

CHAPTER 19

Default Correlation Pricing and Trading

1. Introduction

There are three major issues with the credit sector. First there is the understanding and engineering of the credit risk itself. In other words, how does one strip the default risk component of a bond or a loan and trade it separately? The engineering of a credit default swap (CDS) serves this purpose, and was done in Chapter 5.

The second dimension in studying credit risk is the modeling aspect. Without modeling one cannot implement pricing, hedging, and risk management problems. Modeling helps to go from descriptive or graphical discussion to numbers. In credit risk the modeling has a “novel” component. The risk in question is an *event*, the default. They are zero-one type random variables and are different than risk factors such as interest rates and stock prices which can be approximated by continuous state stochastic processes.¹ Credit risk, being a zero-one event, introduces a new dimensions in modeling.

The third major dimension of the credit sector has to do with the *tranching* of default risks. The main point of Chapter 18 was that tranching credit risk can eventually lead to stripping the *default correlation*. In this chapter we talk about modeling default correlation and the resulting financial engineering of default correlation. We learn how to strip, hedge, and trade it. The recent correlation market provides the real-world background.

The connection between default correlation and tranche values is a complex one and needs to be clarified step by step. We discuss market examples of how this idea can be exploited in setting up *correlation trades* and can be exploited in setting up new structured products. The first issue that we need to introduce is the dependence of the tranche pricing on the default correlation.

¹ Remember that real life indicators are actually not continuous state random variables. Instead the state space is countable. Markets have conventions in terms of decimal points. For example, the EUR/USD rate is quoted up to four decimal points. Thus the minimum tick is .0001 a pips. In a typical trading day the total number of plausible true states of the world is no more than, say, 200–300.

2. Some History

During the 1990s banks moved into securitization of the asset. Loans from credit cards, mortgages, equity, and cars were packaged and sold to investors. A typical strategy was to (1) buy the loan from an *originator*, (2) warehouse it in the bank while the cash flows stabilized and their credit quality was established, (3) hedge the credit exposure during this warehousing period, and (4) finally sell the loan to the investor in the form of an asset backed security (ABS).

This process then was extended to collateralized debt obligations (CDO), collateralized loan obligations (CLO), and their variants. In packaging bonds, mortgages, loans, and ABS securities into CDO-type instruments, banks went through the following practice. The banks decided to keep the *first loss piece*, called the *equity tranche*. For example, the bank took responsibility for the first, say, 3% of the defaults during the CDO maturity. This introduced some *subordination* to the securities sold. The responsibility for the next tranche, the mezzanine took the risk of the defaults between, say, 3% and 6% of the defaults in the underlying portfolio. This was less risky than the equity tranche and the paper could be rated, say, BBB. Institutional investors who are prevented by law to invest in speculative grade securities could then buy the mezzanine tranche and, indirectly, the underlying loans even if the underlying debt was speculative grade.²

Then, the very high quality tranches, called *senior* and *super senior*, were also kept on banks' books because their implied return was too small for many investors.³ As a result, the banks found themselves *long equity tranche* and *long senior and super senior tranches*. They were short the *mezzanine* tranche which paid a good return and was rated investment grade. It turns out, as we will see in this chapter, that the equity tranche value is *positively* affected by the default correlation while the super senior tranche value is *negatively* affected. The sum of these positions formed the correlation books of the banks. Banks had to learn *correlation trading, hedging, and pricing* as a result.

As discussed in the previous chapter, the bespoke tranches evolved later into standard tranches on the credit indices. The *equity tranche* was the first loss piece. A protection seller on the equity tranche bears the risk of the first 0–3% of defaults in *standard tranches*. The *mezzanine tranche* bears the second highest risk. A protection seller on the tranche is responsible, by convention, for 3–6% of the defaults.⁴ The 6–9% tranche is called senior mezzanine. The *senior* and *super senior* iTraxx tranches bear the default risk of 9–12% and 12–22% of names, respectively. In this chapter we study the pricing and the engineering of such tranches.

3. Two Simple Examples

We first discuss two simple cases to illustrate the logic of how default correlation movements affect tranche prices. It is through this logic that observed tranche trading can be used to back out the default correlation. This quantity will be called *implied correlation*.

Let $n = 3$ so that there are only three credit names in the portfolio. With such a portfolio we can consider only three tranches: equity, mezzanine, and senior. In this simple example, the

² This is interesting and we would like to give an example of it. Take 100 bonds all rated B. This is a fairly speculative rating and many institutions by law are not allowed to hold such bonds. Yet, suppose we sell the risk of the first 10% of the defaults to some hedge fund, at which time the default risk on the remaining bonds may in fact become A. Institutional investors can then buy this risk. Hence, credit enhancement made it possible to sell paper that originally was very risky.

³ Super senior tranche pays a spread in the range of 6–40 basis points.

⁴ For the CDX index in the United States the Mezzanine attachment points are 3–7%.

equity tranche bears the risk of the first default (0–33%), the mezzanine tranche bears the risk of the second name defaulting (33–66%), and the senior tranche investors bear the default risk of the third default (66–100%).

In general we follow the same notation as in the previous chapter. As a new parameter, we let the $\rho_t^{i,j}$ be the default correlation between i th and j th names in the portfolio at time t . The p_t^i is the probability of default and the c_t^i is the liquid CDS spread for the i th name, respectively. We make the following assumptions without any loss of generality. The default probabilities for the period $[t_0, T]$ are the same for all names, and they are constant

$$p_t^1 = p_t^2 = p_t^3 = p \quad \text{For all } t \in [t_0, T] \quad (1)$$

We also let the recovery rate be constant and be given by R .

We consider two extreme settings. In the first, default correlation is zero

$$\rho^{i,j} = 0 \quad (2)$$

The second is perfect correlation,

$$\rho^{i,j} = 1 \quad (3)$$

for all $t \in [t_0, T]$, i, j . These two extreme cases will convey the basic idea involved in correlation trading. We study a number of important concepts under these two assumptions. In particular, we obtain the (1) default loss distributions, (2) default correlations, and (3) tranche pricing in this context.

Start with the independence case. In general, with n credit names, the probability distribution of D will be given by the binomial probability distribution. Letting p denote the constant probability of default for each name *and* assuming that defaults are independent, the number of defaults during a period $[t_0, T]$ will be distributed as

$$P(D = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \quad (4)$$

Note that there is no ρ parameter in this distribution since the correlation is zero. Now consider the first numerical example.

3.0.1. Case 1: Independence

With $n = 3$ there are only four possibilities, $D = \{0, 1, 2, 3\}$. For zero default, $D = 0$ we have,

$$P(D = 0) = (1-p)^3 \quad (5)$$

For the remaining probabilities we obtain

$$P(D = 1) = p(1-p)^2 + (1-p)p(1-p) + (1-p)^2 p \quad (6)$$

$$P(D = 2) = p^2(1-p) + p(1-p)p + (1-p)p^2 \quad (7)$$

$$P(D = 3) = p^3 \quad (7)$$

Suppose $p = .05$. Plugging in the formulas above we obtain first the probability that no default occurs,

$$P(D = 0) = (1 - .05)^3 = 0.857375 \quad (8)$$

One default can occur in three different ways:

$$P(D = 1) = (.05)(1 - .05)(1 - .05) + (1 - .05)(.05)(1 - .05) + (1 - .05)(1 - .05)(.05) = 0.135375 \quad (9)$$

Two defaults can occur again in three ways:

$$P(D = 2) = (.05)(.05)(1 - .05) + (.05)(1 - .05)(.05) + (1 - .05)(.05)(.05) = 0.007125 \quad (10)$$

For three defaults there is only one possibility and the probability is:

$$P(D = 3) = (.05)(.05)(.05) = 0.000125 \quad (11)$$

As required, these probabilities sum to one.

