be admitted for early admission or be deferred and later admitted during the regular admission process?

4.4 Conditional Probability

Often, the probability of an event is influenced by whether a related event already occurred. Suppose we have an event A with probability $P(A)$. If we obtain new information and learn that a related event, denoted by B, already occurred, we will want to take advantage of this information by calculating a new probability for event A. This new probability of event A is called a **conditional probability** and is written $P(A \mid B)$. We use the notation $\mid$ to indicate that we are considering the probability of event A *given* the condition that event B has occurred. Hence, the notation $P(A \mid B)$ reads "the probability of A given B."

As an illustration of the application of conditional probability, consider the situation of the promotion status of male and female officers of a major metropolitan police force in the eastern United States. The police force consists of 1200 officers, 960 men and 240 women. Over the past two years, 324 officers on the police force received promotions. The specific breakdown of promotions for male and female officers is shown in Table 4.4.

After reviewing the promotion record, a committee of female officers raised a discrimination case on the basis that 288 male officers had received promotions but only 36 female officers had received promotions. The police administration argued that the relatively low number of promotions for female officers was due not to discrimination, but to the fact that relatively few females are members of the police force. Let us show how conditional probability could be used to analyze the discrimination charge.

Let

$$
\begin{aligned}
M &= \text{event an officer is a man} \\
W &= \text{event an officer is a woman} \\
A &= \text{event an officer is promoted} \\
A^c &= \text{event an officer is not promoted}
\end{aligned}
$$

Dividing the data values in Table 4.4 by the total of 1200 officers enables us to summarize the available information with the following probability values.

$P(M \cap A) = 288/1200 = .24$ probability that a randomly selected officer is a man *and* is promoted

$P(M \cap A^c) = 672/1200 = .56$ probability that a randomly selected officer is a man *and* is not promoted

TABLE 4.4 PROMOTION STATUS OF POLICE OFFICERS OVER THE PAST TWO YEARS

	Men	Women	Total
Promoted	288	36	324
Not Promoted	672	204	876
Total	960	240	1200

TABLE 4.5 JOINT PROBABILITY TABLE FOR PROMOTIONS

Joint probabilities appear in the body of the table.

	Men (M)	Women (W)	Total
Promoted (A)	.24	.03	.27
Not Promoted (A^c)	.56	.17	.73
Total	.80	.20	1.00

Marginal probabilities appear in the margins of the table.

$P(W \cap A) = 36/1200 = .03$ probability that a randomly selected officer is a woman *and* is promoted

$P(W \cap A^c) = 204/1200 = .17$ probability that a randomly selected officer is a woman *and* is not promoted

Because each of these values gives the probability of the intersection of two events, the probabilities are called **joint probabilities**. Table 4.5, which provides a summary of the probability information for the police officer promotion situation, is referred to as a *joint probability table.*

The values in the margins of the joint probability table provide the probabilities of each event separately. That is, $P(M) = .80$, $P(W) = .20$, $P(A) = .27$, and $P(A^c) = .73$. These probabilities are referred to as **marginal probabilities** because of their location in the margins of the joint probability table. We note that the marginal probabilities are found by summing the joint probabilities in the corresponding row or column of the joint probability table. For instance, the marginal probability of being promoted is $P(A) = P(M \cap A) + P(W \cap A) = .24 + .03 = .27$. From the marginal probabilities, we see that 80% of the force is male, 20% of the force is female, 27% of all officers received promotions, and 73% were not promoted.

Let us begin the conditional probability analysis by computing the probability that an officer is promoted given that the officer is a man. In conditional probability notation, we are attempting to determine $P(A \mid M)$. To calculate $P(A \mid M)$, we first realize that this notation simply means that we are considering the probability of the event A (promotion) given that the condition designated as event M (the officer is a man) is known to exist. Thus $P(A \mid M)$ tells us that we are now concerned only with the promotion status of the 960 male officers. Because 288 of the 960 male officers received promotions, the probability of being promoted given that the officer is a man is $288/960 = .30$. In other words, given that an officer is a man, that officer had a 30% chance of receiving a promotion over the past two years.

This procedure was easy to apply because the values in Table 4.4 show the number of officers in each category. We now want to demonstrate how conditional probabilities such as $P(A \mid M)$ can be computed directly from related event probabilities rather than the frequency data of Table 4.4.

We have shown that $P(A \mid M) = 288/960 = .30$. Let us now divide both the numerator and denominator of this fraction by 1200, the total number of officers in the study.

$$P(A \mid M) = \frac{288}{960} = \frac{288/1200}{960/1200} = \frac{.24}{.80} = .30$$

We now see that the conditional probability $P(A \mid M)$ can be computed as .24/.80. Refer to the joint probability table (Table 4.5). Note in particular that .24 is the joint probability of

A and M; that is, $P(A \cap M) = .24$. Also note that .80 is the marginal probability that a randomly selected officer is a man; that is, $P(M) = .80$. Thus, the conditional probability $P(A \mid M)$ can be computed as the ratio of the joint probability $P(A \cap M)$ to the marginal probability $P(M)$.

$$P(A \mid M) = \frac{P(A \cap M)}{P(M)} = \frac{.24}{.80} = .30$$

The fact that conditional probabilities can be computed as the ratio of a joint probability to a marginal probability provides the following general formula for conditional probability calculations for two events A and B.

CONDITIONAL PROBABILITY

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{4.7}$$

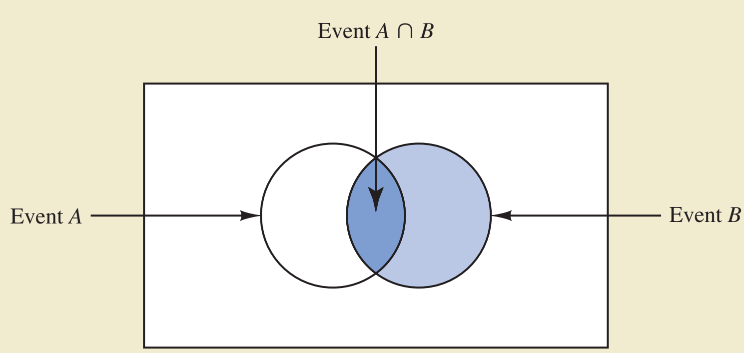
or

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \tag{4.8}$$

The Venn diagram in Figure 4.8 is helpful in obtaining an intuitive understanding of conditional probability. The circle on the right shows that event B has occurred; the portion of the circle that overlaps with event A denotes the event $(A \cap B)$. We know that once event B has occurred, the only way that we can also observe event A is for the event $(A \cap B)$ to occur. Thus, the ratio $P(A \cap B)/P(B)$ provides the conditional probability that we will observe event A given that event B has already occurred.

