

CHAPTER 13

Fixed-Income Engineering

1. Introduction

This chapter extends the discussion of swap type instruments and outlines a simple framework for fixed-income security pricing. Term structure modeling is treated within this framework. The chapter also introduces the recent models that are becoming a benchmark in this sector.

Until recently, short-rate modeling was the most common approach in pricing and risk-managing fixed-income securities. The publication in 1992 of the Heath-Jarrow-Merton (HJM) approach enabled arbitrage-free modeling of multifactor-driven term structure models, but markets continued to use short-rate modeling. Today the situation is changing. The Forward Libor or Brace-Gatarek-Musiela (BGM) model is becoming the market standard for pricing and risk management.

This chapter will approach the issues from a practical point of view using *swap markets* and *swap derivatives* as a background. We are interested in providing a framework for analyzing the mechanics of swaps and swap derivatives, for decomposing them into simpler instruments, and for constructing synthetics. Recent models of fixed income modeling can then be built on this foundation very naturally.

It is worth reviewing the basic principles of swap engineering laid out in Chapter 5. First of all, swaps are almost always designed such that their value at initiation is zero. This is a characteristic of modern swap-type “spread instruments,” and there is no surprise here. Second, what makes the value of the swap equal to zero is a spread or an interest rate that is chosen with the *purpose* that the initial value of the swap vanishes. Third, swaps encompass more than one settlement date. This means that whatever the value of the swap rate or swap spread, these will in the end be some sort of “average of shorter term floating rates or spreads.” This not only imposes simple arbitrage conditions on relevant market rates, but also provides an opportunity to trade the volatility associated with such averages through the use of options on swaps. Since swaps are very liquid, they form an excellent underlying for *swaptions*. Swaptions, in turn, are related to interest rate volatilities for the underlying subperiods, which will relate to cap/floor volatilities. This structure is conducive to designing and understanding more complex swap products such as constant maturity swaps (CMS). The CMS swap is used as an example for showing the advantages of the Forward Libor Model.

Finally, the chapter will further use the developed framework to illustrate the advantages of measure change technology. Switching between various T -forward measures, we show how convexity effects can be calculated.

Most of the discussion will center on a three-period swap first, and then generalize the results. We begin with this simple example, because with a small number of cash flows the analysis becomes more manageable and easier to understand. Next, we lay out a somewhat more technical framework for engineering fixed-income instruments. Eventually, this is developed into the Forward Libor Model. Within our framework, measure changes using Girsanov-type transformations emerge as fundamental tools of financial engineering. The chapter discusses how measures can be changed *sequentially* during a numerical pricing exercise as was done in the simulation of the Forward Libor Model. These tools are then applied to CMS swaps, which are difficult to price with traditional models.

2. A Framework for Swaps

We work with *forward* fixed-payer interest rate swaps and their “spot” equivalent. These are vanilla products in the sense that contracts are predesigned and homogeneous. They are liquid, the bid-ask spreads are tight, and every market player is familiar with their properties and related conventions.

To simplify the discussion we work with a three-period forward swap, shown in Figure 13-1. It is worth repeating the relevant parameters again, given the somewhat more technical approach the chapter will adopt.

1. The notional amount is N , and the tenor of the underlying Libor rate is δ , which represents a proportion of a calendar year. As usual, if a year is denoted by 1, then δ will be $1/4$ in the case of 3-month Libor.
2. The swap *maturity* is three periods. The swap ends at time $T = t_4$. The swap contract is signed at time t_0 but starts at time t_1 , hence the term *forward* swap is used.¹
3. The dates $\{t_1, t_2, t_3\}$ are *reset dates* where the relevant Libor rates L_{t_1}, L_{t_2} , and L_{t_3} will be determined.² These dates are δ time units apart.
4. The dates $\{t_2, t_3, t_4\}$ are *settlement dates* where the Libor rates L_{t_1}, L_{t_2} , and L_{t_3} are used to exchange the floating cash flows, $\delta N L_{t_i}$, against the fixed $\delta N s_{t_0}$ at each t_{i+1} . In this setup, the time that passes until the start of the swap, $t_1 - t_0$, need not equal δ . However, it may be notationally convenient to assume that it does.

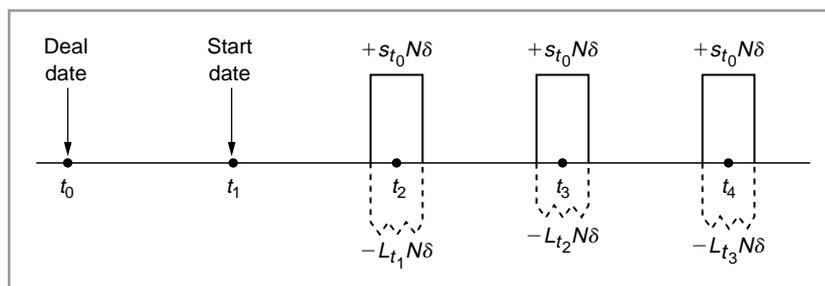


FIGURE 13-1

¹ In the case of the spot swap that we will use, the swap will start at time t_0 and settle three times.

² That is, determined by some objective and predefined authority such as the British Bankers Association.

Our purpose is to provide a systematic framework in which the risk management and pricing of such swaps and the instruments that build on them can be done efficiently. That is, we discuss a technical framework that can be used for running a swap and swap derivatives book.

Swaps are one major component of a general framework for fixed-income engineering. We need *two* additional tools. These we introduce using a simple example again. Consider Figure 13-2, where we show payoff diagrams for three default-free pure *discount bonds*. The current price, $B(t_0, T_i)$, of these bonds is paid at t_0 to receive 1 dollar in the same currency at maturity dates $T_i = t_i$. Given that these bonds are default-free, the time- t_i payoffs are *certain* and the price $B(t_0, T_i)$ can be considered as the value today of 1 dollar to be received at time t_i . This means they are, in fact, the relevant discount factors, or in market language, simply *discounts* for t_i . Note that as

$$T_1 < T_2 < T_3 < T_4, \quad (1)$$

bond prices must satisfy, regardless of the slope of the yield curve:³

$$B(t_0, T_1) > B(t_0, T_2) > B(t_0, T_3) > B(t_0, T_4) \quad (2)$$

These prices can be used as discount factors to calculate *present values* of various cash flows occurring at future settlement dates t_i . They are, therefore, quite useful in successive swap settlements and form the second component in our framework.

The third component of the fixed-income framework is shown in Figure 13-3. Here, we have the cash flow diagrams of three forward rate agreements (FRAs) paid in arrears. The FRAs are,

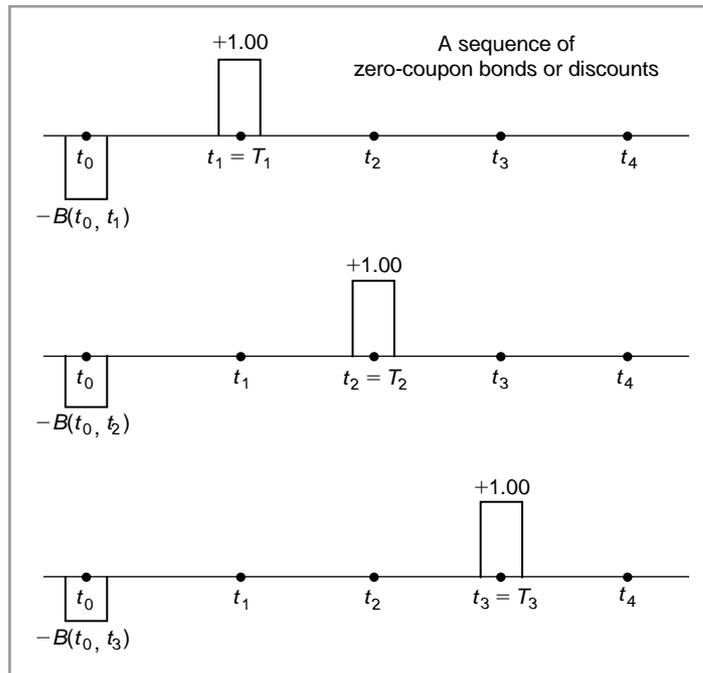


FIGURE 13-2

³ As seen earlier, if we short *one* longer-term bond to fund a long position in *one* short-term bond, we would not have enough money.

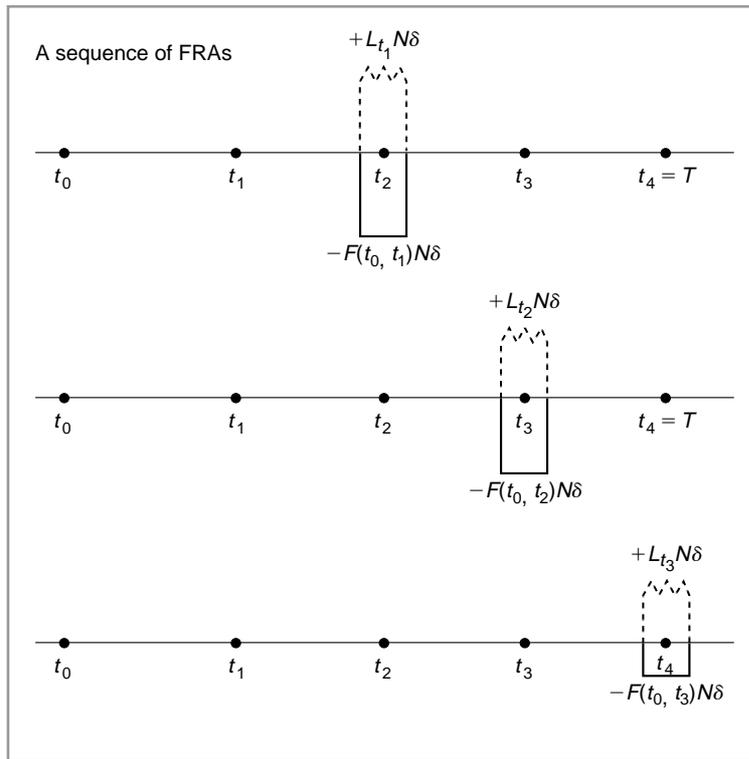


FIGURE 13-3

respectively, $t_1 \times t_2$, $t_2 \times t_3$, and $t_3 \times t_4$. For each FRA, a floating (random) payment is made against a known (fixed) payment for a net cash flow of

$$[L_{t_i} - F(t_0, t_i)]N\delta \tag{3}$$

at time t_{i+1} . Here, the $F(t_0, t_i)$ is the forward rate of a fictitious forward loan contract signed at time t_0 . The forward loan comes into effect at t_i and will be paid back at time $t_{i+1} = t_i + \delta$. We note that the fixed payments $N\delta F(t_0, t_i)$ are not the same across the FRAs. Although all FRA rates are known at time t_0 , they will, in general, not equal each other or equal the payment of the fixed swap leg, $\delta N s_{t_0}$.