The spreads associated with each tranche can be obtained easily from these numbers. Assume for simplicity that defaults occur only at the end of the year. In order to calculate the tranche spreads we first calculate the expected loss for each tranche under a proper working probability. Then the spread is set so that expected cash inflows equal this expected loss. The expected loss for the three tranches are given by

$$\text{Equity} = 0[P(D = 0)] + 1[P(D = 1) + P(D = 2) + P(D = 3)] = .142625 \quad (12)$$

$$\text{Mez} = 0[P(D = 0) + P(D = 1)] + 1[P(D = 2) + P(D = 3)] = .00725 \quad (13)$$

$$\text{Sen} = 0[P(D = 0) + P(D = 1) + P(D = 2)] + 1[P(D = 3)] = .000125 \quad (14)$$

In the case of one-year maturity contracts it is easy to generalize these tranche spread calculations. Let $B(t, T)$ denote the appropriate time- t discount factor for \$1 to be received at time T . Assuming that $N = 1$ and that defaults can occur only on date T , the tranche spreads at time t_0 denoted by $c_{t_0}^j$, $j = e, m, s$ are then given by the following equation,

$$0 = B(t_0, T) [c_{t_0}^e P(D = 0) - [(1 - R) - c_{t_0}^e] P(D \geq 1)] \quad (15)$$

for the equity tranche which is hit with the first default. This can be written as

$$c_{t_0}^e = \frac{P(D = 1) + P(D = 2) + P(D = 3)}{(1 - R)} \quad (16)$$

For the mezzanine tranche we have,

$$0 = B(t_0, T) [c_{t_0}^m (P(D = 0) + P(D = 1)) - [(1 - R) - C_{t_0}^m] (P(D = 2) + P(D = 3))] \quad (17)$$

which gives

$$c_{t_0}^m = \frac{P(D = 2) + P(D = 3)}{(1 - R)} \quad (18)$$

Finally, for the senior tranche we obtain,

$$c_{t_0}^s = \frac{P(D = 3)}{(1 - R)} \quad (19)$$

Note that, in general, with $n > 3$ and the number of tranches being less than n , there will be more than one possible value for R_T . One can obtain numerical values for these spreads by plugging in $p = .05$ and the recovery value $R = 40\%$.

It is interesting to note that for each tranche, the relationship between *spreads* and *probabilities* of making floating payments is similar to the relation between c and p we obtained for a single name CDS,

$$c = \frac{p}{(1 - R)} \quad (20)$$

3.0.2. Case 2: Perfect Correlation

If default correlation increases and $\rho \rightarrow 1$ then all credit names become essentially the same, restricting default probability to be identical. So let

$$p_t^i = .05 \quad (21)$$

for all names and all times $t \in [t_0, T]$. Under these conditions the probability distribution for D will be trivially given by the distribution.

$$P(D = 0) = (1 - .05) \quad (22)$$

$$P(D = 1) \rightarrow 0 \quad (23)$$

$$P(D = 2) \rightarrow 0 \quad (24)$$

$$P(D = 3) = .05 \quad (25)$$

The corresponding expected losses on each tranche can be calculated trivially. For the equity tranche we have, after canceling the $B(t_0, T)$:

$$0 = c_{t_0}^e P(D = 0) - [(1 - R) - c_{t_0}^e](1 - P(D = 0)) \quad (26)$$

The mezzanine tranche spread will be

$$0 = c_{t_0}^m P(D = 0) - [(1 - R) - c_{t_0}^m](1 - P(D = 0)) \quad (27)$$

Finally for the senior tranche we have

$$0 = c_{t_0}^s P(D = 0) - [(1 - R) - c_{t_0}^s](1 - P(D = 0)) \quad (28)$$

We can extract some general conclusions from these two examples. First of all, as $\rho \rightarrow 1$, all three tranche spreads become similar. This is to be expected since under these conditions all names start looking more and more alike. At the limit $\rho = 1$ there are only two possibilities, either nobody defaults, or everybody defaults. There is no risk to “tranche” and sell separately. There is only *one* risk.

Second, we see that as correlation goes from zero to one, the expected loss of equity tranche decreases. The expected loss of the senior tranche, on the other hand, goes up. The mezzanine tranche is somewhere in between: the expected loss goes up as correlation increases, but not as much as the senior tranche.

Finally, note that as default correlation went up the distribution became more skewed with the “two” tails becoming heavier at both ends.

4. The Model

We now show how the distribution of D can be calculated under correlations different than zero and one. We discuss the market model for pricing standard tranches for a portfolio of n names. This model is equivalent to the so-called Gaussian copula model, in a one factor setting.

Let

$$\{S^j\}, \quad j = 1, \dots, n \quad (29)$$

be a sequence of latent variables. Their role is to generate *statistically dependent* zero-one variables.⁵ There is no model of a random variable that can generate dependent zero-one random

⁵ The previous example dealt with an independent default case. Default is a zero-one variable, which made the random variable D , representing total number of defaults during $[t_0, T]$, follow a standard binomial distribution.

variables with a *closed form* density or distribution function. The $\{S^j\}$ are used in a Monte Carlo approach to do this.

It is assumed that the S^j follows a normal distribution and that the default of the i th name occurs the first time S^j falls below a threshold denoted by L_α , where the a is the j th name's default probability. The important step is the way the S^j is structured. Consider the following *one factor* case,

$$S^j = \rho^j F + \sqrt{1 - (\rho^j)^2} \epsilon^j \quad (30)$$

where F is a *common* latent variable independent of ϵ^j , ρ^j is a constant parameter, and ϵ^j is the *idiosyncratic* component. The random variables in this setup have some special characteristics. The F has no superscript, so it is common to all S^j , $j = 1, \dots, n$ and it has a standard normal distribution. The ϵ^j are specific to each i and are also distributed as standard normal,

$$\epsilon^j \sim N(0, 1) \quad (31)$$

$$F \sim N(0, 1) \quad (32)$$

Finally, the common factor and the idiosyncratic components are uncorrelated.

$$E[\epsilon^j F] = 0 \quad (33)$$

It turns out that the ϵ^j and the F may have any desired distribution.⁶ If this distribution is assumed to be normal, then the model described becomes similar to a Gaussian copula model. Note this case is in fact a one factor latent variable model.