Let us return to the issue of discrimination against the female officers. The marginal probability in row 1 of Table 4.5 shows that the probability of promotion of an officer is $P(A) = .27$ (regardless of whether that officer is male or female). However, the critical issue in the discrimination case involves the two conditional probabilities $P(A \mid M)$ and $P(A \mid W)$. That is, what is the probability of a promotion *given* that the officer is a man, and what is the probability of a promotion *given* that the officer is a woman? If these two probabilities are equal, a discrimination argument has no basis because the chances of a promotion are the same for male and female officers. However, a difference in the two conditional probabilities will support the position that male and female officers are treated differently in promotion decisions.

FIGURE 4.8 CONDITIONAL PROBABILITY $P(A \mid B) = P(A \cap B)/P(B)$



We already determined that $P(A \mid M) = .30$. Let us now use the probability values in Table 4.5 and the basic relationship of conditional probability in equation (4.7) to compute the probability that an officer is promoted given that the officer is a woman; that is, $P(A \mid W)$. Using equation (4.7), with W replacing B, we obtain

$$P(A \mid W) = \frac{P(A \cap W)}{P(W)} = \frac{.03}{.20} = .15$$

What conclusion do you draw? The probability of a promotion given that the officer is a man is .30, twice the .15 probability of a promotion given that the officer is a woman. Although the use of conditional probability does not in itself prove that discrimination exists in this case, the conditional probability values support the argument presented by the female officers.

Independent Events

In the preceding illustration, $P(A) = .27$, $P(A \mid M) = .30$, and $P(A \mid W) = .15$. We see that the probability of a promotion (event A) is affected or influenced by whether the officer is a man or a woman. Particularly, because $P(A \mid M) \neq P(A)$, we would say that events A and M are dependent events. That is, the probability of event A (promotion) is altered or affected by knowing that event M (the officer is a man) exists. Similarly, with $P(A \mid W) \neq P(A)$, we would say that events A and W are *dependent events*. However, if the probability of event A is not changed by the existence of event M—that is, $P(A \mid M) = P(A)$—we would say that events A and M are **independent events**. This situation leads to the following definition of the independence of two events.

INDEPENDENT EVENTS

Two events A and B are independent if

$$P(A \mid B) = P(A) \quad \textbf{(4.9)}$$

or

$$P(B \mid A) = P(B) \quad \textbf{(4.10)}$$

Otherwise, the events are dependent.

Multiplication Law

Whereas the addition law of probability is used to compute the probability of a union of two events, the multiplication law is used to compute the probability of the intersection of two events. The multiplication law is based on the definition of conditional probability. Using equations (4.7) and (4.8) and solving for $P(A \cap B)$, we obtain the **multiplication law**.

MULTIPLICATION LAW

$$P(A \cap B) = P(B)P(A \mid B) \quad \textbf{(4.11)}$$

or

$$P(A \cap B) = P(A)P(B \mid A) \quad \textbf{(4.12)}$$

To illustrate the use of the multiplication law, consider a newspaper circulation department where it is known that 84% of the households in a particular neighborhood subscribe to the daily edition of the paper. If we let D denote the event that a household subscribes to the daily edition, $P(D) = .84$. In addition, it is known that the probability that a household that already holds a daily subscription also subscribes to the Sunday edition (event S) is .75; that is, $P(S \mid D) = .75$. What is the probability that a household subscribes to both the Sunday and daily editions of the newspaper? Using the multiplication law, we compute the desired $P(S \cap D)$ as

$$P(S \cap D) = P(D)P(S \mid D) = .84(.75) = .63$$

We now know that 63% of the households subscribe to both the Sunday and daily editions.

Before concluding this section, let us consider the special case of the multiplication law when the events involved are independent. Recall that events A and B are independent whenever $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$. Hence, using equations (4.11) and (4.12) for the special case of independent events, we obtain the following multiplication law.

MULTIPLICATION LAW FOR INDEPENDENT EVENTS

$$P(A \cap B) = P(A)P(B) \qquad \textbf{(4.13)}$$

To compute the probability of the intersection of two independent events, we simply multiply the corresponding probabilities. Note that the multiplication law for independent events provides another way to determine whether A and B are independent. That is, if $P(A \cap B) = P(A)P(B)$, then A and B are independent; if $P(A \cap B) \neq P(A)P(B)$, then A and B are dependent.

As an application of the multiplication law for independent events, consider the situation of a service station manager who knows from past experience that 80% of the customers use a credit card when they purchase gasoline. What is the probability that the next two customers purchasing gasoline will each use a credit card? If we let

A = the event that the first customer uses a credit card

B = the event that the second customer uses a credit card

then the event of interest is $A \cap B$. Given no other information, we can reasonably assume that A and B are independent events. Thus,

$$P(A \cap B) = P(A)P(B) = (.80)(.80) = .64$$

To summarize this section, we note that our interest in conditional probability is motivated by the fact that events are often related. In such cases, we say the events are dependent and the conditional probability formulas in equations (4.7) and (4.8) must be used to compute the event probabilities. If two events are not related, they are independent; in this case neither event's probability is affected by whether the other event occurred.

NOTES AND COMMENTS

Do not confuse the notion of mutually exclusive events with that of independent events. Two events with nonzero probabilities cannot be both mutually exclusive and independent. If one mutually exclusive event is known to occur, the other cannot occur; thus, the probability of the other event occurring is reduced to zero. They are therefore dependent.

Exercises

Methods

SELF test

30. Suppose that we have two events, A and B, with $P(A) = .50$, $P(B) = .60$, and $P(A \cap B) = .40$.
    a. Find $P(A \mid B)$.
    b. Find $P(B \mid A)$.
    c. Are A and B independent? Why or why not?

31. Assume that we have two events, A and B, that are mutually exclusive. Assume further that we know $P(A) = .30$ and $P(B) = .40$.
    a. What is $P(A \cap B)$?
    b. What is $P(A \mid B)$?
    c. A student in statistics argues that the concepts of mutually exclusive events and independent events are really the same, and that if events are mutually exclusive they must be independent. Do you agree with this statement? Use the probability information in this problem to justify your answer.
    d. What general conclusion would you make about mutually exclusive and independent events given the results of this problem?

Applications

32. The automobile industry sold 657,000 vehicles in the United States during January 2009 (*The Wall Street Journal,* February 4, 2009). This volume was down 37% from January 2008 as economic conditions continued to decline. The Big Three U.S. automakers—General Motors, Ford, and Chrysler—sold 280,500 vehicles, down 48% from January 2008. A summary of sales by automobile manufacturer and type of vehicle sold is shown in the following table. Data are in thousands of vehicles. The non-U.S. manufacturers are led by Toyota, Honda, and Nissan. The category Light Truck includes pickup, minivan, SUV, and crossover models.