We can now use this framework to develop some important results and then apply them in financial engineering.

2.1. Equivalence of Cash Flows

The first financial engineering rule that we discuss in this chapter is associated with the perceived *equivalence* of cash flows. In Figure 13-3, there is a strip of floating cash flows:

$$\{L_{t_1}N\delta, L_{t_2}N\delta, L_{t_3}N\delta\} \tag{4}$$

and, given observed liquid prices, the market is willing to exchange these random cash flows against the known (fixed) cash flows:

$$\{F(t_0, t_1)N\delta, F(t_0, t_2)N\delta, F(t_0, t_3)N\delta\} \tag{5}$$

According to this, if these FRAs are liquid at time t_0 , the known cash flow sequence in equation (5) is perceived by the markets as the *correct* exchange against the unknown, floating payments in equation (4). If we then consider the swap cash flows shown in Figure 13-1, we notice that exactly the same floating cash flow sequence as in equation (4) is exchanged for the known and fixed swap leg

$$\{s_{t_0}N\delta, s_{t_0}N\delta, s_{t_0}N\delta\} \quad (6)$$

The settlement dates are the same as well. In both exchanges, neither party makes any upfront payments at time t_0 . We can therefore combine the two exchanges at time t_0 , and obtain the following result.

The market is willing to exchange the fixed and *known* cash flows

$$\{s_{t_0}N\delta, s_{t_0}N\delta, s_{t_0}N\delta\} \quad (7)$$

against the variable *known* cash flows:

$$\{F(t_0, t_1)N\delta, F(t_0, t_2)N\delta, F(t_0, t_3)N\delta\} \quad (8)$$

at no additional time- t_0 compensation.

This has an important implication. It means that the time- t_0 values of the two cash flow sequences are the same. Otherwise, one party would demand an initial cash payment. Given that the cash flows are known as of time t_0 , their equivalence provides an *equation* that can be used in pricing, as we will see next. This argument will be discussed further using the forward Libor model.

2.2. Pricing the Swap

We have determined two *known* cash flow sequences the market is willing to exchange at no additional cost. Using this information, we now calculate the time- t_0 values of the two cash flows. To do this, we use the second component of our framework, namely, the discount bond prices given in Figure 13-2.

Suppose the pure discount bonds with arbitrage-free prices $B(t_0, t_i)$, $i = 1, 2, 3, 4$ are liquid and actively traded. We can then use $\{B(t_0, t_2), B(t_0, t_3), B(t_0, t_4)\}$ to value cash flows settled at times t_2, t_3 , and t_4 respectively.⁴ In fact, the time- t_0 value of the sequence of cash flows,

$$\{F(t_0, t_1)N\delta, F(t_0, t_2)N\delta, F(t_0, t_3)N\delta\} \quad (9)$$

is given by multiplying each cash flow by the *discount* factor that corresponds to that particular settlement date and then adding. We use the default-free bond prices as our discount factors, and obtain the value of the fixed FRA cash flows

$$\begin{aligned} & B(t_0, t_2)F(t_0, t_1)N\delta + B(t_0, t_3)F(t_0, t_2)N\delta + B(t_0, t_4)F(t_0, t_3)N\delta \\ & = [B(t_0, t_2)F(t_0, t_1) + B(t_0, t_3)F(t_0, t_2) + B(t_0, t_4)F(t_0, t_3)]N\delta \end{aligned} \quad (10)$$

The time- t_0 value of the fixed swap cash flows can be calculated similarly

$$\begin{aligned} & B(t_0, t_2)s_{t_0}N\delta + B(t_0, t_3)s_{t_0}N\delta + B(t_0, t_4)s_{t_0}\delta N \\ & = [B(t_0, t_2) + B(t_0, t_3) + B(t_0, t_4)]\delta N s_{t_0} \end{aligned} \quad (11)$$

⁴ The fact that we are using default-free discount bonds to value a private party cash flow indicates that we are abstracting from all counterparty or credit risk.

Now, according to the argument in the previous section, the values of the two cash flows must be the same.

$$\begin{aligned} [B(t_0, t_2)F(t_0, t_1) + B(t_0, t_3)F(t_0, t_2) + B(t_0, t_4)F(t_0, t_3)]\delta N \\ = [B(t_0, t_2) + B(t_0, t_3) + B(t_0, t_4)]\delta N s_{t_0} \end{aligned} \quad (12)$$

This equality has at least two important implications. First, it implies that the value of the swap at time t_0 is zero. Second, note that equality can be used as an *equation* to determine the value of *one* unknown. As a matter of fact, pricing the swap means determining a value for s_{t_0} such that the equation is satisfied. Taking s_{t_0} as the unknown, we can rearrange equation (12), simplify, and obtain

$$s_{t_0} = \frac{B(t_0, t_2)F(t_0, t_1) + B(t_0, t_3)F(t_0, t_2) + B(t_0, t_4)F(t_0, t_3)}{B(t_0, t_2) + B(t_0, t_3) + B(t_0, t_4)} \quad (13)$$

This pricing formula can easily be generalized by moving from the three-period setting to a vanilla (forward) swap that makes n payments starting at time t_2 . We obtain

$$s_{t_0} = \frac{\sum_{i=1}^n B(t_0, t_{i+1})F(t_0, t_i)}{\sum_{i=1}^n B(t_0, t_{i+1})} \quad (14)$$

This is a compact formula that ties together the three important components of the fixed-income framework we are using in this chapter.

2.2.1. Interpretation of the Swap Rate

The formula that gives the arbitrage-free value of the (forward) swap has a nice interpretation. For simplicity, revert to the three-period case. Rewrite equation (13) as

$$\begin{aligned} s_{t_0} = & \frac{B(t_0, t_2)}{[B(t_0, t_2) + B(t_0, t_3) + B(t_0, t_4)]} F(t_0, t_1) \\ & + \frac{B(t_0, t_3)}{[B(t_0, t_2) + B(t_0, t_3) + B(t_0, t_4)]} F(t_0, t_2) \end{aligned} \quad (15)$$

$$+ \frac{B(t_0, t_4)}{[B(t_0, t_2) + B(t_0, t_3) + B(t_0, t_4)]} F(t_0, t_3) \quad (16)$$

According to this expression, we see that the “correct” (forward) swap rate is a *weighted average* of the FRA paid-in-arrears rates during the life of the swap:

$$s_{t_0} = \omega_1 F(t_0, t_1) + \omega_2 F(t_0, t_2) + \omega_3 F(t_0, t_3) \quad (17)$$

The weights are given by

$$\omega_i = \frac{B(t_0, t_{i+1})}{[B(t_0, t_2) + B(t_0, t_3) + B(t_0, t_4)]} \quad (18)$$

and add up to one:

$$\omega_1 + \omega_2 + \omega_3 = 1 \quad (19)$$

This can again be generalized for a (forward) swap that makes n payments:

$$s_{t_0} = \sum_{i=1}^n \omega_i F(t_0, t_i) \quad (20)$$

with

$$\sum_{i=1}^n \omega_i = 1 \quad (21)$$

Thus, the (forward) swap rate is an *average paid-in-arrears FRA rate*. We emphasize that this is true as long as the FRAs under consideration are paid-in-arrears. There are, on the other hand, so-called *Libor-in-arrears* FRAs where a convexity adjustment needs to be made for the argument to hold.⁵

It is important to realize that the weights $\{\omega_i\}$ are obtained from pure discount bond prices, which, as shown in Chapters 4 and 12, are themselves functions of forward rates:

$$B(t_0, t_i) = \frac{1}{\prod_{j=0}^{i-1} (1 + \delta F(t_0, t_j))} \quad (22)$$

According to these formulas, three important components of our pricing framework—the swap market, the FRA market, and the bond market—are interlinked through nonlinear functions of forward rates. The important role played by the forward rates in these formulas suggests that obtaining *arbitrage-free dynamics* of the latter is required for the pricing of all swap and swap-related derivatives. The Forward Libor Model does exactly this. Because this model is set up in a way as to fit market conventions, it is also practical.

However, before we discuss these more advanced concepts, it is best to look at an example. In practice, swap and FRA markets are liquid and market makers readily quote the relevant rates. The real-world equivalents of the pure discount bonds $\{B(t_0, t_i)\}$, on the other hand, are not that liquid, even when they exist.⁶ In the following example, we sidestep this point and assume that such quotes are available at all desired maturities. Even then, some important technical issues emerge, as the example illustrates.

EXAMPLE:

Suppose we observe the following paid-in-arrears FRA quotes:

Term	Bid-Ask
0 × 6	4.05–4.07
6 × 12	4.15–4.17
12 × 18	4.32–4.34
18 × 24	4.50–4.54

Also, suppose the following treasury strip prices are observed:

⁵ We repeat the difference in terminology. One instrument is *paid-in-arrears*, the other is *Libor-in-arrears*. Here, the Libor of the settlement time t_i is used to determine the time- t_i cash flows. With paid-in-arrear FRAs, the Libor of the previous settlement date, t_{i-1} , is used.