We obtain the *mean* and *variance* of a typical S^j as follows,

$$\begin{aligned} E[S^j] &= \rho E[F] + E[\sqrt{1 - \rho^2} \epsilon^j] \\ &= 0 \end{aligned} \quad (34)$$

and

$$\begin{aligned} E[(S^j)^2] &= \rho^2 E[F^2] + E[(1 - \rho^2)(\epsilon^j)^2] \\ &= 1 + \rho^2 - \rho^2 = 1 \end{aligned} \quad (35)$$

since both random variables on the right side have zero mean and unit second moment by assumption and since the F is uncorrelated with ϵ^j . The model above has an important characteristic that we will use in stripping default correlation. It turns out that the correlation between defaults can be conveniently modeled using the common factor variable and the associated ρ^i . Calculate the correlation

$$\begin{aligned} E[S^i S^k] &= E[\rho F + \sqrt{1 - \rho^2} \epsilon^i][\rho F + \sqrt{1 - \rho^2} \epsilon^k] \\ &= \rho^2 E[F^2] + E[2(1 - \rho^2) \epsilon^i \epsilon^k] + E[\rho \sqrt{(1 - \rho^2)} \epsilon^i F] + E[\rho \sqrt{(1 - \rho^2)} \epsilon^k F] \end{aligned} \quad (36)$$

This gives

$$\text{Corr}[S^i S^j] = \rho^i \rho^j \quad (37)$$

⁶ One caveat here is the following. The sum of two normal distributions is normal; yet, the sum of two arbitrary distributions may not necessarily belong to the same family. In fact, any closed form distribution to model such a sum may not exist. One example is the sum of a normally distributed variable and a student's t distribution variable which cannot be represented by a closed form distribution formula. Such exercises require the use of Monte Carlo and will generate the distributions numerically.

The case where

$$\rho^i = \rho^j \quad (38)$$

for all i, j , is called the *compound correlation* and is the market convention. The compound default correlation coefficients is then given by ρ^2 .

The i th name default probability is defined as

$$p^i = P\left(\left(\rho F + \sqrt{1 - \rho^2} \epsilon^i\right) \leq L_\alpha\right) \quad (39)$$

The probability is that the value of S^i will fall below the level L_α .⁷ This setup provides an agenda one can follow to price and hedge the tranches. It will also help us to back out an implied default correlation once we observe liquid tranche spreads in the market.

The agenda is as follows: First, observe the CDS rate c_t^i for each name in the markets. Using this, calculate the risk-adjusted default probability p^i . Using this find the corresponding L_α^i . Generate pseudo-random numbers for the F and the ϵ^i . Next, assume a value for ρ and calculate the implied S^i . Check to see if the value of S^i obtained this way is less than L_α^i . This way, obtain a simulated default processes:

$$d^i = \begin{cases} 1 & \text{if } S^i < L_\alpha \\ 0 & \text{otherwise} \end{cases} \quad (40)$$

Finally calculate the number of defaults for the trial as

$$D = \sum_{i=1}^n d^i \quad (41)$$

Repeating this procedure m times will yield m replicas of the random variable D . If m is large, we can use the resulting histogram as the default loss distribution on the n reference names and then calculate the spreads for each tranche. This distribution will depend on a certain level of default correlation due to the choice of ρ . The *implied correlation* is that level of ρ which yields a calculated tranche spread that equals the observed spread in the markets. Below we have a simple example that shows this process.

EXAMPLE:

Let $n = 3$. Assume that the default probability is 1%; and let the default correlation be $\rho^2 = .36$. We generate 10,000 replicas for each $\{S^1, S^2, S^3\}$ using

$$S^1 = .25F + \sqrt{64}\epsilon^1 \quad (42)$$

$$S^2 = .25F + \sqrt{64}\epsilon^2 \quad (43)$$

$$S^3 = .25F + \sqrt{64}\epsilon^3 \quad (44)$$

For example, we select four standard pseudo-random variables

$$F = -1.12615 \quad (45)$$

$$\epsilon^1 = -2.17236 \quad (46)$$

$$\epsilon^2 = 0.64374 \quad (47)$$

$$\epsilon^3 = -0.326163 \quad (48)$$

⁷ Merton (1976) has a structural credit model where the default occurs when the value of the firm falls below the debt issued by the firm. Hence, at the outset it appears that the market convention is similar to Merton's model. This is somewhat misleading, since the S^i has no structural interpretation here. It will be mainly used to generate correlated binomial variables.

Using these we obtain the S^i as

$$S^1 = -2.41358 \quad (49)$$

$$S^2 = -0.160698 \quad (50)$$

$$S^3 = -0.936621 \quad (51)$$

We then calculate the corresponding S^j and see if they are less than $L_{.01} = -2.32$, where the latter is the threshold that gives a 1% tail probability in a standard normal distribution.

If $S^1 < -2.32$, $S^2 < -2.32$, $S^3 < -2.32$, then we let the corresponding $d^i = 1$. Otherwise they are zero. We then add the three d^i to get the D for that Monte Carlo run.

In this particular case $d^1 = 1$, $d^2 = 0$, $d^3 = 0$. So the first Monte Carlo run gives $D = 1$.

Next we would like to show an example dealing with implied correlation calculations. The example again starts with a Monte Carlo sample on the D , obtains the tranche spreads, then extends this to three functions, calculated numerically, that show the mapping between tranche spreads and correlation.

EXAMPLE: Implied Correlation

Suppose we are given a Monte Carlo sample of correlated defaults from a reference portfolio,

$$\text{Sample} = \{D_1, \dots, D_m\} \quad (52)$$

Then, we can obtain the histogram of the number of defaults.

Assume $\rho = .3$, $n = 100$, and $m = 1,000$. Running the procedure above we obtain 1,000 replicas of D_i . We can do at least two analyses with the distribution of D . First we can calculate the fair value of the tranches of interest. For example, with recovery 40% and zero interest rates, we can compute the value of the equity tranche in this case as

$$c_e = [.05(0) + .12(33.33).6 + .15(66).6 + (1 - .05 - .12 - .15)].6 = .68 \quad (53)$$

This one-year spread is based on the fact that the party that sells protection with nominal $N = 100$ on the first 3 defaults will lose nothing if there are no defaults, will lose a third of the investment if there is one default, will lose two-thirds if there are two defaults, and finally will lose all investments if there are three defaults.

Repeating this for all values of $\rho^2 \in [0, 1]$ we can get three surfaces. These graphs plot the value of the equity, mezzanine, and senior tranches against the fair value of the tranche.

Note that the value of the equity tranche goes up as correlation increases. The value of the mezzanine tranche is a U-shaped curve, whereas the value of the senior tranche is again monotonic.

4.1. The Central Limit effect

Why does the default loss distribution change as a function of the correlation? This is an important technical problem that has to do with the central limit theorem and the assumptions behind it. Now define the *correlated* zero-one stochastic process z_i :

$$z_j = \begin{cases} 1 & \text{If } S^j \leq L_\alpha \\ 0 & \text{Otherwise} \end{cases} \quad (54)$$

Next calculate a *sum* of these random variables as

$$Z^n = \sum_{j=1}^n z^j \quad (55)$$

According to the central limit theorem, even if the z^j are individually very far from being normally distributed *and* are correlated, then the distribution of the sum will approach a normal distribution if the underlying processes have finite means and variances as in $n \rightarrow \infty$.

$$\frac{\sum_i^n z_i - \sum_i^n \mu_i}{\sqrt{\sum_i^n \sigma_i^2}} \rightarrow N(\mu^d, \sigma^d) \quad (56)$$

Thus if we had a very high number of names in the reference portfolio the distribution of D will look like normal. On the other hand, with finite n *and* highly correlated z^j , this convergence effect will be slow. The higher the “correlation” ρ , the slower will the convergence be. In particular, with n around 100, the distribution of the D will be heavily dependent on the size of ρ and will be far from normal. This is where the relationship between Index tranches and correlation comes in. In the extreme case when correlation is perfect, the *sum* will involve the *same* z^j .