		Type of Vehicle	
		Car	**Light Truck**
Manufacturer	**U.S.**	87.4	193.1
	Non-U.S.	228.5	148.0

    a. Develop a joint probability table for these data and use the table to answer the remaining questions.
    b. What are the marginal probabilities? What do they tell you about the probabilities associated with the manufacturer and the type of vehicle sold?
    c. If a vehicle was manufactured by one of the U.S. automakers, what is the probability that the vehicle was a car? What is the probability it was a light truck?
    d. If a vehicle was not manufactured by one of the U.S. automakers, what is the probability that the vehicle was a car? What is the probability it was a light truck?
    e. If the vehicle was a light truck, what is the probability that it was manufactured by one of the U.S. automakers?
    f. What does the probability information tell you about sales?

SELF test

33. In a survey of MBA students, the following data were obtained on "students' first reason for application to the school in which they matriculated."

		Reason for Application			
		School Quality	**School Cost or Convenience**	**Other**	**Totals**
Enrollment Status	**Full Time**	421	393	76	890
	Part Time	400	593	46	1039
	Totals	821	986	122	1929

    a. Develop a joint probability table for these data.
    b. Use the marginal probabilities of school quality, school cost or convenience, and other to comment on the most important reason for choosing a school.

c. If a student goes full time, what is the probability that school quality is the first reason for choosing a school?
d. If a student goes part time, what is the probability that school quality is the first reason for choosing a school?
e. Let A denote the event that a student is full time and let B denote the event that the student lists school quality as the first reason for applying. Are events A and B independent? Justify your answer.

34. The U.S. Department of Transportation reported that during November, 83.4% of Southwest Airlines' flights, 75.1% of US Airways' flights, and 70.1% of JetBlue's flights arrived on time (*USA Today,* January 4, 2007). Assume that this on-time performance is applicable for flights arriving at concourse A of the Rochester International Airport, and that 40% of the arrivals at concourse A are Southwest Airlines flights, 35% are US Airways flights, and 25% are JetBlue flights.
    a. Develop a joint probability table with three rows (airlines) and two columns (on-time arrivals vs. late arrivals).
    b. An announcement has just been made that Flight 1424 will be arriving at gate 20 in concourse A. What is the most likely airline for this arrival?
    c. What is the probability that Flight 1424 will arrive on time?
    d. Suppose that an announcement is made saying that Flight 1424 will be arriving late. What is the most likely airline for this arrival? What is the least likely airline?

35. According to the Ameriprise Financial Money Across Generations study, 9 out of 10 parents with adult children ages 20 to 35 have helped their adult children with some type of financial assistance ranging from college, a car, rent, utilities, credit-card debt, and/or down payments for houses (*Money,* January 2009). The following table with sample data consistent with the study shows the number of times parents have given their adult children financial assistance to buy a car and to pay rent.

		Pay Rent	
		Yes	**No**
Buy a Car	**Yes**	56	52
	No	14	78

    a. Develop a joint probability table and use it to answer the remaining questions.
    b. Using the marginal probabilities for buy a car and pay rent, are parents more likely to assist their adult children with buying a car or paying rent? What is your interpretation of the marginal probabilities?
    c. If parents provided financial assistance to buy a car, what it the probability that the parents assisted with paying rent?
    d. If parents did not provide financial assistance to buy a car, what is the probability the parents assisted with paying rent?
    e. Is financial assistance to buy a car independent of financial assistance to pay rent? Use probabilities to justify your answer.
    f. What is the probability that parents provided financial assistance for their adult children by either helping buy a car or pay rent?

36. Jerry Stackhouse of the National Basketball Association's Dallas Mavericks is the best free-throw shooter on the team, making 89% of his shots (ESPN website, July, 2008). Assume that late in a basketball game, Jerry Stackhouse is fouled and is awarded two shots.
    a. What is the probability that he will make both shots?
    b. What is the probability that he will make at least one shot?
    c. What is the probability that he will miss both shots?

d. Late in a basketball game, a team often intentionally fouls an opposing player in order to stop the game clock. The usual strategy is to intentionally foul the other team's worst free-throw shooter. Assume that the Dallas Mavericks' center makes 58% of his free-throw shots. Calculate the probabilities for the center as shown in parts (a), (b), and (c), and show that intentionally fouling the Dallas Mavericks' center is a better strategy than intentionally fouling Jerry Stackhouse.

37. Visa Card USA studied how frequently young consumers, ages 18 to 24, use plastic (debit and credit) cards in making purchases (Associated Press, January 16, 2006). The results of the study provided the following probabilities.
- The probability that a consumer uses a plastic card when making a purchase is .37.
- Given that the consumer uses a plastic card, there is a .19 probability that the consumer is 18 to 24 years old.
- Given that the consumer uses a plastic card, there is a .81 probability that the consumer is more than 24 years old.

U.S. Census Bureau data show that 14% of the consumer population is 18 to 24 years old.

a. Given the consumer is 18 to 24 years old, what is the probability that the consumer use a plastic card?
b. Given the consumer is over 24 years old, what is the probability that the consumer uses a plastic card?
c. What is the interpretation of the probabilities shown in parts (a) and (b)?
d. Should companies such as Visa, MasterCard, and Discover make plastic cards available to the 18 to 24 year old age group before these consumers have had time to establish a credit history? If no, why? If yes, what restrictions might the companies place on this age group?

38. A Morgan Stanley Consumer Research Survey sampled men and women and asked each whether they preferred to drink plain bottled water or a sports drink such as Gatorade or Propel Fitness water (*The Atlanta Journal-Constitution*, December 28, 2005). Suppose 200 men and 200 women participated in the study, and 280 reported they preferred plain bottled water. Of the group preferring a sports drink, 80 were men and 40 were women.

Let

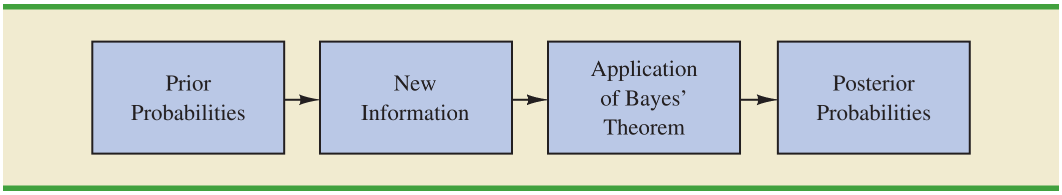
M = the event the consumer is a man
W = the event the consumer is a woman
B = the event the consumer preferred plain bottled water
S = the event the consumer preferred sports drink

a. What is the probability a person in the study preferred plain bottled water?
b. What is the probability a person in the study preferred a sports drink?
c. What are the conditional probabilities $P(M \mid S)$ and $P(W \mid S)$?
d. What are the joint probabilities $P(M \cap S)$ and $P(W \cap S)$?
e. Given a consumer is a man, what is the probability he will prefer a sports drink?
f. Given a consumer is a woman, what is the probability she will prefer a sports drink?
g. Is preference for a sports drink independent of whether the consumer is a man or a woman? Explain using probability information.