⁶ In the United States, the instruments that come closest to these discount bonds are *treasury strips*. These are cash flows stripped from existing U.S. treasuries, and there are a fair number of them. However, they are not very liquid and, in general, the market quoted prices cannot be used as substitutes for $B(t_0, t_i)$, for various technical reasons.

Maturity	Bid-Ask
12 months	96.00–96.02
18 months	93.96–93.99
24 months	91.88–91.92

We can ask two questions. First, are these data arbitrage-free so that they can be used in obtaining an arbitrage-free swap rate? Second, if they are, what is the implied forward swap rate for the period that starts in six months and ends in 24 months?

The answer to the first question can be checked by using the following arbitrage equality, written for discount bonds with par value \$100, as market convention suggests:

$$B(t_0, t_i) = \frac{100}{\prod_{j=0}^{i-1} (1 + \delta F(t_0, t_j))} \tag{23}$$

where the value of δ will be 1/2 in this example. Substituting the relevant forward rates from the preceding table, we indeed find that the given discount bond prices satisfy this equality. For example, for $B(0, 2)^{ask}$ we have

$$B(0, 2)^{ask} = \frac{100}{(1 + .5(.0405))(1 + .5(.0415))} = 96.02 \tag{24}$$

The relevant equalities hold for other discount bond prices as well. This means that the data are arbitrage-free and can be used in finding an arbitrage-free swap rate for the above-mentioned forward start swap.

Replacing straightforwardly in

$$s_{t_0}^{ask} = \omega_1^{ask} F(t_0, t_1)^{ask} + \omega_2^{ask} F(t_0, t_2)^{ask} + \omega_3^{ask} F(t_0, t_3)^{ask} \tag{25}$$

$$\omega_i^{ask} = \frac{B(t_0, t_{i+1})^{ask}}{[B(t_0, t_2)^{ask} + B(t_0, t_3)^{ask} + B(t_0, t_4)^{ask}]}, \tag{26}$$

we find

$$\omega_1^{ask} = \frac{96.02}{[96.02 + 93.99 + 91.92]} = .341, \tag{27}$$

$$\omega_2^{ask} = \frac{93.99}{[96.02 + 93.99 + 91.92]} = .333, \tag{28}$$

$$\omega_3^{ask} = \frac{91.92}{[96.02 + 93.99 + 91.92]} = .326 \tag{29}$$

The asking swap rate is

$$s_{t_0}^{ask} = (.341)4.17 + (.333)4.34 + (.326)4.54 = 4.34 \tag{30}$$

Similarly, we can calculate the bid rate:

$$s_{t_0}^{bid} = (.341)4.15 + (.333)4.32 + (.326)4.50 = 4.32 \tag{31}$$

It is worth noting that the weights have approximately the same size.

We now consider further financial engineering applications of the fixed-income framework outlined in this section.

2.3. Some Applications

The first step is to consider the synthetic creation of swaps within our new framework. Our purpose is to obtain an alternative synthetic for swaps by manipulating the formulas derived in the previous section. In Chapter 5 we discussed one way of replicating swaps. We showed that a potential synthetic is the simultaneous shorting of a particular coupon bond and buying of a proper floating rate bond. This embodies the classical approach to synthetic swap creation, and it will be the starting point of the following discussion.

2.3.1. Another Formula

We have already derived a formula for the (forward) swap rate, s_{t_0} , that gives an arbitrage-free swap value:

$$s_{t_0} = \frac{[B(t_0, t_2)F(t_0, t_1) + B(t_0, t_3)F(t_0, t_2) + B(t_0, t_4)F(t_0, t_3)]}{[B(t_0, t_2) + B(t_0, t_3) + B(t_0, t_4)]} \quad (32)$$

Or, in the general form,

$$s_{t_0} = \frac{\sum_{i=1}^n B(t_0, t_{i+1})F(t_0, t_i)}{\sum_{i=1}^n B(t_0, t_{i+1})} \quad (33)$$

Now, we would like to obtain an alternative way of looking at the same swap rate by modifying the formula. We start the discussion with the arbitrage relation between the discount bond prices, $B(t_0, t_i)$, and the forward rates, $F(t_0, t_i)$, obtained earlier in Chapter 4:

$$1 + \delta F(t_0, t_i) = \frac{B(t_0, t_i)}{B(t_0, t_{i+1})} \quad (34)$$

Rearranging

$$F(t_0, t_i) = \frac{1}{\delta} \left[\frac{B(t_0, t_i)}{B(t_0, t_{i+1})} - 1 \right] \quad (35)$$

We now substitute this expression in equation (32) to obtain

$$s_{t_0} = \frac{1}{\delta[B(t_0, t_2) + B(t_0, t_3) + B(t_0, t_4)]} \left\{ B(t_0, t_2) \left[\frac{B(t_0, t_1)}{B(t_0, t_2)} - 1 \right] \right. \quad (36)$$

$$\left. + B(t_0, t_3) \left[\frac{B(t_0, t_2)}{B(t_0, t_3)} - 1 \right] + B(t_0, t_4) \left[\frac{B(t_0, t_3)}{B(t_0, t_4)} - 1 \right] \right\} \quad (37)$$

Simplifying the common $B(t_0, t_i)$ terms on the right-hand side, we get

$$s_{t_0} = \frac{1}{\delta \sum_{i=1}^3 B(t_0, t_{i+1})} ([B(t_0, t_1) - B(t_0, t_2)] \quad (38)$$

$$+ [B(t_0, t_2) - B(t_0, t_3)] + [B(t_0, t_3) - B(t_0, t_4)])$$

$$= \frac{1}{\delta \sum_{i=1}^3 B(t_0, t_{i+1})} [B(t_0, t_1) - B(t_0, t_4)] \quad (39)$$

We can try to recognize what this formula means by first rearranging,

$$\delta s_{t_0} [B(t_0, t_2) + B(t_0, t_3) + B(t_0, t_4)] = [B(t_0, t_1) - B(t_0, t_4)] \quad (40)$$

and then regrouping:

$$B(t_0, t_1) - [s_{t_0} \delta B(t_0, t_2) + s_{t_0} \delta B(t_0, t_3) + B(t_0, t_4)(1 + \delta s_{t_0})] = 0 \quad (41)$$

The equation equates two cash flows. $B(t_0, t_1)$ is the value of 1 dollar to be received at time t_1 . Thus, the position needs to be *long* a t_1 -maturity discount bond. Second, there appear to be coupon *payments* of constant size, δs_{t_0} at times t_2, t_3, t_4 and then a payment of 1 dollar at time t_4 .⁷ Thus, this seems to be a short (forward) position in a t_4 -maturity coupon bond with coupon rate s_{t_0} .

To summarize, this particular forward fixed-receiver interest rate swap is equivalent to

$$\text{Fixed-payer forward swap} = \{\text{Buy } t_1 \text{ discount bond, forward sell } t_4\text{-maturity coupon bond}\} \quad (42)$$

This synthetic will replicate the value of the forward swap. Note that the floating cash flows do not have to be replicated. This is because, in a forward swap, the floating cash flows are related to deposits (loans) that will be made in the future, at interest rates to be determined then.

2.3.2. Marking to Market

We can use the same framework for discussing mark-to-market practices. Start at time t_0 . As discussed earlier, the market is willing to pay the known cash flows

$$\{s_{t_0} N \delta, s_{t_0} N \delta, s_{t_0} N \delta\} \quad (43)$$

against the random cash flows

$$\{L_{t_1} N \delta, L_{t_2} N \delta, L_{t_3} N \delta\} \quad (44)$$

Now, let a short but noninfinitesimal time, Δ , pass. There will be a *new* swap rate $s_{t_0+\Delta}$, which, in all probability, will be different than s_{t_0} . This means that the market is now willing to pay the *new* known cash flows

$$\{s_{t_0+\Delta} N \delta, s_{t_0+\Delta} N \delta, s_{t_0+\Delta} N \delta\} \quad (45)$$

against the *same* random cash flows:

$$\{L_{t_1} N \delta, L_{t_2} N \delta, L_{t_3} N \delta\} \quad (46)$$

This implies that the value of the original swap, written at time t_0 , is nonzero and is given by the difference:

$$[s_{t_0+\Delta} N \delta - s_{t_0} N \delta][B(t_0 + \Delta, t_2) + B(t_0 + \Delta, t_3) + B(t_0 + \Delta, t_4)] \quad (47)$$

This can be regarded as the profit and loss for the fixed *payer*. At time $t_0 + \Delta$, the floating payment to be received has a value given by equation (47), and the actual floating payments would cancel out.⁸ We can apply the same reasoning using the FRA rates and calculate the mark-to-market value of the original swap from the difference:

$$\left(N \delta \left[\sum_{i=1}^n \omega_{it_0} F(t_0, t_i) \right] - N \delta \left[\sum_{i=1}^n \omega_{i(t_0+\Delta)} F(t_0 + \Delta, t_i) \right] \right) \sum_{i=1}^n B(t_0 + \Delta, t_{i+1}) \quad (48)$$

⁷ These payments are discounted to the present and this introduces the corresponding bond prices to the expression in the brackets.

⁸ What one “plugs in” for unknown Libor rates in equation (46) does change. But we are valuing the swap from the fixed leg and we consider the fixed payments as compensation for random, unknown floating rates. These random variables remain the same.