5. Default Correlation and Trading

Correlation impacts risk assessment and valuation of CDO tranches. Each tranche value reflects correlation in a different way. Perhaps the most interesting correlation relationship shows in the equity (0–3%) tranche. This may first appear counterintuitive. It turns out that the higher the correlation, the lower the equity tranche risk, and the higher the value of the tranche. The reasoning behind this is as follows. Higher correlation makes extreme cases of very few defaults more likely. Everything else being the same, the more correlation, the lower the risk the investor takes on and the lower premium the investor is going to receive over the lifetime of the tranche.⁸

The influence of default correlation on the mezzanine tranche is not as clear. In fact, the value of the mezzanine tranche is less sensitive to default correlation. For the senior tranches, higher correlation of default implies a higher probability that losses will wipe out the equity and mezzanine tranches and inflict losses on the senior tranche as well. Thus as default correlation rises, the value of the senior tranche falls. A similar reasoning applies to the super senior tranche.

Correlation trading is based on this different dependence of the tranche spreads on default correlation. One of the most popular trades of 2004 and the first half of 2005 was to sell the equity tranche and hedge the default probability movements (i.e., the market risk) by going long the mezzanine index.

This is a *long correlation* trade and it has significant positive carry.⁹ The trade also has significant (positive) convexity exposure. In fact, as the position holder adjusts the *delta* hedge, the hedging gains would lead to a gain directly tied to index volatility. Finally, if the carry and convexity are higher, the position’s exposure to correlation changes will be higher.

A long correlation position has two major risks that are in fact related. First, if a single name credit event occurs, long correlation positions would realize a loss. This loss will depend on the

⁸ Note that when we talk about lower correlation, we keep the default probabilities the same. The sensitivity to correlation changes is conditional on this assumption. Otherwise, when default probabilities increase, all tranches would lose value.

⁹ Around 300 bps on the average during 2005–2007.

way the position is structured and will be around 5 to 15%. The recovery value of the defaulted bonds will also affect this number.

Second, a change in correlation will lead to mark-to-market gains and losses. In fact, the change in correlation is equivalent to markets changing their view on an idiosyncratic event happening. It is generally believed that a 100 basis point drop in correlation will lead to a change in the value of a *delta*-hedged equity tranche by around 1%. For short equity tranche protection, long mezzanine tranche protection position this loss would be even larger. This means that the position may suffer significant mark-to-market losses if expected correlation declines. Sometimes these positions need to be liquidated.

6. Delta Hedging and Correlation Trading

The *delta* is the sensitivity of the individual tranche spreads toward movements in the underlying index, I_t .¹⁰

Let the tranche spreads at time t be denoted by c_t^j , $j = e, m, s, sup$ as before. The superscript represents the equity, mezzanine, senior, and super senior tranches, respectively. There will then be (at least) two variables affecting the tranche spread: the probability of default and the default correlation. Let the average probability of default be denoted by p_t and the compound default correlation be ρ_t . We can write the index spread as a function of these two variables.

$$c^i = f^i(p_t, \rho_t) \quad (57)$$

It is important to remember that with changes in p_t the *sign* of the sensitivity is the same for *all* tranches. As probability of default goes up, the tranche spreads will all go up, albeit in different degrees. The sensitivities with respect to the index (or probability of default) will be given by:

$$\Delta^i = \frac{\partial c_t^i}{\partial I_t} > 0 \quad (58)$$

for all i . These constitute the tranche *deltas* with respect to the index itself. It is natural, given the level of subordination in higher seniority tranches, to find that

$$\Delta^e > \Delta^m > \Delta^s > \Delta^{sup} \quad (59)$$

Yet, the *correlation sensitivity* of the tranches are very different. As discussed earlier, even the sign changes.

$$\frac{\partial c^e}{\partial \rho} < 0 \quad (60)$$

$$\frac{\partial c^m}{\partial \rho} \cong 0 \quad (61)$$

$$\frac{\partial c^s}{\partial \rho} > 0 \quad (62)$$

$$\frac{\partial c^{sup}}{\partial \rho} > 0 \quad (63)$$

¹⁰ *Deltas* can be calculated with respect to each other and reported individually. But the procedure outlined below comes to the same thing. We report the *deltas* with respect to the index for two reasons. First, this is what the market reports, and second, the positions are hedged with respect to the index and not by buying and selling equity or mezzanine tranches with direct hedging.

According to this, the equity protection seller benefits if the compound default correlation goes up. The super senior protection seller loses under these circumstances. The middle tranches are more or less neutral.

Suppose the market participant desires to take a position positively responsive to ρ_t but more or less neutral toward changes in p_t . This is clearly a *long correlation* exposure discussed earlier. How would one put such a correlation trade on in practice? To do this the market practitioner would use the *delta* of the tranches with respect to the index. The position will consist of *two* hedging efforts. First, a right amount of the equity and mezzanine tranches have to be purchased so that the portfolio is immune to changes in the default correlation. Second, each tranche should be hedged separately with respect to the changes in the index, as time passes.¹¹

Let N^e and N^m be the two notional amounts. N^e is exposure on equity, while N^m is the exposure on the mezzanine tranche. What we want is a change in the index to not lead to a change in the total value of the position on these two tranches.

Thus the portfolio P_t will consist of selling default protection by the new amount

$$P_t = N^e - N^m \quad (64)$$

we let

$$N^m = \lambda N^e \quad (65)$$

where the λ is the hedge ratio to be selected. Substituting we obtain,

$$P_t = N^e - \lambda_t N^e \quad (66)$$

The λ_t is the hedge ratio selected as,

$$\lambda_t = \frac{\Delta_t^e}{\Delta_t^m} \quad (67)$$

With this selection the portfolio value will be immune to changes in the index value at time t :

$$\frac{\partial}{\partial I_t} P_t = \frac{\partial}{\partial I_t} N^e - \frac{\partial}{\partial I_t} N^m \quad (68)$$

Replacing from equation (67)

$$\frac{\partial}{\partial I_t} P_t = \Delta^e - \frac{\Delta^e}{\Delta^m} \Delta^m = 0. \quad (69)$$

Hence the portfolio of selling N^E units of equity protection and buying simultaneously λN^E units of the mezzanine protection is *delta hedged*. As the underlying index moves, the portfolio would be neutral to the first order of approximation. Also, as the market moves, the hedge ratio λ_t would change and the *delta* hedge would need to be adjusted. This means that there will be *gamma* gains (losses). Figures 19-1 and 19-2 show how to take these positions.