4.5 Bayes' Theorem

In the discussion of conditional probability, we indicated that revising probabilities when new information is obtained is an important phase of probability analysis. Often, we begin the analysis with initial or **prior probability** estimates for specific events of interest. Then, from sources such as a sample, a special report, or a product test, we obtain additional information about the events. Given this new information, we update the prior probability values by calculating revised probabilities, referred to as **posterior probabilities**. **Bayes' theorem** provides a means for making these probability calculations. The steps in this probability revision process are shown in Figure 4.9.

FIGURE 4.9 PROBABILITY REVISION USING BAYES' THEOREM



As an application of Bayes' theorem, consider a manufacturing firm that receives shipments of parts from two different suppliers. Let A_1 denote the event that a part is from supplier 1 and A_2 denote the event that a part is from supplier 2. Currently, 65% of the parts purchased by the company are from supplier 1 and the remaining 35% are from supplier 2. Hence, if a part is selected at random, we would assign the prior probabilities $P(A_1) = .65$ and $P(A_2) = .35$.

The quality of the purchased parts varies with the source of supply. Historical data suggest that the quality ratings of the two suppliers are as shown in Table 4.6. If we let G denote the event that a part is good and B denote the event that a part is bad, the information in Table 4.6 provides the following conditional probability values.

$$P(G \mid A_1) = .98 \quad P(B \mid A_1) = .02$$
$$P(G \mid A_2) = .95 \quad P(B \mid A_2) = .05$$

The tree diagram in Figure 4.10 depicts the process of the firm receiving a part from one of the two suppliers and then discovering that the part is good or bad as a two-step experiment. We see that four experimental outcomes are possible; two correspond to the part being good and two correspond to the part being bad.

Each of the experimental outcomes is the intersection of two events, so we can use the multiplication rule to compute the probabilities. For instance,

$$P(A_1, G) = P(A_1 \cap G) = P(A_1)P(G \mid A_1)$$

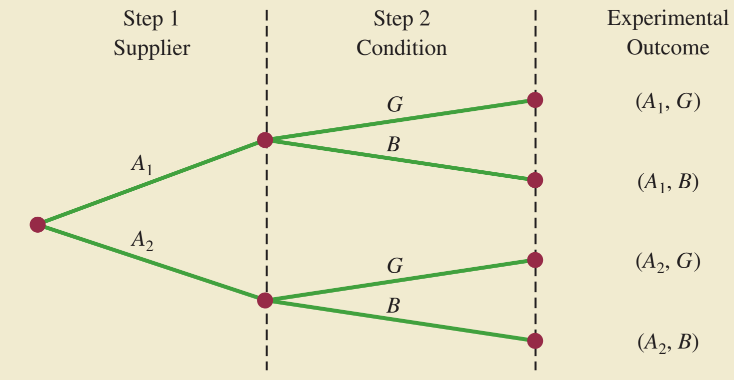
The process of computing these joint probabilities can be depicted in what is called a probability tree (see Figure 4.11). From left to right through the tree, the probabilities for each branch at step 1 are prior probabilities and the probabilities for each branch at step 2 are conditional probabilities. To find the probabilities of each experimental outcome, we simply multiply the probabilities on the branches leading to the outcome. Each of these joint probabilities is shown in Figure 4.11 along with the known probabilities for each branch.

Suppose now that the parts from the two suppliers are used in the firm's manufacturing process and that a machine breaks down because it attempts to process a bad part. Given the information that the part is bad, what is the probability that it came from supplier 1 and

TABLE 4.6 HISTORICAL QUALITY LEVELS OF TWO SUPPLIERS

	Percentage Good Parts	Percentage Bad Parts
Supplier 1	98	2
Supplier 2	95	5

FIGURE 4.10 TREE DIAGRAM FOR TWO-SUPPLIER EXAMPLE



Note: Step 1 shows that the part comes from one of two suppliers, and step 2 shows whether the part is good or bad.

what is the probability that it came from supplier 2? With the information in the probability tree (Figure 4.11), Bayes' theorem can be used to answer these questions.

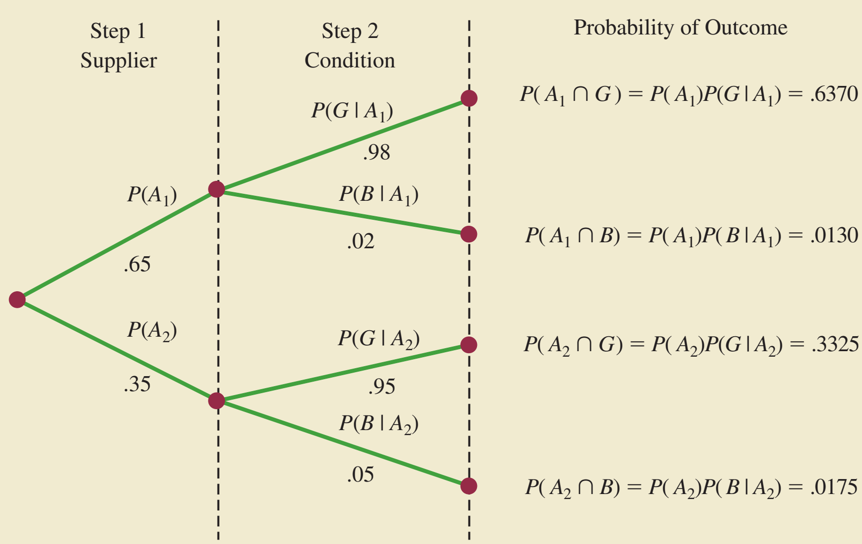
Letting B denote the event that the part is bad, we are looking for the posterior probabilities $P(A_1 \mid B)$ and $P(A_2 \mid B)$. From the law of conditional probability, we know that

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} \tag{4.14}$$

Referring to the probability tree, we see that

$$P(A_1 \cap B) = P(A_1)P(B \mid A_1) \tag{4.15}$$

FIGURE 4.11 PROBABILITY TREE FOR TWO-SUPPLIER EXAMPLE



To find $P(B)$, we note that event B can occur in only two ways: $(A_1 \cap B)$ and $(A_2 \cap B)$. Therefore, we have

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) \\ &= P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) \end{aligned} \quad \textbf{(4.16)}$$

Substituting from equations (4.15) and (4.16) into equation (4.14) and writing a similar result for $P(A_2 \mid B)$, we obtain Bayes' theorem for the case of two events.