This way of writing the expression shows the profit and loss from the point of view of a fixed receiver. It should be noted that here the weights w_i have time subscripts, since they will change as time passes. Thus, managing a swap book will depend nonlinearly on the forward rate dynamics.⁹

3. Term Structure Modeling

The framework outlined in this chapter demonstrates the links between swap, bond, and FRA markets. We will now discuss the term structure implications of the derived formulas. The set of formulas we studied implies that, given the necessary information from *two* of these markets, we can, in principle, obtain arbitrage-free prices for the remaining market.¹⁰ We discuss this briefly, after noting the following small, but significant, modification. Term structure models concern forward rates *as well as* spot rates. As a matter of fact, traditional yield curve construction is done by first obtaining the spot yields and then moving to forward rates. (The appendix at the end of this chapter provides a short review of traditional yield curve analysis.)

Following this tradition, and noting that *spot swaps* are more liquid than *forward swaps*, in this section we let $s_{t_0}^n$ denote the *spot* swap rate with maturity n years. Without loss of generality, we can assume that swap maturities are across years $n = 1, \dots, 30$, so that the longest dated swap is for 30 years.¹¹ The discussion will be conducted in terms of spot swap rates.

3.1. Determining the Forward Rates from Swaps

Given a sufficient number of arbitrage-free values of observed spot swap rates $\{s_{t_0}^n\}$ and using the equalities

$$s_{t_0}^n = \frac{\sum_{i=0}^{n-1} B(t_0, t_{i+1}) F(t_0, t_i)}{\sum_{i=0}^{n-1} B(t_0, t_{i+1})} \quad (49)$$

and

$$\frac{B(t_0, t_i)}{B(t_0, t_{i+1})} = (1 + \delta F(t_0, t_i)) \quad (50)$$

we can obtain all forward rates, for the case $\delta = 1$. By substituting the $B(t_0, t_i)$ out from the first set of equations, we obtain n equations in n forward rates.¹² In the case of $\delta = \frac{1}{4}$ or $\delta = \frac{1}{2}$, there are more unknown $F(t_0, t_i)$ than equations, if traded swap maturities are in years. Under these conditions the t_i would run over quarters whereas the superscript in $s_{t_0}^n$, $n = 1, 2, \dots$ will be in years. This is due to the fact that swap rates are quoted for annual intervals, whereas the settlement dates would be quarterly or semiannual. Some type of interpolation of swap rates or modeling will be required, which is common even in traditional yield curve calculations.

⁹ Again, these forward rates need to be associated with paid-in-arrears FRAs or forward loans.

¹⁰ This assumes that all maturities of the underlying instruments trade actively, which is, in general, not the case.

¹¹ Swaps start to trade from two years and on. A 1-year swap against 1-year Libor would, in fact, be equivalent to a trivial FRA.

¹² Remember that $F(t_0, t_0)$ equals the current Libor for that tenor and is a trivial forward rate.

3.2. Determining the $B(t_0, t_i)$ from Forward Rates

Now, if the forward rates $\{F(t_0, t_i)\}$ and the current Libor curve are provided by markets or are obtained from $\{s_{t_0}^n\}$ as in our case, we can use the formula

$$B(t_0, t_{i+1}) = \frac{1}{\prod_{j=0}^i (1 + \delta F(t_0, t_j))} \quad (51)$$

to calculate the arbitrage-free values of the relevant pure discount bond prices. In each case, we can derive the values of $B(t_0, t_i)$ from the observed $\{F(t_0, t_i)\}$ and $\{s_{t_0}^n\}$. This procedure would price the FRAs and bonds *off* the swap markets. It is called the *curve algorithm*.

3.3. Determining the Swap Rate

We can proceed in the opposite direction as well. Given arbitrage-free values of forward rates, we can, in principle, use the same formulas to determine the swap rates. All we need to do is (1) calculate the discount bond prices from the forward rates and (2) substitute these bond prices and the appropriate forward rates in our formula,

$$s_{t_0}^n = \frac{\sum_{i=0}^{n-1} B(t_0, t_{i+1}) F(t_0, t_i)}{\sum_{i=0}^{n-1} B(t_0, t_{i+1})} \quad (52)$$

Repeating this for all available $s_{t_0}^n$, $n = 1, \dots, 30$, we obtain the arbitrage-free *swap curve* and discounts. In this case, we would be going from the spot and forward Libor curve to the (spot) swap curve.

3.4. Real-World Complications

There are, of course, several real-world complications to going back and forth between the forward rates, discount bond prices, and swap rates. Let us mention three of these. First, as mentioned in the previous section, in reality swaps are traded for *yearly* intervals and the FRAs or Eurodollar contracts are traded for three-month or six-month tenors. This means that if we desired to go from swap quotes to quotes on forward rates using these formulas, there will be the need to interpolate the swap rates for portions of a year.

Second, observed quotes on forward rates do *not* necessarily come from paid-in-arrears FRAs. Market-traded FRAs settle at the time the Libor rate is observed, not at the end of the relevant period. The FRA rates generated by these markets *will* be consistent with the formulas introduced earlier. On the other hand, some traders use interest rate futures, and, specifically, Eurocurrency futures, in hedging their swap books. Futures markets are more *transparent* than the FRA markets, and have a great deal of liquidity. But the forward rates determined in futures markets require *convexity adjustments* before they can be used in the swap formulas discussed in this chapter.

Third, the liquidity of FRA and swap rates depends on the maturity under consideration. As mentioned earlier, FRAs are more liquid for the shorter end of the curve, whereas swaps are more liquid at the longer end. This means that it may not be possible to go from FRA rates to swap rates for maturities over five years. Similarly, for very short maturities there will be no observed quotes for swaps.

3.4.1. Remark

Another important point needs to be mentioned here. In practice, Libor rates L_{t_i} apply to AA-rated credits¹³. This is implicit in the fixing process of the BBA Libor. The banks that form the BBA panels have, in general, ratings of AA or AA−, and the interest rate that they pay reflects this level of credit risk. Our treatment has followed the general convention in academic work of using the term “Libor” as if it relates to a default-free loan.

Thus, if a financial engineer follows the procedures described here, the resulting curve will be the *swap curve* and not the treasury or sovereign curve. This swap curve will be “above” the sovereign or treasury curve, and the difference will be the curve for the swap spreads.

4. Term Structure Dynamics

In the remainder of this chapter, we will see that the Forward Libor Model is the correct way to approach term structure dynamics. The model is based on the idea of converting the dynamics of each forward rate into a Martingale using some properly chosen forward measure. According to the linkages between sectors shown in this chapter, once such dynamics are obtained, we can use them to generate dynamics for other fixed-income instruments.

Most of the derivation associated with the Forward Libor Model is an application of the fundamental theorem of asset pricing discussed in Chapter 11. Thus, we continue to use the same finite state world discussed in Chapter 11. The approach is mostly straightforward. There is only *one* aspect of forward Libor or swap models that makes them potentially difficult to follow. Depending on the instruments, arbitrage-free dynamics of *different* forward rates may have to be expressed under the *same* forward measure. The methodology then becomes more complicated. It requires a judicious sequence of Girsanov-style measure changes to be applied to forward rate dynamics in some recursive fashion. Otherwise, arbitrage-free dynamics of individual forward rates would not be correctly represented.

The Girsanov theorem is a powerful tool, but it is not easy to conceive such successive measure changes. Doing this within a discrete framework, in a discrete setting, provides a great deal of *motivation* and facilitates understanding of arbitrage-free dynamics. This is the purpose behind the second part of this chapter.

4.1. The Framework

We adopt a simple discrete framework and then extend it to general formulas. Consider a market where instruments can be priced and risk-managed in discrete times that are δ apart:

$$t_0 < t_1 < \dots < t_n = T \quad (53)$$

with

$$t_i - t_{i-1} = \delta \quad (54)$$

Initially, we concentrate on the first three times, t_0 , t_1 , and t_2 that are δ apart. In this framework we consider four simple fixed-income securities:

1. A default-free zero-coupon bond $B(t_0, t_2)$ that matures at time t_2 .

¹³ During the credit crisis of 2007–2008 Libor rates were quite unstable as AA− entities. Sometimes they behaved as if they had a rating of A or lower.

2. A default-free zero-coupon bond that matures one period later, at time t_3 . Its current price is expressed as $B(t_0, t_3)$.
3. A savings account that pays (in-arrears) the discrete-time simple rate L_{t_i} observed at time t_i . Therefore, the savings account payoff at t_2 will be

$$R_{t_2} = (1 + \delta L_{t_0})(1 + \delta L_{t_1}) \tag{55}$$

Note that the L_{t_1} observed from the initial time t_0 will be a random variable.

4. An FRA contracted at time t_0 and settled at time t_2 , where the buyer receives/pays the differential between the fixed-rate $F(t_0, t_1)$ and the floating rate L_{t_1} at time t_2 . We let the notional amount of this instrument equal 1 and abbreviate the forward rate to F_{t_0} . The final payoff can be written as

$$(L_{t_1} - F_{t_0})\delta \tag{56}$$

These assets can be organized in the following payoff matrix D for time t_2 as in Chapter 11, assuming that at every t_i , from every node there are only two possible movements for the underlying random process. Denoting these movements by u, d , we can write¹⁴

$$D = \begin{bmatrix} R_{t_2}^{uu} & R_{t_2}^{ud} & R_{t_2}^{du} & R_{t_2}^{dd} \\ 1 & 1 & 1 & 1 \\ B_{t_2}^{uu} & B_{t_2}^{ud} & B_{t_2}^{du} & B_{t_2}^{dd} \\ \delta(F_{t_0} - L_{t_1}^u) & \delta(F_{t_0} - L_{t_1}^u) & \delta(F_{t_0} - L_{t_1}^d) & \delta(F_{t_0} - L_{t_1}^d) \end{bmatrix} \tag{57}$$

where the $B_{t_2}^{ij}$ is the (random) value of the t_3 maturity discount bond at time t_2 . This value will be state-dependent at t_2 because the bond matures one period later, at time t_3 . Looked at from time t_0 , this value will be random. Clearly, with this D matrix we have simplified the notation significantly. We are using only four states of the world, expressing the forward rate $F(t_0, t_2)$ simply as F_{t_0} , and the $B(t_2, t_3)^{ij}$ simply as $B_{t_2}^{ij}$.