¹¹ *Deltas* can be calculated with respect to changes in other tranche spreads as well. But the market reports the *deltas* with respect to the index mainly because tranche positions are hedged with respect to the index and not by buying and selling equity or mezzanine tranches and doing direct hedging.

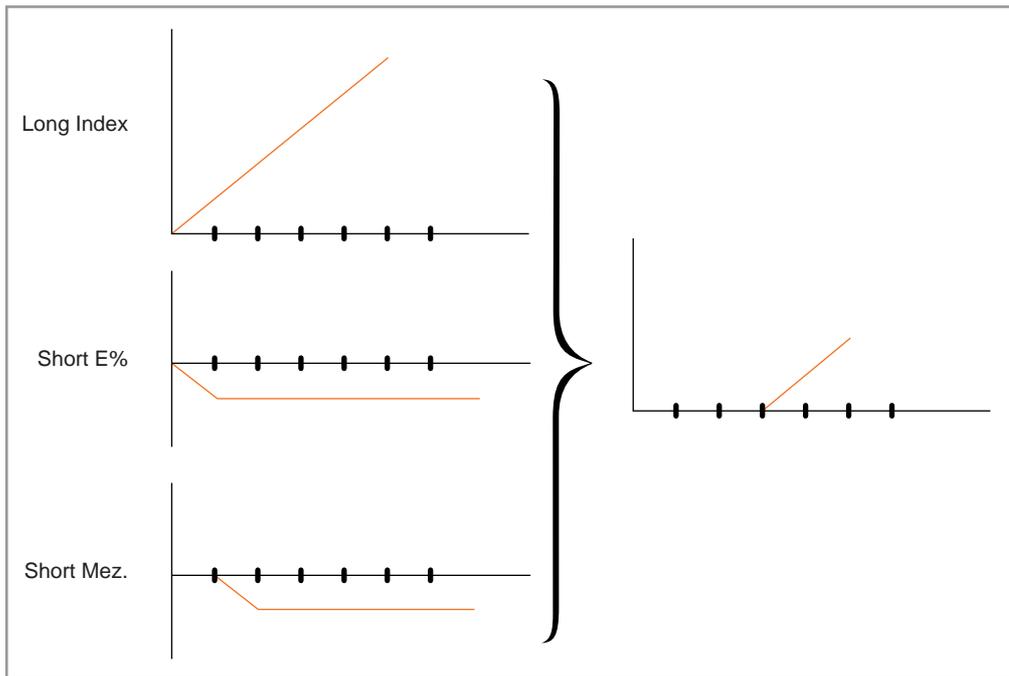


FIGURE 19-1

6.1. How to Calculate Deltas

In the Black-Scholes world, *deltas* and other sensitivity factors are calculated by taking the appropriate derivatives of a function. This is often not possible in the credit analysis. In the case of tranche *deltas*, the approach is one of numerical calculation of sensitivity factors or of obtaining closed form solutions by approximations.

One way is to use the Monte Carlo approach and one factor latent variable model to generate a sample $\{S_i^j\}$, and then determine the tranche spread. These results would depend on an initially assumed index spread or default probability. To obtain *delta* one would divide the original value of the index by an amount ΔI and then repeat the valuation exercise. The difference in tranche spread divided by ΔI will provide a numerical estimate for *delta*.

6.2. Gamma Sensitivity

Volatility in the spread movements is an important factor. Actively managing *delta* positions suggests that there may be *gamma gains*. The *gamma* effect seems to be most pronounced in the equity and senior tranches. There exist differences in *gamma* exposures depending on the particular tranche.

Unlike options, one can distinguish three different types of *gamma* in the credit sector.

- *Gamma* is defined as the portfolio convexity corresponding to a uniform relative shift in all the underlying CDS spreads.
- *iGamma* is the individual *gamma* defined as the portfolio convexity resulting from one CDS spread moving independently of the others; i.e., one spread moves and the others remain constant.

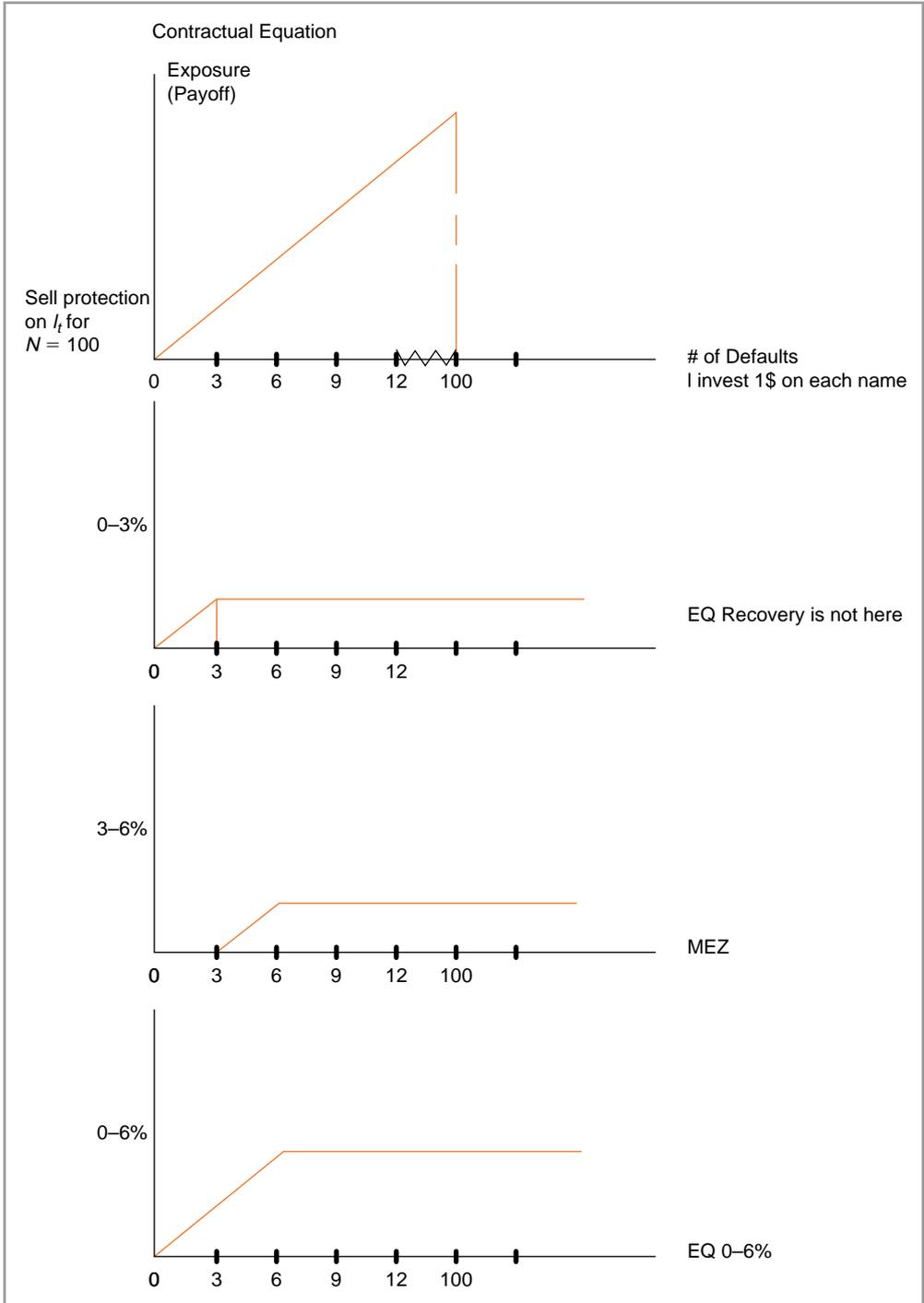


FIGURE 19-2

- *nGamma* is the negative *gamma* defined as the portfolio convexity corresponding to a uniform relative shift in underlying CDS spreads, with half of the credits widening and half of the credits tightening by a uniform amount.