The Reverend Thomas Bayes (1702–1761), a Presbyterian minister, is credited with the original work leading to the version of Bayes' theorem in use today.

BAYES' THEOREM (TWO-EVENT CASE)

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)} \quad \textbf{(4.17)}$$

$$P(A_2 \mid B) = \frac{P(A_2)P(B \mid A_2)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)} \quad \textbf{(4.18)}$$

Using equation (4.17) and the probability values provided in the example, we have

$$\begin{aligned} P(A_1 \mid B) &= \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)} \\ &= \frac{(.65)(.02)}{(.65)(.02) + (.35)(.05)} = \frac{.0130}{.0130 + .0175} \\ &= \frac{.0130}{.0305} = .4262 \end{aligned}$$

In addition, using equation (4.18), we find $P(A_2 \mid B)$.

$$\begin{aligned} P(A_2 \mid B) &= \frac{(.35)(.05)}{(.65)(.02) + (.35)(.05)} \\ &= \frac{.0175}{.0130 + .0175} = \frac{.0175}{.0305} = .5738 \end{aligned}$$

Note that in this application we started with a probability of .65 that a part selected at random was from supplier 1. However, given information that the part is bad, the probability that the part is from supplier 1 drops to .4262. In fact, if the part is bad, it has better than a 50–50 chance that it came from supplier 2; that is, $P(A_2 \mid B) = .5738$.

Bayes' theorem is applicable when the events for which we want to compute posterior probabilities are mutually exclusive and their union is the entire sample space.[2] For the case of n mutually exclusive events $A_1, A_2, \ldots, A_n$, whose union is the entire sample space, Bayes' theorem can be used to compute any posterior probability $P(A_i \mid B)$ as shown here.

BAYES' THEOREM

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n)} \quad \textbf{(4.19)}$$

[2] If the union of events is the entire sample space, the events are said to be *collectively exhaustive*.

With prior probabilities $P(A_1), P(A_2), \ldots, P(A_n)$ and the appropriate conditional probabilities $P(B \mid A_1), P(B \mid A_2), \ldots, P(B \mid A_n)$, equation (4.19) can be used to compute the posterior probability of the events $A_1, A_2, \ldots, A_n$.

Tabular Approach

A tabular approach is helpful in conducting the Bayes' theorem calculations. Such an approach is shown in Table 4.7 for the parts supplier problem. The computations shown there are done in the following steps.

Step 1. Prepare the following three columns:
Column 1—The mutually exclusive events A_i for which posterior probabilities are desired
Column 2—The prior probabilities $P(A_i)$ for the events
Column 3—The conditional probabilities $P(B \mid A_i)$ of the new information B given each event

Step 2. In column 4, compute the joint probabilities $P(A_i \cap B)$ for each event and the new information B by using the multiplication law. These joint probabilities are found by multiplying the prior probabilities in column 2 by the corresponding conditional probabilities in column 3; that is, $P(A_i \cap B) = P(A_i)P(B \mid A_i)$.

Step 3. Sum the joint probabilities in column 4. The sum is the probability of the new information, $P(B)$. Thus we see in Table 4.7 that there is a .0130 probability that the part came from supplier 1 and is bad and a .0175 probability that the part came from supplier 2 and is bad. Because these are the only two ways in which a bad part can be obtained, the sum .0130 + .0175 shows an overall probability of .0305 of finding a bad part from the combined shipments of the two suppliers.

Step 4. In column 5, compute the posterior probabilities using the basic relationship of conditional probability.

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)}$$

Note that the joint probabilities $P(A_i \cap B)$ are in column 4 and the probability $P(B)$ is the sum of column 4.

TABLE 4.7 TABULAR APPROACH TO BAYES' THEOREM CALCULATIONS FOR THE TWO-SUPPLIER PROBLEM

(1) Events A_i	(2) Prior Probabilities $P(A_i)$	(3) Conditional Probabilities $P(B \mid A_i)$	(4) Joint Probabilities $P(A_i \cap B)$	(5) Posterior Probabilities $P(A_i \mid B)$
A_1	.65	.02	.0130	.0130/.0305 = .4262
A_2	.35	.05	.0175	.0175/.0305 = .5738
	1.00		$P(B)$ = .0305	1.0000

NOTES AND COMMENTS

1. Bayes' theorem is used extensively in decision analysis. The prior probabilities are often subjective estimates provided by a decision maker. Sample information is obtained and posterior probabilities are computed for use in choosing the best decision.
2. An event and its complement are mutually exclusive, and their union is the entire sample space. Thus, Bayes' theorem is always applicable for computing posterior probabilities of an event and its complement.

Exercises

Methods

SELF test

39. The prior probabilities for events A_1 and A_2 are $P(A_1) = .40$ and $P(A_2) = .60$. It is also known that $P(A_1 \cap A_2) = 0$. Suppose $P(B \mid A_1) = .20$ and $P(B \mid A_2) = .05$.
    a. Are A_1 and A_2 mutually exclusive? Explain.
    b. Compute $P(A_1 \cap B)$ and $P(A_2 \cap B)$.
    c. Compute $P(B)$.
    d. Apply Bayes' theorem to compute $P(A_1 \mid B)$ and $P(A_2 \mid B)$.
40. The prior probabilities for events A_1, A_2, and A_3 are $P(A_1) = .20$, $P(A_2) = .50$, and $P(A_3) = .30$. The conditional probabilities of event B given A_1, A_2, and A_3 are $P(B \mid A_1) = .50$, $P(B \mid A_2) = .40$, and $P(B \mid A_3) = .30$.
    a. Compute $P(B \cap A_1)$, $P(B \cap A_2)$, and $P(B \cap A_3)$.
    b. Apply Bayes' theorem, equation (4.19), to compute the posterior probability $P(A_2 \mid B)$.
    c. Use the tabular approach to applying Bayes' theorem to compute $P(A_1 \mid B)$, $P(A_2 \mid B)$, and $P(A_3 \mid B)$.

Applications

41. A consulting firm submitted a bid for a large research project. The firm's management initially felt they had a 50–50 chance of getting the project. However, the agency to which the bid was submitted subsequently requested additional information on the bid. Past experience indicates that for 75% of the successful bids and 40% of the unsuccessful bids the agency requested additional information.
    a. What is the prior probability of the bid being successful (that is, prior to the request for additional information)?
    b. What is the conditional probability of a request for additional information given that the bid will ultimately be successful?
    c. Compute the posterior probability that the bid will be successful given a request for additional information.