If the FRA, the savings account, and the two bonds do not admit any arbitrage opportunities, the fundamental theorem of asset pricing permits the following linear representation:

$$\begin{bmatrix} 1 \\ B(t_0, t_2) \\ B(t_0, t_3) \\ 0 \end{bmatrix} = \begin{bmatrix} R_{t_2}^{uu} & R_{t_2}^{ud} & R_{t_2}^{du} & R_{t_2}^{dd} \\ 1 & 1 & 1 & 1 \\ B_{t_2}^{uu} & B_{t_2}^{ud} & B_{t_2}^{du} & B_{t_2}^{dd} \\ \delta(F_{t_0} - L_{t_1}^u) & \delta(F_{t_0} - L_{t_1}^u) & \delta(F_{t_0} - L_{t_1}^d) & \delta(F_{t_0} - L_{t_1}^d) \end{bmatrix} \begin{bmatrix} Q^{uu} \\ Q^{ud} \\ Q^{du} \\ Q^{dd} \end{bmatrix} \tag{58}$$

where $\{Q^{ij}, i, j = u, d\}$ are the four state prices for period t_2 . Under the no-arbitrage condition, the latter exist and are *positive*

$$Q^{ij} > 0 \tag{59}$$

for all states i, j .¹⁵

This matrix equation incorporates the ideas that (1) the fair market value of an FRA is zero at initiation, (2) 1 dollar is to be invested in the savings account originally, and (3) the bonds are default-free. They mature at times t_2 and t_3 . The $R_{t_2}^{i,j}$, finally, represent the gross returns to the savings account as of time t_2 . Because the interest rate that applies to time t_i is paid in arrears,

¹⁴ This table can be regarded as the second step in a non-recombining binomial tree.

¹⁵ As usual, we are eliminating the time subscript on the state prices, since it is clear by now that we are dealing with time- t_2 payoffs.

at time $t_i + \delta$, we can express these gross returns as functions of the underlying Libor rates in the following way:

$$R_{t_2}^{uu} = R_{t_2}^{ud} = (1 + \delta L_{t_0})(1 + \delta L_{t_1}^u) \quad (60)$$

and

$$R_{t_2}^{dd} = R_{t_2}^{du} = (1 + \delta L_{t_0})(1 + \delta L_{t_1}^d) \quad (61)$$

We now present the Libor market model and the associated measure change methodology within this simple framework. The framework can be used to conveniently display most of the important tools and concepts that we need for fixed-income engineering. The first important concept that we need is the *forward measure* introduced in Chapter 11.

4.2. Normalization and Forward Measure

To obtain the t_2 and the t_3 forward measures, it is best to begin with a *risk-neutral probability*, and show why it is not a good working measure in the fixed-income environment described earlier. We can then show how to *convert* the risk-neutral probability to a desired forward measure explicitly.

4.2.1. Risk-Neutral Measure Is Inconvenient

As usual, define the risk-neutral measure $\{\tilde{p}_{ij}\}$ using the *first row* of the matrix equation:

$$1 = R_{t_2}^{uu}Q^{uu} + R_{t_2}^{ud}Q^{ud} + R_{t_2}^{du}Q^{du} + R_{t_2}^{dd}Q^{dd} \quad (62)$$

Relabel

$$\tilde{p}_{uu} = R_{t_2}^{uu}Q^{uu} \quad (63)$$

$$\tilde{p}_{ud} = R_{t_2}^{ud}Q^{ud} \quad (64)$$

$$\tilde{p}_{du} = R_{t_2}^{du}Q^{du} \quad (65)$$

$$\tilde{p}_{dd} = R_{t_2}^{dd}Q^{dd} \quad (66)$$

The $\{\tilde{p}_{ij}\}$ then have the characteristics of a probability distribution, and they can be exploited with the associated Martingale equality.

We know from Chapter 11 that, under the condition that every asset's price is arbitrage-free, $\{Q^{ij}, i, j = u, d\}$ exist and are all positive, and \tilde{p}_{ij} will be the risk-neutral probabilities. Then, by using the last row of the system in equation (58) we can write the following equality:

$$\begin{aligned} 0 = & \left[\delta(F_{t_0} - L_{t_1}^u) \frac{1}{R_{t_2}^{uu}} \right] \tilde{p}_{uu} + \left[\delta(F_{t_0} - L_{t_1}^u) \frac{1}{R_{t_2}^{ud}} \right] \tilde{p}_{ud} + \left[\delta(F_{t_0} - L_{t_1}^d) \frac{1}{R_{t_2}^{du}} \right] \tilde{p}_{du} \\ & + \left[\delta(F_{t_0} - L_{t_1}^d) \frac{1}{R_{t_2}^{dd}} \right] \tilde{p}_{dd} \end{aligned} \quad (67)$$

Here, $(F_{t_0} - L_{t_1}^i), i = u, d$ are “normalized” so that Q^{ij} can be replaced by the respective \tilde{p}_{ij} . Note that in this equation, F_{t_0} is determined at time t_0 , and can be factored out. Grouping and rearranging, we get

$$F_{t_0} = \frac{\left(\left[L_{t_1}^u \frac{1}{R_{t_2}^{uu}} \right] \tilde{p}_{uu} + \left[L_{t_1}^u \frac{1}{R_{t_2}^{ud}} \right] \tilde{p}_{ud} + \left[L_{t_1}^d \frac{1}{R_{t_2}^{du}} \right] \tilde{p}_{du} + \left[L_{t_1}^d \frac{1}{R_{t_2}^{dd}} \right] \tilde{p}_{dd} \right)}{\left(\frac{1}{R_{t_2}^{uu}} \tilde{p}_{uu} + \frac{1}{R_{t_2}^{ud}} \tilde{p}_{ud} + \frac{1}{R_{t_2}^{du}} \tilde{p}_{du} + \frac{1}{R_{t_2}^{dd}} \tilde{p}_{dd} \right)} \quad (68)$$

This can be written using the expectation operator

$$F_{t_0} = \frac{1}{E_{t_0}^{\tilde{P}} \left[\frac{1}{R_{t_2}} \right]} E_{t_0}^{\tilde{P}} \left[\frac{L_{t_1}}{R_{t_2}} \right] \quad (69)$$

According to this last equality, if R_{t_2} is a random variable and is not independent of L_{t_1} ,¹⁶ it cannot be moved outside the expectation operator. In other words, for general t ,

$$F(t, t_i) \neq E_t^{\tilde{P}} [L_{t_i}] \quad t < t_i \quad (70)$$

That is to say, under the risk-neutral measure, \tilde{P} , the forward rate for time t_i is a *biased* “forecast” of the future Libor rate L_{t_i} . In fact, it is not very difficult to see that the $E_t^{\tilde{P}} [L_{t_i}]$ is the *futures* rate that will be determined by, say, a Eurodollar contract at time t . The “bias” in the forward rate, therefore, is associated with the convexity adjustment.

Another way of putting it is that F_t is *not* a Martingale with respect to the risk-neutral probability \tilde{P} , and that a discretized stochastic difference equation that represents the dynamics of F_t will, in general, have a trend:

$$F_{t+\Delta} - F_t = a(F_t, t)F_t\Delta + \sigma(F_t, t)F_t[W_{t+\Delta} - W_t] \quad (71)$$

where $a(F_t, t)$ is the nonzero expected rate of change of the forward rate under the probability \tilde{P} .

The fact that F_t is *not* a Martingale with respect to probability \tilde{P} makes the risk-neutral measure an inconvenient working tool for pricing and risk management in the fixed-income sector. Before we can use equation (71), we need to *calibrate* the drift factor $a(\cdot)$. This requires first obtaining a functional form for the drift under the probability \tilde{P} . The original HJM article *does* develop a functional form for such drifts using *continuously compounded* instantaneous forward rates. But, this creates an environment quite different from Libor-driven markets and the associated actuarial rates L_{t_i} used here.¹⁷

On the other hand, we will see that in the interest rate sector, arbitrage-free drifts become much easier to calculate if we use the Forward Libor Model and switch to appropriate forward measures.

4.2.2. The Forward Measure

Consider defining a new set of probabilities for the states under consideration by using the default-free zero-coupon bond that matures at time t_2 . First, we present the simple case. Use the second row of the system in equation (58):

$$B(t_0, t_2) = Q^{uu} + Q^{ud} + Q^{du} + Q^{dd} \quad (72)$$

and then divide every element by $B(t_0, t_2)$. Renaming, we get the forward t_2 -measure \tilde{P}^{t_2}

$$1 = \tilde{p}_{uu}^{t_2} + \tilde{p}_{ud}^{t_2} + \tilde{p}_{du}^{t_2} + \tilde{p}_{dd}^{t_2} \quad (73)$$

where the probability of each state is obtained by scaling the corresponding Q^{ij} using the time t_0 price of the corresponding bond:

$$\tilde{p}_{ij}^{t_2} = \frac{Q^{ij}}{B(t_0, t_2)} \quad (74)$$

¹⁶ Although we know, in general, that this will not be the case since these are returns to short-term investments in Eurocurrency markets.

¹⁷ Further, new technical problems appear that make the continuous compounding numerically unstable.

It is important to index the forward measure with the superscript, t_2 , in these fixed-income models, as *other* forward measures would be needed for other forward rates. The superscript is a nice way of keeping track of the measure being used. For some instruments, these measures have to be switched sequentially.