Delta-hedged investors who are *long* correlation, benefiting if the correlation increases, will be *long gamma* while being *short iGamma* and *nGamma*.

6.3. Correlation Trade and Gamma Gains

Suppose one expects the compound default correlation to go up. This would be advantageous to the equity tranche protection seller and more or less neutral toward the mezzanine protection seller.¹² Thus, one would take a long correlation exposure by selling N^e units of equity protection while simultaneously buying N^m units of mezzanine exposure. The latter notional amount is determined according to the

$$N^m = \lambda N^e \quad (70)$$

What would be the gains from such a position? What would be the risks? First, consider the gains.

It turns out that such long correlation positions have positive carry, meaning that, even if the anticipated change in correlation does not materialize and the status quo continues, the position holder will make money. In fact, this positive carry is significant at around 200–350 bp depending on the prevailing levels of the index I_t and of the correlation ρ_t .

The position will also have positive *gamma* gains. As the *delta* hedge is adjusted, the hedge adjustments will monetize the I_t volatility because the long correlation position has, in fact, positive convexity with respect to the I_t . In addition, the position will gain if the correlation goes up as expected.

The risks are related to *iGamma* effects and to the declining correlation. In both cases the position will suffer some losses, although these may not be big enough to make the overall return negative.

Below we see an example of the *gamma* gains and dynamic *delta* hedging.

EXAMPLE: Delta Hedging and Gamma Gains

We are given the following information concerning *iTraxx* Index quotes (I_t) and the deltas of the equity and mezzanine tranches at various levels of I . For example if the index is at 30 bps, the equity tranche delta is 18.5, whereas the mezzanine tranche delta is 8.9.

I_t	Delta of 0–3 tranche	Delta of 0–6 tranche
30 bps	18.5	8.9
35 bps	17.6	9.4
40 bps	15.1	10.1

Now we use these to generate a dynamically adjusted series of delta hedges. Suppose we observe the following index movements across four days,

$$I_1 = 30, I_2 = 40, I_3 = 30 \quad (71)$$

¹² Depending on the level of the index and the level of correlation at time t_0 .

And suppose our notional is $N = 100$. Then a long correlation position will sell 100 units of equity protection and hedge this with $\frac{18.5}{8.9} 100 =$ units of mezzanine protection. This hedge can be taken with the index.

The investor will buy $18.5100 =$ units of protection on the index and sell $8.9100 =$ units of index protection for the mezzanine. On the net side, the investor is buying $(18.5 - 8.9)$ units of protection on the index. The resulting position will be delta neutral with respect to changes in I .

During day 2, initially the position is not delta neutral since I_t has moved. The correct hedge ratio is smaller, at $\frac{15.1}{10.1} 100$. The investor needs to reduce the long protection position on I_t by $(18.5 - 8.9) - (15.1 - 10.1)$. This means the investor will sell $[X]$ units of protection on the index.

During day 3 this position reverts back to the original position on the index.

Look what happened as a result of dynamic *delta* hedging of the tranche portfolio using the index. The investor sold protection when the I_t widened and bought it back when the I_t tightened. Clearly, these will lead to convexity gains similar to the case of options or long bonds.

7. Real-World Complications

The discussion of how to model and value tranche spreads has been a very simplified one. Real-world trading has several complications and also requires further modeling effort. Several additional questions also need to be addressed. We briefly discuss them below.

7.1. Base Correlations

The calculation of the implied compound correlation leads to what is known as *correlation smile*. In fact, if one uses five standard tranche spreads to back out five implied correlations, the resulting implied correlations would now be the same and would in general have a skew. Further, the implied correlations can sometimes not be unique, or may not even exist in mezzanine tranches.

One solution provided by the industry is to calculate and trade the so-called *base correlation*. Base correlation will also give a skew, but this skew will always exist and be unique.¹³ See Figure 19-3.

7.2. The Dispersion Effect

The *dispersion* effect refers to how different individual spreads are from the index spread. For example, if individual CDS spreads are more or less the same as the overall index spread, then we say that there is low dispersion.

The dispersion of individual CDS spreads in the portfolio versus the spread of the portfolio itself is an important factor influencing the tranche valuation. The effect varies with individual tranches. For the equity tranche, the higher the dispersion of spreads the higher the number of basis points. This is due to the increased riskiness of the tranche. In the mezzanine tranche,

¹³ A base correlation first calculates the market implied spreads for successive equity tranches with attachment points, 0–3%, 0–6%, 0–9%, and so on. In fact, using the 0–3% equity tranche and the 3–6% mezzanine tranche spreads one can calculate an arbitrage-free 0–6% equity tranche spread.

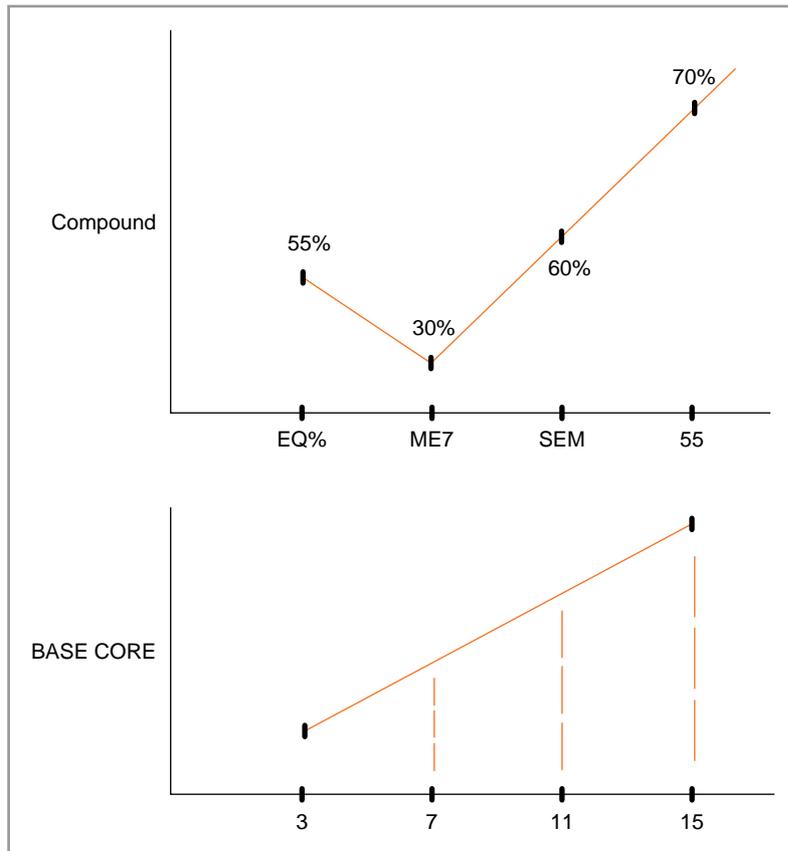


FIGURE 19-3

the lower the dispersion, the higher the number of bps indicating increased risk. The results of senior and super senior tranches are similar to the mezzanine.