SELF test

42. A local bank reviewed its credit card policy with the intention of recalling some of its credit cards. In the past approximately 5% of cardholders defaulted, leaving the bank unable to collect the outstanding balance. Hence, management established a prior probability of .05 that any particular cardholder will default. The bank also found that the probability of missing a monthly payment is .20 for customers who do not default. Of course, the probability of missing a monthly payment for those who default is 1.
    a. Given that a customer missed one or more monthly payments, compute the posterior probability that the customer will default.
    b. The bank would like to recall its card if the probability that a customer will default is greater than .20. Should the bank recall its card if the customer misses a monthly payment? Why or why not?

43. Small cars get better gas mileage, but they are not as safe as bigger cars. Small cars accounted for 18% of the vehicles on the road, but accidents involving small cars led to 11,898 fatalities during a recent year (*Reader's Digest,* May 2000). Assume the probability a small car is involved in an accident is .18. The probability of an accident involving a small car leading to a fatality is .128 and the probability of an accident not involving a small car leading to a fatality is .05. Suppose you learn of an accident involving a fatality. What is the probability a small car was involved? Assume that the likelihood of getting into an accident is independent of car size.

44. The American Council of Education reported that 47% of college freshmen earn a degree and graduate within five years (Associated Press, May 6, 2002). Assume that graduation records show women make up 50% of the students who graduated within five years, but only 45% of the students who did not graduate within five years. The students who had not graduated within five years either dropped out or were still working on their degrees.
    a. Let A_1 = the student graduated within five years
       A_2 = the student did not graduate within five years
       W = the student is a female student
       Using the given information, what are the values for $P(A_1)$, $P(A_2)$, $P(W \mid A_1)$, and $P(W \mid A_2)$?
    b. What is the probability that a female student will graduate within five years?
    c. What is the probability that a male student will graduate within five years?
    d. Given the preceding results, what are the percentage of women and the percentage of men in the entering freshman class?

45. In an article about investment alternatives, *Money* magazine reported that drug stocks provide a potential for long-term growth, with over 50% of the adult population of the United States taking prescription drugs on a regular basis. For adults age 65 and older, 82% take prescription drugs regularly. For adults age 18 to 64, 49% take prescription drugs regularly. The age 18–64 age group accounts for 83.5% of the adult population (*Statistical Abstract of the United States*, 2008).
    a. What is the probability that a randomly selected adult is 65 or older?
    b. Given an adult takes prescription drugs regularly, what is the probability that the adult is 65 or older?

Summary

In this chapter we introduced basic probability concepts and illustrated how probability analysis can be used to provide helpful information for decision making. We described how probability can be interpreted as a numerical measure of the likelihood that an event will occur. In addition, we saw that the probability of an event can be computed either by summing the probabilities of the experimental outcomes (sample points) comprising the event or by using the relationships established by the addition, conditional probability, and multiplication laws of probability. For cases in which additional information is available, we showed how Bayes' theorem can be used to obtain revised or posterior probabilities.

Glossary

Probability A numerical measure of the likelihood that an event will occur.
Experiment A process that generates well-defined outcomes.
Sample space The set of all experimental outcomes.
Sample point An element of the sample space. A sample point represents an experimental outcome.
Tree diagram A graphical representation that helps in visualizing a multiple-step experiment.

Basic requirements for assigning probabilities Two requirements that restrict the manner in which probability assignments can be made: (1) for each experimental outcome E_i we must have $0 \le P(E_i) \le 1$; (2) considering all experimental outcomes, we must have $P(E_1) + P(E_2) + \cdots + P(E_n) = 1.0$.

Classical method A method of assigning probabilities that is appropriate when all the experimental outcomes are equally likely.

Relative frequency method A method of assigning probabilities that is appropriate when data are available to estimate the proportion of the time the experimental outcome will occur if the experiment is repeated a large number of times.

Subjective method A method of assigning probabilities on the basis of judgment.

Event A collection of sample points.

Complement of *A* The event consisting of all sample points that are not in A.

Venn diagram A graphical representation for showing symbolically the sample space and operations involving events in which the sample space is represented by a rectangle and events are represented as circles within the sample space.

Union of *A* and *B* The event containing all sample points belonging to A or B or both. The union is denoted $A \cup B$.

Intersection of *A* and *B* The event containing the sample points belonging to both A and B. The intersection is denoted $A \cap B$.

Addition law A probability law used to compute the probability of the union of two events. It is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. For mutually exclusive events, $P(A \cap B) = 0$; in this case the addition law reduces to $P(A \cup B) = P(A) + P(B)$.

Mutually exclusive events Events that have no sample points in common; that is, $A \cap B$ is empty and $P(A \cap B) = 0$.

Conditional probability The probability of an event given that another event already occurred. The conditional probability of A given B is $P(A \mid B) = P(A \cap B)/P(B)$.

Joint probability The probability of two events both occurring; that is, the probability of the intersection of two events.

Marginal probability The values in the margins of a joint probability table that provide the probabilities of each event separately.

Independent events Two events A and B where $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$; that is, the events have no influence on each other.

Multiplication law A probability law used to compute the probability of the intersection of two events. It is $P(A \cap B) = P(B)P(A \mid B)$ or $P(A \cap B) = P(A)P(B \mid A)$. For independent events it reduces to $P(A \cap B) = P(A)P(B)$.

Prior probabilities Initial estimates of the probabilities of events.

Posterior probabilities Revised probabilities of events based on additional information.

Bayes' theorem A method used to compute posterior probabilities.

Key Formulas

Counting Rule for Combinations

$$C_n^N = \binom{N}{n} = \frac{N!}{n!(N-n)!} \tag{4.1}$$

Counting Rule for Permutations

$$P_n^N = n!\binom{N}{n} = \frac{N!}{(N-n)!} \tag{4.2}$$

Computing Probability Using the Complement

$$P(A) = 1 - P(A^c) \tag{4.5}$$

Addition Law

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{4.6}$$

Conditional Probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{4.7}$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \tag{4.8}$$