As discussed in Chapter 11, using the t_2 -forward measure we can price any asset C_t , with time- t_2 payoffs $C_{t_2}^{ij}$

$$C_{t_0} = [B(t_0, t_2)C_{t_2}^{uu}] \tilde{p}_{uu}^{t_2} + [B(t_0, t_2)C_{t_2}^{ud}] \tilde{p}_{ud}^{t_2} + [B(t_0, t_2)C_{t_2}^{du}] \tilde{p}_{du}^{t_2} + [B(t_0, t_2)C_{t_2}^{dd}] \tilde{p}_{dd}^{t_2} \quad (75)$$

This implies that, for an asset that settles at time T and has no other payouts, the *general* pricing equation is given by

$$C_t = B(t, T) E_t^{\tilde{P}^T} [C_T] \quad (76)$$

where \tilde{P}^T is the associated T -forward measure and where C_T is the time- T payoff. According to this equality, it is the ratio

$$Z_t = \frac{C_t}{B(t, T)} \quad (77)$$

which is a Martingale with respect to the measure \tilde{P}^T . In fact, $B(t, T)$ being the *discount* factor for time T , and, hence, being less than one, Z_t is nothing more than the T -forward value of the C_t . This means that the forward measure \tilde{P}^T operates in terms of Martingales that are measured in time- T dollars. The advantage of the forward \tilde{P}^T measure becomes clear if we apply the same transformation to price the FRA as was done earlier for the case the of risk-neutral measure.

4.2.3. Arbitrage-Free SDEs for Forward Rates

To get arbitrage-free dynamics for forward rates, we now go back to the simple model in equation (58). Divide the fourth row of the system by $B(t_0, t_2)$ and rearrange,

$$\begin{aligned} \frac{F_{t_0}}{B(t_0, t_2)} [Q^{uu} + Q^{ud} + Q^{du} + Q^{dd}] &= \frac{L_{t_1}^u}{B(t_0, t_2)} Q^{uu} + \frac{L_{t_1}^u}{B(t_0, t_2)} Q^{ud} \\ &+ \frac{L_{t_1}^d}{B(t_0, t_2)} Q^{du} + \frac{L_{t_1}^d}{B(t_0, t_2)} Q^{dd} \end{aligned} \quad (78)$$

Now, as done in Chapter 11, substitute the t_2 -forward measure into this equation using the equality:

$$\tilde{p}_{ij}^{t_2} = \frac{1}{B(t_0, t_2)} Q^{ij}$$

The equation becomes

$$F_{t_0} = [L_{t_1}^u] \tilde{p}_{uu}^{t_2} + [L_{t_1}^u] \tilde{p}_{ud}^{t_2} + [L_{t_1}^d] \tilde{p}_{du}^{t_2} + [L_{t_1}^d] \tilde{p}_{dd}^{t_2} \quad (79)$$

Extending this to the general case of m discrete states

$$F_{t_0} = \sum_{i=1}^m L_{t_1}^i \tilde{p}_i^{t_2} \quad (80)$$

This is clearly the expectation

$$F_{t_0} = E_{t_0}^{\tilde{P}^{t_2}} [L_{t_1}] \quad (81)$$

This means that, under the measure \tilde{P}^{t_2} , the forward rate for the period $[t_1, t_2]$ will be an *unbiased estimate* of the corresponding Libor rate.

Consequently, switching to the general notation of (t, T) , the process

$$F_t = F(t, T, T + \delta) \quad (82)$$

will be a Martingale under the $(T + \delta)$ -forward measure $\tilde{P}^{T+\delta}$. Assuming that the errors due to discretization are small, its dynamics can be described by a (discretized) SDE over small intervals of length Δ ¹⁸

$$F_{t+\Delta} - F_t = \sigma_t F_t \Delta W_t \quad (83)$$

where W_t is a Wiener process under the measure $\tilde{P}^{T+\delta}$. ΔW_t is the Wiener process increment:

$$\Delta W_t = W_{t+\Delta} - W_t \quad (84)$$

This (approximate) equation has *no* drift component since, by arbitrage arguments, and writing for the general t, T , we have

$$1 + \delta F(t, T) = \frac{B(t, T)}{B(t, T + \delta)} \quad (85)$$

It is clear from the normalization arguments of Chapter 11 that, under the measure $\tilde{P}^{T+\delta}$ and normalization by $B(t, T + \delta)$, the ratio on the right-hand side of this equation is a Martingale with respect to $\tilde{P}^{T+\delta}$. This makes the corresponding forward rate a Martingale, so that the implied SDE will have no drift.

However, note that the forward rate for the period $[T - \delta, T]$ given by

$$1 + \delta F(t, T - \delta) = \frac{B(t, T - \delta)}{B(t, T)} \quad (86)$$

is *not* a Martingale under the *same* forward measure $\tilde{P}^{T+\delta}$. Instead, this forward rate is a Martingale under its *own* measure \tilde{P}^T which requires normalization by $B(t, T)$. Thus, we get a critical result for the Forward Libor Model:

Each forward rate $F(t, T_i)$ admits a Martingale representation under *its own* forward measure $\tilde{P}^{T_i+\delta}$.

This means that each forward rate dynamics can be approximated individually by a difference equation with no drift given the proper normalization. The only parameter that would be needed to characterize such dynamics is the corresponding forward rate volatility.

4.3. Arbitrage-Free Dynamics

The previous section discussed the dynamics of forward rates under their *own* forward measure. We now show what happens when we use *one* forward measure for *two* forward rates that apply to two consecutive periods. Then, one of the forward rates has to be evaluated under a measure different from its own, and the Martingale dynamics will be broken. Yet, we will be able to obtain the new drift.

¹⁸ See Suggested Reading at the end of the chapter for a source on discretization errors and their relevance here.

To keep the issue as simple as possible, we continue with the basic model in equation (58), except we add one more time period so that we can work with *two* nontrivial forward rates and their respective forward measures. This is the simplest setup within which we can show how *measure-change* technology can be implemented. Using the forward measures introduced earlier and shown in Figure 13-4, we can now define the following forward rate dynamics for the *two* forward Libor processes $\{F(t_0, t_1), F(t_0, t_2)\}$ under consideration. The first will be a Martingale under the normalization with $B(t_0, t_2)$, whereas the second will be a Martingale under the normalization with $B(t_0, t_3)$. This means that \tilde{P}^{t_2} and \tilde{P}^{t_3} are the forward Libor processes' "own" measures.

Altogether, it is important to realize that during the following discussion we are working with a very simple example involving only four time periods, t_0, t_1, t_2 , and t_3 . We start with the arbitrage-free "dynamics" of the forward rate $F(t_0, t_2)$. In our simplified setup, we will observe only two future values of this forward rate at times t_1 and t_2 . These are given by

$$\begin{aligned} F(t_1, t_2) - F(t_0, t_2) &= \sigma_2 F(t_0, t_2) \Delta W_{t_1}^{t_3} \\ F(t_2, t_2) - F(t_1, t_2) &= \sigma_2 F(t_1, t_2) \Delta W_{t_2}^{t_3} \end{aligned} \tag{87}$$

The superscript in $\Delta W_{t_i}^{t_3}$, $i = 1, 2$, indicates that the Wiener process increments have zero mean under the probability \tilde{P}^{t_3} . These equations show how the "current" value of the forward rate $F(t_0, t_2)$ first changes to become $F(t_1, t_2)$ and then ends up as $F(t_2, t_2)$. The latter is also L_{t_2} .

For the "nearer" forward rate $F(t_0, t_1)$, we need only one equation¹⁹ defined under the normalization with the bond $B(t_0, t_2)$ (i.e., the \tilde{P}^{t_2} measure) and the associated zero drift:

$$F(t_1, t_1) - F(t_0, t_1) = \sigma_1 F(t_0, t_1) \Delta W_{t_1}^{t_2} \tag{88}$$

Similarly, the superscript in $\Delta W_{t_1}^{t_2}$ indicates that this Wiener process increment has zero mean under the probability \tilde{P}^{t_2} . Here, the $F(t_1, t_1)$ is also the Libor rate L_{t_1} . We reemphasize that each dynamic is defined under a different probability measure. Under these *different* forward

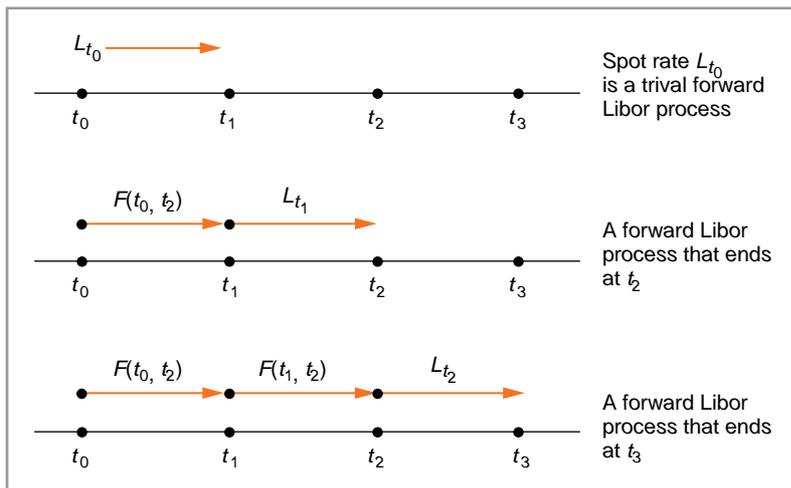


FIGURE 13-4

¹⁹ This forward rate process will terminate at t_2 .

measures, each forward Libor process behaves like a Martingale.²⁰ Consequently, there are no drift terms in either equation.