7.3. The Time Effect

Time is also a variable affecting the tranche spreads. As the time to maturity gets shorter, tranche values tend to decline. In other words, the tranche spread curves are, in general, upward sloping.

7.4. Do Deltas Add Up to One?

Suppose there are two tranches and only two names in the reference portfolio. We can define two tranches: the equity tranche and, say, the senior tranche. Can we say that the *deltas* of the entire reference portfolio weighted by tranche size should sum to one?

Suppose one default occurs. If we sold protection on the index itself, the index spread we would receive in the future would stay the same. However, a weighted average of tranche spreads would be significantly affected, since with the first default equity protection the seller would receive no further premiums. And, it turns out that the equity tranche spreads would be the highest.

This effect is due to the fact that tranche spreads are unevenly distributed whereas the index has a single spread. With the first default, the high spread tranche is triggered and the average spread of the remaining tranche portfolio decreases significantly. Clearly, if all tranches and the index traded on an upfront basis this effect would disappear.

8. Conclusions

This chapter discussed only the beginnings of index and index tranche modeling. The idea was to introduce the Monte Carlo approach to stripping and valuing default correlation positions which were put together using index tranches. Needless to say, this discussion should be taken much further before one can implement these ideas in real-world tranche pricing and trading. This is the case, since, more than any other sector in financial markets, tranche valuation and hedging depends on which model the practitioner chooses.

Suggested Reading

*The best references to continue with the discussion given in this chapter are the handbooks prepared by broker-dealers themselves. The best among these is the JP Morgan's Credit Derivatives Handbook (2007). The two-volume Merrill Lynch set published in 2006 is also excellent and contains many examples. The Credit Suisse (2007) Handbook on Credit Risk Modeling is a very good modeling source. For an academic approach **Schonbucher** (2004) and **Duffie and Singleton** (2002) are the best.*

APPENDIX 19–1: Some Basic Statistical Concepts

Consider again the table discussed in the text, shown below, which gives the joint distribution of two random variables.

This simple table is an example of marginal and joint distribution functions associated with the two random variables d^A, d^B representing the default possibilities for the two references named A, B , respectively.

It is best to start the discussion with a small credit portfolio, then generalize to the, say, iTraxx index. Suppose we have an equally weighted CDS portfolio of $n = 2$ names denoted by $j = A, B$. We are interested in a horizon (i.e., maturity) of one year. According to this, there are two random variables d^A and d^B , each representing the credits $j = A$ and $j = B$. The d^j assumes the value of 1 if the j th name defaults, otherwise it is equal to zero.

Suppose the probabilities of default by A and B are given as in the table below.

	A defaults ($d^A = 1$)	A survives ($d^A = 0$)	Sum of Rows
B Defaults ($d^B = 1$)	.02	.03	.05
B Survives ($d^B = 0$)	.01	.94	.95
Sum of columns	.03	.97	1

Now we can discuss the most relevant point concerning this table. If the default-related random variables d^A and d^B were *independent*, then we would have

$$p^{11} = (1 - p^A)(1 - p^B) \quad (72)$$

$$p^{22} = (p^A)(p^B) \quad (73)$$

$$p^{12} = (1 - p^A)(p^B) \quad (74)$$

$$p^{21} = (p^A)(1 - p^B) \quad (75)$$

Otherwise, if any of these conditions were not satisfied, then the default random variables would be correlated. In this particular case a quick calculation would reveal that none of these conditions are satisfied. For example for p^{11} we have

$$p^{11} = .02 \neq p^A p^B = .0015 \quad (76)$$

As a matter of fact, the joint probability of default is more than ten times greater than the simple minded (and in this case, wrong) calculation by simple multiplication of individual default probabilities.

Let us calculate the expected values, variances, and covariances of the two random variables d^A and d^B . For the expected values

$$\begin{aligned} E[d^A] &= 1p^A + 0(1 - p^A) = p^A \\ &= .03 \end{aligned} \quad (77)$$

$$\begin{aligned} E[d^B] &= 1p^B + 0(1 - p^B) = p^B \\ &= .05 \end{aligned} \quad (78)$$

For the variances we have

$$\begin{aligned} E[(d^A - E[d^A])^2] &= (1 - p^A)^2 p^A + (0 - p^A)^2 (1 - p^A) = (1 - p^A) p^A \\ &= (.03)(.97) \end{aligned} \quad (79)$$

$$\begin{aligned} E[(d^B - E[d^B])^2] &= (1 - p^B)^2 p^B + (0 - p^B)^2 (1 - p^B) = (1 - p^B) p^B \\ &= (.05)(.95) \end{aligned} \quad (80)$$

Finally, the more important moment, the covariance between d^A and d^B is given by

$$E[(d^B - E[d^B])(d^A - E[d^A])] = (1 - p^B)(1 - p^A)p^{11} + (0 - p^B)(0 - p^A)p^{22} \\ + (1 - p^B)(0 - p^A)p^{12} + (1 - p^A)(0 - p^B)p^{21} \quad (81)$$

Where the p^{ik} is the joint probability that the corresponding events happen jointly. Remember that we must have

$$p^{11} + p^{12} + p^{21} + p^{22} = 1 \quad (82)$$

To convert this into a correlation, we need to divide this by the square root of the variances of d^A and d^B .

Exercises

1. We consider a reference portfolio of three investment grade names with the following one-year CDS rates:

$$c(1) = 15$$

$$c(2) = 11$$

$$c(3) = 330$$

The recovery rate is the same for all names at $R = 40$.

The notional amount invested in every CDO tranche is \$1.50. Consider the questions:

- What are the corresponding default probabilities?
 - How would you use this information in predicting actual defaults?
 - Suppose the defaults are uncorrelated. What is the distribution of the number of defaults during one year?
 - How much would a 0–66% tranche lose under these conditions?
 - Suppose there are two tranches: 0–50% and 50–100%. How much would each tranche pay over a year if you sell protection?
 - Suppose all CDS rates are now equal and that we have $c(1) = c(2) = c(3) = 100$. Also, all defaults are correlated with a correlation of one. What is the loss distribution? What is the spread of the 0–50% tranche?
2. The iTraxx equity tranche spread followed the path given below during three successive time periods:

$$\{14, 15, 5, 16\}$$

Assume that there are 30 reference names in this portfolio.

- You decide to select a leverage ratio of 2 and structure a *five-year* CPPI note on iTraxx crossover index. Libor rates are 5%. Describe your general strategy and, more important, show your initial portfolio composition.
- Given the path above, calculate your portfolio adjustments for the three periods.
- In period four, iTraxx becomes 370 and one company defaults. Show your portfolio adjustments. (Assume a recovery of 40%. Reminder: Do not forget that there are 30 names in the portfolio.)