Multiplication Law

$$P(A \cap B) = P(B)P(A \mid B) \tag{4.11}$$

$$P(A \cap B) = P(A)P(B \mid A) \tag{4.12}$$

Multiplication Law for Independent Events

$$P(A \cap B) = P(A)P(B) \tag{4.13}$$

Bayes' Theorem

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n)} \tag{4.19}$$

Supplementary Exercises

46. *The Wall Street Journal*/Harris Personal Finance poll asked 2082 adults if they owned a home (All Business website, January 23, 2008). A total of 1249 survey respondents answered Yes. Of the 450 respondents in the 18–34 age group, 117 responded Yes.
    a. What is the probability that a respondent to the poll owned a home?
    b. What is the probability that a respondent in the 18–34 age group owned a home?
    c. What is the probability that a respondent to the poll did not own a home?
    d. What is the probability that a respondent in the 18–34 age group did not own a home?
47. A financial manager made two new investments—one in the oil industry and one in municipal bonds. After a one-year period, each of the investments will be classified as either successful or unsuccessful. Consider the making of the two investments as an experiment.
    a. How many sample points exist for this experiment?
    b. Show a tree diagram and list the sample points.
    c. Let O = the event that the oil industry investment is successful and M = the event that the municipal bond investment is successful. List the sample points in O and in M.
    d. List the sample points in the union of the events $(O \cup M)$.
    e. List the sample points in the intersection of the events $(O \cap M)$.
    f. Are events O and M mutually exclusive? Explain.
48. In early 2003, President Bush proposed eliminating the taxation of dividends to shareholders on the grounds that it was double taxation. Corporations pay taxes on the earnings that are later paid out in dividends. In a poll of 671 Americans, TechnoMetrica Market Intelligence found that 47% favored the proposal, 44% opposed it, and 9% were not sure (*Investor's Business Daily,* January 13, 2003). In looking at the responses across party lines

the poll showed that 29% of Democrats were in favor, 64% of Republicans were in favor, and 48% of Independents were in favor.

a. How many of those polled favored elimination of the tax on dividends?
b. What is the conditional probability in favor of the proposal given the person polled is a Democrat?
c. Is party affiliation independent of whether one is in favor of the proposal?
d. If we assume people's responses were consistent with their own self-interest, which group do you believe will benefit most from passage of the proposal?

49. A study of 31,000 hospital admissions in New York State found that 4% of the admissions led to treatment-caused injuries. One-seventh of these treatment-caused injuries resulted in death, and one-fourth were caused by negligence. Malpractice claims were filed in one out of 7.5 cases involving negligence, and payments were made in one out of every two claims.

a. What is the probability a person admitted to the hospital will suffer a treatment-caused injury due to negligence?
b. What is the probability a person admitted to the hospital will die from a treatment-caused injury?
c. In the case of a negligent treatment-caused injury, what is the probability a malpractice claim will be paid?

50. A telephone survey to determine viewer response to a new television show obtained the following data.

Rating	Frequency
Poor	4
Below average	8
Average	11
Above average	14
Excellent	13

a. What is the probability that a randomly selected viewer will rate the new show as average or better?
b. What is the probability that a randomly selected viewer will rate the new show below average or worse?

51. The following crosstabulation shows household income by educational level of the head of household (*Statistical Abstract of the United States*, 2008).

	Household Income ($1000s)					
Education Level	**Under 25**	**25.0–49.9**	**50.0–74.9**	**75.0–99.9**	**100 or more**	**Total**
Not H.S. Graduate	4,207	3,459	1,389	539	367	9,961
H.S. Graduate	4,917	6,850	5,027	2,637	2,668	22,099
Some College	2,807	5,258	4,678	3,250	4,074	20,067
Bachelor's Degree	885	2,094	2,848	2,581	5,379	13,787
Beyond Bach. Deg.	290	829	1,274	1,241	4,188	7,822
Total	13,106	18,490	15,216	10,248	16,676	73,736

a. Develop a joint probability table.
b. What is the probability of a head of household not being a high school graduate?
c. What is the probability of a head of household having a bachelor's degree or more education?
d. What is the probability of a household headed by someone with a bachelor's degree earning $100,000 or more?

e. What is the probability of a household having income below $25,000?
f. What is the probability of a household headed by someone with a bachelor's degree earning less than $25,000?
g. Is household income independent of educational level?

52. An MBA new-matriculants survey provided the following data for 2018 students.

		Applied to More Than One School	
		Yes	**No**
Age Group	**23 and under**	207	201
	24–26	299	379
	27–30	185	268
	31–35	66	193
	36 and over	51	169

a. For a randomly selected MBA student, prepare a joint probability table for the experiment consisting of observing the student's age and whether the student applied to one or more schools.
b. What is the probability that a randomly selected applicant is 23 or under?
c. What is the probability that a randomly selected applicant is older than 26?
d. What is the probability that a randomly selected applicant applied to more than one school?

53. Refer again to the data from the MBA new-matriculants survey in exercise 52.
a. Given that a person applied to more than one school, what is the probability that the person is 24–26 years old?
b. Given that a person is in the 36-and-over age group, what is the probability that the person applied to more than one school?
c. What is the probability that a person is 24–26 years old or applied to more than one school?
d. Suppose a person is known to have applied to only one school. What is the probability that the person is 31 or more years old?
e. Is the number of schools applied to independent of age? Explain.

54. An IBD/TIPP poll conducted to learn about attitudes toward investment and retirement (*Investor's Business Daily,* May 5, 2000) asked male and female respondents how important they felt level of risk was in choosing a retirement investment. The following joint probability table was constructed from the data provided. "Important" means the respondent said level of risk was either important or very important.

	Male	**Female**	**Total**
Important	.22	.27	.49
Not Important	.28	.23	.51
Total	.50	.50	1.00

a. What is the probability a survey respondent will say level of risk is important?
b. What is the probability a male respondent will say level of risk is important?
c. What is the probability a female respondent will say level of risk is important?
d. Is the level of risk independent of the gender of the respondent? Why or why not?
e. Do male and female attitudes toward risk differ?

55. A large consumer goods company ran a television advertisement for one of its soap products. On the basis of a survey that was conducted, probabilities were assigned to the following events.

B = individual purchased the product
S = individual recalls seeing the advertisement
$B \cap S$ = individual purchased the product and recalls seeing the advertisement

The probabilities assigned were $P(B) = .20$, $P(S) = .40$, and $P(B \cap S) = .12$.

a. What is the probability of an individual's purchasing the product given that the individual recalls seeing the advertisement? Does seeing the advertisement increase the probability that the individual will purchase the product? As a decision maker, would you recommend continuing the advertisement (assuming that the cost is reasonable)?
b. Assume that individuals who do not purchase the company's soap product buy from its competitors. What would be your estimate of the company's market share? Would you expect that continuing the advertisement will increase the company's market share? Why or why not?
c. The company also tested another advertisement and assigned it values of $P(S) = .30$ and $P(B \cap S) = .10$. What is $P(B \mid S)$ for this other advertisement? Which advertisement seems to have had the bigger effect on customer purchases?

56. Cooper Realty is a small real estate company located in Albany, New York, specializing primarily in residential listings. They recently became interested in determining the likelihood of one of their listings being sold within a certain number of days. An analysis of company sales of 800 homes in previous years produced the following data.