Fortunately, as long as we can work with these equations *separately*, no arbitrage-free *drift terms* need to be estimated or calibrated. The only parameters we need to determine are the volatilities of the two forward rates: σ_2 for the forward rate $F(t_0, t_2)$, and σ_1 for the forward rate $F(t_0, t_1)$.²¹

In fact, each Wiener increment has a zero expectation under its own measure. For example, the Wiener increments of the two forward rates will satisfy, for time $t_0 < t_1$

$$E_{t_0}^{\tilde{P}^{t_2}} [\Delta W_{t_1}^{t_2}] = 0 \quad (89)$$

and

$$E_{t_0}^{\tilde{P}^{t_3}} [\Delta W_{t_1}^{t_3}] = 0 \quad (90)$$

Yet, when we evaluate the expectations under \tilde{P}^{t_2} , we get

$$E_{t_0}^{\tilde{P}^{t_2}} [\Delta W_{t_1}^{t_2}] = 0 \quad (91)$$

and

$$E_{t_0}^{\tilde{P}^{t_2}} [\Delta W_{t_1}^{t_3}] = \lambda_{t_0}^{t_2} \Delta \neq 0 \quad (92)$$

Here, $\lambda_{t_0}^{t_2}$ is a *mean correction* that needs to be made because we are evaluating the Wiener increment under a measure different from its own forward measure \tilde{P}^{t_3} . This, in turn, means that the dynamics for $F(t_0, t_2)$ lose their Martingale characteristic.

We will now comment on the second moments, variances, and covariances. Each Wiener increment is assumed to have the same variance under the two measures. The Girsanov theorem ensures that this is true in continuous time. In discrete time, this holds as an approximation. Finally, we are operating in an environment where there is only *one* factor.²² So, the Wiener process increments defined under the two forward measures *will* be exactly correlated if they belong to the same time period. In other words, although their means are different, we can assume that, approximately, their *covariance* would be Δ :

$$E^{\tilde{P}^{t_3}} [\Delta W_t^{t_3} \Delta W_t^{t_2}] = E^{\tilde{P}^{t_2}} [\Delta W_t^{t_3} \Delta W_t^{t_2}] = \Delta \quad (93)$$

Similar equalities will hold for the variances as well.²³

4.3.1. Review

The results thus far indicate that for the pricing and risk managing of equity-linked assets, the risk-neutral measure \tilde{P} may be quite convenient since it is easily adaptable to lognormal

²⁰ Again, we are assuming that the discretization bias is negligible.

²¹ Note that according to the characterization here, the volatility parameters are not allowed to vary over time. This assumption can be relaxed somewhat, but we prefer this simple setting, since most market applications are based on constant volatility characterization as well.

²² As a reminder, a one-factor model assumes that all random processes under consideration have the same unpredictable component up to a factor of proportionality. In other words, the correlation coefficients between these processes would be one.

²³ These relations will hold as Δ goes to zero.

models where the arbitrage-free drifts are simple and known functions of the risk-free interest rate. As far as equity products are concerned, the assumption that short rates are constant is a reasonable approximation, especially for short maturities. Yet, for contracts written on future values of interest rates (rather than on asset prices), the use of the \tilde{P} leads to complex arbitrage-free dynamics that cannot be captured easily by Martingales and, hence, the corresponding arbitrage-free drift terms may be difficult to calibrate.

Appropriate forward measures, on the other hand, result in Martingale equalities and lead to dynamics convenient for the calculation of arbitrage-free drifts, even when they are not zero. Forward (and swap) measures are the proper working probabilities for fixed-income environments.

4.4. A Monte Carlo Implementation

Suppose we want to generate Monte Carlo “paths” from the two discretized SDEs for two forward rates, $F(t_i, t_1)$ and $F(t_i, t_2)$,

$$F(t_i, t_1) - F(t_{i-1}, t_1) = \sigma_1 F(t_{i-1}, t_1) \Delta W_{t_i}^{t_2} \quad (94)$$

$$F(t_i, t_2) - F(t_{i-1}, t_2) = \sigma_2 F(t_{i-1}, t_2) \Delta W_{t_i}^{t_3} \quad (95)$$

where $i = 1, 2$ for the second equation, and $i = 1$ for the first.

It is easy to generate *individual* paths for the two forward rates separately by using these Martingale equations defined under their own forward measures. Consider the following approach.

Suppose volatilities σ_1 and σ_2 can be observed in the market. We select two random variables $\{\Delta W_1^3, \Delta W_2^3\}$ from the distribution

$$\Delta W_i^3 \sim N(0, \Delta) \quad (96)$$

with a pseudo-random number generator, and then calculate, sequentially, the randomly generated forward rates in the following order, starting with the *observed* $F(t_0, t_2)$

$$F(t_1, t_2)^1 = F(t_0, t_2) + \sigma_2 F(t_0, t_2) \Delta W_1^3 \quad (97)$$

$$F(t_2, t_2)^1 = F(t_1, t_2)^1 + \sigma_2 F(t_1, t_2)^1 \Delta W_2^3 \quad (98)$$

where the superscript on the left-hand side indicates that these values are for the first Monte Carlo trajectory. Proceeding sequentially, all the terms on the right-hand side will be known. This gives the first simulated “path” $\{F(t_0, t_2), F(t_1, t_2)^1, F(t_2, t_2)^1\}$. We can repeat this algorithm to obtain M such paths for potential use in pricing.

What does this imply for the other forward Libor process $F(t_0, t_1)$? Can we use the *same* randomly generated random variable ΔW_1^3 in the Martingale equation for $F(t, t_1)$, and obtain the first “path” $\{F(t_0, t_1), F(t_1, t_1)^1\}$ from

$$F(t_1, t_1) = F(t_0, t_1) + \sigma_1 F(t_0, t_1) \Delta W_1^3 \quad (99)$$

The answer is no. As mentioned earlier, the Wiener increments $\{\Delta W_{t_1}^{t_2}\}$ have zero mean only under the probability \tilde{P}^{t_2} . But, the first set of random variables was selected using the measure \tilde{P}^{t_3} . Under \tilde{P}^{t_2} , these random variables do not have zero mean, but are distributed as

$$N(\lambda_{t_0}^{t_2} \Delta, \Delta) \quad (100)$$

Thus, if we use the same ΔW_1^3 in equation (99), then we need to correct for the term $\lambda_{t_0}^{t_2} \Delta$. To do this, we need to calculate the numerical value of $\lambda_{t_0}^{t_2}$.

Once this is done, the dynamics for $F(t_0, t_1)$ can be written as

$$F(t_1, t_1) = F(t_0, t_1) - \sigma_1 F(t_0, t_1)(\lambda_{t_0}^{t_2} \Delta) + \sigma_1 F(t_0, t_1) \Delta W_{t_1}^{t_3} \quad (101)$$

To see why this is so, take the expectation under \tilde{P}^{t_2} on the right-hand side and use the information in equation (100):

$$E_{t_1}^{\tilde{P}^{t_2}} [F(t_0, t_1) - \sigma_1 F(t_0, t_1)(\lambda_{t_0}^{t_2} \Delta) + \sigma_1 F(t_0, t_1) \Delta W_{t_1}^{t_3}] = F(t_0, t_1) \quad (102)$$

Thus, we get the correct result under the \tilde{P}^{t_2} , after the mean correction. It is obvious that we need to determine these correction factors before the randomly generated Brownian motion increments can be used in all equations.

Yet, notice the following simple case. If the instrument under consideration has additive cash flows where each cash flow depends on a single forward rate, then individual zero-drift equations *can* be used separately to generate paths. This applies for several liquid instruments. For example, FRAs and especially swaps have payment legs that depend on one Libor rate only. Individual zero-drift equations can be used for valuing each leg separately, and then these values can be added using *observed* zero-coupon bond prices, $B(t, T_i)$. However, this cannot be done in the case of constant maturity swaps, for example, because *each settlement leg will depend nonlinearly on more than one forward rate*.

We now discuss further how mean corrections can be conducted so that all forward rates are projected using a single forward measure. This will permit pricing instruments where individual cash flows depend on more than one forward rate.

5. Measure Change Technology

We introduce a relatively general framework and then apply the results to the simple example shown previously. Basically, we need three previously developed relationships. We let t_i obey

$$t_0 < t_1 < \dots < t_n = T \quad (103)$$

with

$$t_i - t_{i-1} = \delta \quad (104)$$

denote settlement dates of some basic interest rate swap structure and limit our attention to forward rates for successive forward loans contracted to begin at t_i , and paid at t_{i+1} . An example is shown in Figure 13-4.

- *Result 1*

The forward rate at time t , for a Libor-based forward loan that starts at time t_i and ends at time $t_i + \delta$, denoted by $F(t, t_i)$, admits the following arbitrage relationship:

$$1 + F(t, t_i)\delta = \frac{B(t, t_i)}{B(t, t_{i+1})} \quad t < t_i \quad (105)$$

where, as usual, $B(t, t_i)$ and $B(t, t_{i+1})$ are the time- t prices of default-free zero coupon bonds that mature at times t_i and t_{i+1} , respectively.

The left side of this equality is a gross forward return. The right side, on the other hand, is a traded asset price, $B(t, t_i)$, normalized by another asset price, $B(t, t_{i+1})$. Hence, the ratio will be a Martingale under a proper measure—here, the forward measure denoted by $\tilde{P}^{t_{i+1}}$.

- *Result 2*

In a discrete state setting with k states of the world, assuming that all asset prices are arbitrage-free, and that the time- t_i state prices $Q^j, j = 1, \dots, k$, with $0 < Q^j$ exist, the time- t_i values of the forward measure \tilde{P}^{t_i} are given by²⁴

$$\tilde{p}_1^{t_i} = \frac{1}{B(t, t_i)} Q^1, \quad \tilde{p}_2^{t_i} = \frac{1}{B(t, t_i)} Q^2, \dots, \tilde{p}_k^{t_i} = \frac{1}{B(t, t_i)} Q^k \quad (106)$$

These probabilities satisfy:

$$\tilde{p}_1^{t_i} + \tilde{p}_2^{t_i} + \dots + \tilde{p}_k^{t_i} = 1 \quad (107)$$

and

$$0 < \tilde{p}_j^{t_i} \quad \forall j$$

Note that the proportionality factors used to convert Q^j into $\tilde{p}_j^{t_i}$ are equal across j .