CASE STUDY: May 2005 Volatility

During May 2005 credit markets witnessed an intriguing event which looked like a puzzle and created a significant amount of discussion as to the correctness of the credit markets' standard models. The events are worth reviewing briefly since they are a good example of the way cash derivatives markets influence each other in the newly developing credit markets.

Essentially, the May 2005 events were triggered by General Motors and Ford being downgraded by rating agencies. These credits that had massive amount of debt were investment grade and were downgraded to speculative grade, which meant that many institutional players would be prevented from holding them in their portfolios. The ensuing sales led to significant credit volatility in credit markets.

The biggest volatility happened in CDX and iTraxx tranche markets, since several players anticipating these events had put long correlation trades in place. The following IFR report shows the series of events as they happened in May 2005.

EXAMPLE: May 2005 Events

Speculation about losses in the hedge fund industry sent a shudder through credit markets earlier this month. Fund managers were wrong-footed by going long on the equity piece of synthetic CDOs, while funding that position by shorting the mezzanine tranche on the view that spreads would move in the way predicted by their correlation assumptions.

This positive carry trade was popular with banks and hedge funds because the trade made money if spreads of the related pieces moved in a parallel fashion, as well as creating a hedge against large losses.

The risk was exposure to unexpected default in the portfolio, but with default rates at historic lows, investors were happy to assume this risk.

The ratings downgrades on GM and Ford, which are often included in these CDO structures because they are such large issuers of debt, made the banks change their outlook on default rates and prompted them to begin unwinding massive correlation books.

While equity spreads jumped higher, those on the mezzanine tranches headed in the other direction, ensuring these trades significantly underperformed. As a result, investors suffered on both legs of the trade.

Initially, the underperformance was most acutely felt in the US, however during the last couple of weeks the European market has been hit by hedge fund unwinding of iTraxx CDO tranche trades. It is estimated that between 20 to 30 hedge funds executed correlation trades in Europe.

"Selling is a result of the negative returns that these types of trades have generated in the US recently," said a European credit strategist.

The iTraxx spreads should not be affected by the sell-off in correlation positions. "Fundamentally, a risk re-distribution between different tranches, as reflected by a change in correlation, should have a very limited effect on the underlying market," said a credit strategist. However, from a sentimental point of view the impact was much more significant and the rush to unwind the same trade caused massive volatility in the DJ iTraxx crossover index last week.

The index ballooned to 475bp in five-year from 330bp the previous week.

However, hedge funds putting on large shorts in a widening market coupled with a rally in US treasuries caused a violent snap back in the index to 380bp towards the end of the week.

Attention also turned last week to the banking sector and the size of potential losses. Most of the fall-out will be felt in London and the US where the majority of international banks keep their correlation books, whereas Asia is less exposed.

That is partly because synthetic tranching of purely Asian credit has not really taken off. Facilitating the interest in structured credit products in Europe and the US has been the development and liquidity of the credit default swap indices; they have given transparency to implied correlation in those markets. Not so in Asia, where the regional indices have not proved hugely popular and there has been even less take-up of index tranching.

The end of June is critical for the high-yield market because most funds have quarterly liquidity, and there is the threat of huge redemptions. Many are unwinding positions in expectation that clients will redeem their cash.

We can now study this case in more detail. According to the series of events, the status quo before the volatility was as follows.

Several hedge funds and bank proprietary trading desks were long correlation, meaning that they had sold protection on the equity tranche by a notional amount N^E and bought protection on the mezzanine tranche by a notional amount N^M . These two notional amounts were related to each other according to

$$N_t^M = \lambda_t N_t^E \quad (83)$$

where the λ_t is the hedge ratio selected so that the sensitivity of the portfolio to movements in the underlying index, or, in average default probability P_t is approximately zero. Letting V^E and V^M represent the value of the \$1 invested in the equity and mezzanine tranches, respectively, we can write the value

$$\frac{\partial}{\partial p_t} [N^E V_t^E - \lambda N^E V_t^M] = 0 \quad (84)$$

These positions were taken with the view that the compound default correlation coefficient denoted by ρ_t would go up. Since we had

$$\frac{\partial}{\partial \rho_t} V_t^E > 0 \quad (85)$$

and

$$\frac{\partial}{\partial \rho_t} V_t^M \cong 0 \quad (86)$$

The position would benefit as correlation increased,

$$\frac{\partial}{\partial \rho_t} V_t^E - \frac{\partial}{\partial \rho_t} V_t^M > 0 \quad (87)$$

in fact, as the reading above indicates, with the GM and Ford downgrades the reverse happened. One possible explanation of the May 2005 events is as follows.¹⁴ As GM and Ford were downgraded, the equity spreads increased and the V_t^E decreased, which led to a sudden collapse of

¹⁴ Remember that the Gaussian copula model is not a structural model that can provide a true “explanation” for these events. It is a mathematical construction that is used to calibrate various parameters we need during pricing and hedging. Hence, the discussion of May 2005 events can only be heuristic here.

the default correlation. Meanwhile, the mezzanine spreads decreased and the V_t^M increased. This meant severe mark-to-market losses of the long correlation trades

$$\frac{\partial}{\partial \rho_t} V_t^E - \frac{\partial}{\partial \rho_t} V_t^M < 0 \quad (88)$$

since both legs of the trades started to have severe mark-to-market losses.

The dynamics of the tranche spreads observed during May 2005 contradicted what the theoretical models implied. The observed real-world relationships had the opposite sign of what the market standard models would imply. What was the reason behind this puzzle?

One explanation is, of course, the models themselves. If the market convention used the “wrong” model then it is natural that sometimes the real-world development would contradict the model predictions. This is possible, but we know of no satisfactory new theoretical model that would explain the observed discrepancy at that time.

There is, on the other hand, a plausible structural explanation of the puzzle and it relates to the cash CDO market. Over the years, starting from the mid-1990s, banks had sold mezzanine tranches of cash CDOs (the banks were paying mezzanine spreads). Some of these positions could have been unhedged. As GM and Ford were downgraded, and as these names were heavily used in cash CDOs, the banks who were short the mezzanine tranche tried to cover these positions in the index market. This means that they had to sell protection on the mezzanine index tranche.¹⁵ It is plausible that this dynamic created a downward spiral where the attempt to hedge the cash mezzanine position via selling mezzanine protection in the index market resulted in even further mezzanine spread tightening.

At the same time, since the correlation trade had gone in the opposite direction, hedge funds tried to close their position in the equity tranche. They were selling equity protection before, but closing the position meant buying equity protection and such a rush by institutions would create exactly the type of dynamic observed at that time. The equity spreads increased and the mezzanine spreads declined and, at the end, correlation declined. The downgrade by GM and Ford had created the opposite effect.

¹⁵ This means that the banks who were paying the mezzanine spreads would now be receiving the mezzanine spreads. Obviously as mezzanine spreads declined, this would create mark-to-market losses and even negative P&L for those banks that were short the mezzanine tranche.