		Days Listed Until Sold			
		Under 30	**31–90**	**Over 90**	**Total**
Initial Asking Price	**Under $150,000**	50	40	10	100
	$150,000–$199,999	20	150	80	250
	$200,000–$250,000	20	280	100	400
	Over $250,000	10	30	10	50
	Total	100	500	200	800

a. If A is defined as the event that a home is listed for more than 90 days before being sold, estimate the probability of A.
b. If B is defined as the event that the initial asking price is under $150,000, estimate the probability of B.
c. What is the probability of $A \cap B$?
d. Assuming that a contract was just signed to list a home with an initial asking price of less than $150,000, what is the probability that the home will take Cooper Realty more than 90 days to sell?
e. Are events A and B independent?

57. A company studied the number of lost-time accidents occurring at its Brownsville, Texas, plant. Historical records show that 6% of the employees suffered lost-time accidents last year. Management believes that a special safety program will reduce such accidents to 5% during the current year. In addition, it estimates that 15% of employees who had lost-time accidents last year will experience a lost-time accident during the current year.

a. What percentage of the employees will experience lost-time accidents in both years?
b. What percentage of the employees will suffer at least one lost-time accident over the two-year period?

58. A survey showed that 8% of Internet users age 18 and older report keeping a blog. Referring to the 18–29 age group as young adults, the survey showed that for bloggers 54% are young adults and for nonbloggers 24% are young adults (Pew Internet & American Life Project, July 19, 2006).
    a. Develop a joint probability table for these data with two rows (bloggers vs. non-bloggers) and two columns (young adults vs. older adults).
    b. What is the probability that an Internet user is a young adult?
    c. What is the probability that an Internet user keeps a blog and is a young adult?
    d. Suppose that in a follow-up phone survey we contact someone who is 24 years old. What is the probability that this person keeps a blog?
59. An oil company purchased an option on land in Alaska. Preliminary geologic studies assigned the following prior probabilities.

$$P(\text{high-quality oil}) = .50$$
$$P(\text{medium-quality oil}) = .20$$
$$P(\text{no oil}) = .30$$

    a. What is the probability of finding oil?
    b. After 200 feet of drilling on the first well, a soil test is taken. The probabilities of finding the particular type of soil identified by the test follow.

$$P(\text{soil} \mid \text{high-quality oil}) = .20$$
$$P(\text{soil} \mid \text{medium-quality oil}) = .80$$
$$P(\text{soil} \mid \text{no oil}) = .20$$

    How should the firm interpret the soil test? What are the revised probabilities, and what is the new probability of finding oil?
60. Companies that do business over the Internet can often obtain probability information about website visitors from previous websites visited. The article "Internet Marketing" (*Interfaces,* March/April 2001) described how clickstream data on websites visited could be used in conjunction with a Bayesian updating scheme to determine the gender of a website visitor. Par Fore created a website to market golf equipment and apparel. Management would like a certain offer to appear for female visitors and a different offer to appear for male visitors. From a sample of past website visits, management learned that 60% of the visitors to the website ParFore are male and 40% are female.
    a. What is the prior probability that the next visitor to the website will be female?
    b. Suppose you know that the current visitor to the website ParFore previously visited the Dillard's website, and that women are three times as likely to visit the Dillard's website as men. What is the revised probability that the current visitor to the website ParFore is female? Should you display the offer that appeals more to female visitors or the one that appeals more to male visitors?

Case Problem Hamilton County Judges

Hamilton County judges try thousands of cases per year. In an overwhelming majority of the cases disposed, the verdict stands as rendered. However, some cases are appealed, and of those appealed, some of the cases are reversed. Kristen DelGuzzi of *The Cincinnati Enquirer* conducted a study of cases handled by Hamilton County judges over a three-year period. Shown in Table 4.8 are the results for 182,908 cases handled (disposed) by 38 judges in Common Pleas Court, Domestic Relations Court, and Municipal Court. Two of the judges (Dinkelacker and Hogan) did not serve in the same court for the entire three-year period.

TABLE 4.8 TOTAL CASES DISPOSED, APPEALED, AND REVERSED IN HAMILTON COUNTY COURTS

WEB file
Judge

Common Pleas Court

Judge	Total Cases Disposed	Appealed Cases	Reversed Cases
Fred Cartolano	3,037	137	12
Thomas Crush	3,372	119	10
Patrick Dinkelacker	1,258	44	8
Timothy Hogan	1,954	60	7
Robert Kraft	3,138	127	7
William Mathews	2,264	91	18
William Morrissey	3,032	121	22
Norbert Nadel	2,959	131	20
Arthur Ney, Jr.	3,219	125	14
Richard Niehaus	3,353	137	16
Thomas Nurre	3,000	121	6
John O'Connor	2,969	129	12
Robert Ruehlman	3,205	145	18
J. Howard Sundermann	955	60	10
Ann Marie Tracey	3,141	127	13
Ralph Winkler	3,089	88	6
Total	43,945	1762	199

Domestic Relations Court

Judge	Total Cases Disposed	Appealed Cases	Reversed Cases
Penelope Cunningham	2,729	7	1
Patrick Dinkelacker	6,001	19	4
Deborah Gaines	8,799	48	9
Ronald Panioto	12,970	32	3
Total	30,499	106	17

Municipal Court

Judge	Total Cases Disposed	Appealed Cases	Reversed Cases
Mike Allen	6,149	43	4
Nadine Allen	7,812	34	6
Timothy Black	7,954	41	6
David Davis	7,736	43	5
Leslie Isaiah Gaines	5,282	35	13
Karla Grady	5,253	6	0
Deidra Hair	2,532	5	0
Dennis Helmick	7,900	29	5
Timothy Hogan	2,308	13	2
James Patrick Kenney	2,798	6	1
Joseph Luebbers	4,698	25	8
William Mallory	8,277	38	9
Melba Marsh	8,219	34	7
Beth Mattingly	2,971	13	1
Albert Mestemaker	4,975	28	9
Mark Painter	2,239	7	3
Jack Rosen	7,790	41	13
Mark Schweikert	5,403	33	6
David Stockdale	5,371	22	4
John A. West	2,797	4	2
Total	108,464	500	104

The purpose of the newspaper's study was to evaluate the performance of the judges. Appeals are often the result of mistakes made by judges, and the newspaper wanted to know which judges were doing a good job and which were making too many mistakes. You are called in to assist in the data analysis. Use your knowledge of probability and conditional probability to help with the ranking of the judges. You also may be able to analyze the likelihood of appeal and reversal for cases handled by different courts.

Managerial Report

Prepare a report with your rankings of the judges. Also, include an analysis of the likelihood of appeal and case reversal in the three courts. At a minimum, your report should include the following:

1. The probability of cases being appealed and reversed in the three different courts.
2. The probability of a case being appealed for each judge.
3. The probability of a case being reversed for each judge.
4. The probability of reversal given an appeal for each judge.
5. Rank the judges within each court. State the criteria you used and provide a rationale for your choice.