- *Result 3*

In the same setting, the time t_i -probabilities associated with the t_{i+1} forward measure $\tilde{P}^{t_{i+1}}$ are given by:

$$\tilde{p}_1^{t_{i+1}} = \frac{B(t_i, t_{i+1})^1}{B(t, t_{i+1})} Q^1, \quad \tilde{p}_2^{t_{i+1}} = \frac{B(t_i, t_{i+1})^2}{B(t, t_{i+1})} Q^2, \dots, \tilde{p}_k^{t_{i+1}} = \frac{B(t_i, t_{i+1})^k}{B(t, t_{i+1})} Q^k \quad (108)$$

where the $B(t_i, t_{i+1})^j$ are the state dependent values of the t_{i+1} -maturity bond at time t_i . Here, the bond that matures at time t_{i+1} is used to normalize the cash flows for time t_i . Since the maturity date is t_{i+1} , the $B(t_i, t_{i+1})^j$ are not constant at t_i . The factors used to convert $\{Q^j\}$ into $\tilde{P}^{t_{i+1}}$ cease to be constant as well.

We use these results in discussing the mechanics of measure changes. Suppose we need to price an instrument whose value depends on *two* forward Libor processes, $F(t, t_i)$ and $F(t, t_{i+1})$, simultaneously. We know that each process is a Martingale and obeys an SDE with zero-drift under its *own* forward measure.

Consider a one-factor setting, where a single Wiener process causes fluctuations in the two forward rates. Suppose that in this setting, starting from time t , with $t < t_i, i = 1, \dots, n$, a small time interval denoted by h passes with $t + h < t_i$. By imposing a Gaussian volatility structure, we can write down the *individual* discretized arbitrage-free dynamics for two successive forward rates $F(t, t_i)$ and $F(t, t_{i+1})$ as

$$F(t + h, t_i) - F(t, t_i) = \sigma^i F(t, t_i) \Delta W_{t+h}^1 \quad (109)$$

and

$$F(t + h, t_{i+1}) - F(t, t_{i+1}) = \sigma^{i+1} F(t, t_{i+1}) \Delta W_{t+h}^2 \quad (110)$$

Changes in these forward rates have zero mean under their own forward measure and, hence, are written with zero drift. This means that the unique *real world* Wiener process W_{t+h} is now denoted by ΔW_{t+h}^1 and ΔW_{t+h}^2 in the two equations. These are normally distributed, with mean zero and variance h *only* under their own forward measures, $\tilde{P}^{t_{i+1}}$ and the $\tilde{P}^{t_{i+2}}$.

²⁴ The reader will note the slight change in notation, which is dictated by the environment relevant in this section.

The superscript in W_{t+h}^1 and W_{t+h}^2 expresses the t_{i+1} and t_{i+2} forward probability measures, respectively.²⁵ Finally, note how we simplify the characterization of volatilities and assume that they are constant over time.

The individual Martingale dynamics are very convenient from a financial engineering point of view. The respective drift components are zero and, hence, they need not be modeled during pricing. The only major task of the market practitioner is to get the respective volatilities σ^i and σ^{i+1} .

However, some securities' prices may depend on more than one forward rate in a nonlinear fashion and their value may have to be calculated as an expectation under *one single* measure. For example, suppose a security's price, S_t , depends on $F(t, t_i)$ and $F(t, t_{i+1})$ through a pricing relation such as:

$$S_t = E_t^{\tilde{P}} [g(F(t, t_i), F(t, t_{i+1}))] \tag{111}$$

where $g(\cdot)$ is a known nonlinear function. Then, the expectation has to be calculated under *one* measure only. This probability can be *either* the time- t_{i+1} , or the time- t_{i+2} forward measure. We then have to choose a forward rate equation with Martingale dynamics and carry out a mean correction to get the correct arbitrage-free dynamics for the other. The forward measure of one of the Martingale relationships is set as the *working probability distribution*, and the other equation(s) is obtained in terms of this unique probability by going through successive measure changes. We discuss this in detail below.

5.1. The Mechanics of Measure Changes

We have the following expectations concerning ΔW_{t+h}^1 and ΔW_{t+h}^2 , defined in (109) and (110)

$$E_t^{\tilde{P}^{t_{i+1}}} [\Delta W_{t+h}^1] = 0 \tag{112}$$

$$E_t^{\tilde{P}^{t_{i+2}}} [\Delta W_{t+h}^2] = 0 \tag{113}$$

Under their own forward measure, each Wiener increment has zero expectation. If we select $\tilde{P}^{t_{i+2}}$ as our working measure, one of these equalities has to change. We would have²⁶

$$E_t^{\tilde{P}^{t_{i+2}}} [\Delta W_{t+h}^1] = \lambda_t h \tag{114}$$

$$E_t^{\tilde{P}^{t_{i+2}}} [\Delta W_{t+h}^2] = 0 \tag{115}$$

The value of λ_t gives the correction factor that we need to use in order to obtain the correct arbitrage-free dynamics, if the working measure is $\tilde{P}^{t_{i+2}}$. Calculating this factor implies that we can change measures in the dynamics of $F(t, t_i)$.

We start with the original expectation:

$$E_t^{\tilde{P}^{t_{i+1}}} [\Delta W_{t+h}^1] = \sum_{j=1}^k \Delta W_{t+h}^{1j} \tilde{p}_j^{t_{i+1}} = 0 \tag{116}$$

where the $\tilde{p}_j^{t_{i+1}}$ are the probabilities associated with the individual states $j = 1, \dots, k$. Now, using the identity,

$$\frac{B(t, t_{i+2})B(t_i, t_{i+2})^j}{B(t, t_{i+2})B(t_i, t_{i+2})^j} \equiv 1 \tag{117}$$

²⁵ Remember that the time- t_i forward rate will have a time- t_{i+1} forward measure as its own measure.

²⁶ In the general case where there are m forward rates, all equations except one will change.

we rewrite the expectation as

$$E_t^{\tilde{P}^{t_{i+1}}} [\Delta W_{t+h}^1] = \sum_{j=1}^k (\Delta W_{t+h}^{1j}) \left[\frac{B(t, t_{i+2}) B(t_i, t_{i+2})^j}{B(t, t_{i+2}) B(t_i, t_{i+2})^j} \right] \tilde{p}_j^{t_{i+1}} \quad (118)$$

We regroup and use the definition of the t_{i+1} and t_{i+2} forward measures as implied by Result 3

$$\tilde{p}_j^{t_{i+1}} = \frac{B(t_i, t_{i+1})^j}{B(t, t_{i+1})} Q^j \quad (119)$$

and

$$\tilde{p}_j^{t_{i+2}} = \frac{B(t_i, t_{i+2})^j}{B(t, t_{i+2})} Q^j \quad (120)$$

Rescaling the Q^j using appropriate factors, equation (118) becomes

$$\sum_{j=1}^k (\Delta W_{t+h}^{1j}) \left[\frac{B(t, t_{i+2})}{B(t, t_{i+1})} \frac{B(t_i, t_{i+1})^j}{B(t_i, t_{i+2})^j} \right] \tilde{p}_j^{t_{i+2}} = 0 \quad (121)$$

Note that the probabilities switch as the factors that were applied to the Q^j changed. The superscript in W_{t+h}^1 does not change.

The next step in the derivation is to try to “recognize” the elements in this expectation. Using Result 1, we recognize the equality

$$1 + \delta F(t_i, t_{i+1})^j = \frac{B(t_i, t_{i+1})^j}{B(t_i, t_{i+2})^j} \quad (122)$$

Replacing, eliminating the j -independent terms, and rearranging gives

$$\sum_{j=1}^k (\Delta W_{t+h}^{1j}) (1 + \delta F(t_i, t_{i+1})^j) \tilde{p}_j^{t_{i+2}} = 0 \quad (123)$$

Now, multiplying through, this leads to

$$\sum_{j=1}^k (\Delta W_{t+h}^{1j}) \tilde{p}_j^{t_{i+2}} = - \left(\sum_{j=1}^k (\Delta W_{t+h}^{1j}) F(t_i, t_{i+1})^j \tilde{p}_j^{t_{i+2}} \right) \delta \quad (124)$$

We can write this using the conditional expectation operator,

$$E_t^{\tilde{P}^{t_{i+2}}} [\Delta W_{t+h}^1] = -E_t^{\tilde{P}^{t_{i+2}}} [\Delta W_{t+h}^1 F(t_i, t_{i+1})] \delta. \quad (125)$$

In the last expression, the left-hand side is the desired expectation of the ΔW_{t+h}^1 under the new probability $\tilde{P}^{t_{i+2}}$. This expectation will *not* equal zero if the right-hand side random variables are correlated. This correlation is nonzero as long as forward rates are correlated. To evaluate the mean of ΔW_{t+h}^1 under the new probability $\tilde{P}^{t_{i+2}}$, we then have to calculate the covariance.

Let the covariance be given by $-\lambda_t h$. We have,

$$\delta E_t^{\tilde{P}^{t_{i+2}}} [\Delta W_{t+h}^1 F(t_i, t_{i+1})] = -\lambda_t h \quad (126)$$

Using the λ_t we can switch probabilities in the $F(t, t_i)$ dynamics. We start with the original Martingale dynamics:

$$F(t+h, t_i) = F(t, t_i) + \sigma^i F(t, t_i) \Delta W_{t+h}^1 \quad (127)$$

Switch by adding and subtracting $\sigma^i F(t, t_i) \lambda_t h$ to the right-hand side and regroup:

$$F(t+h, t_i) = F(t, t_i) - \sigma^i F(t, t_i) \lambda_t h + \sigma^i F(t, t_i) [\lambda_t h + \Delta W_{t+h}^1] \quad (128)$$

Let

$$\Delta W_{t+h}^2 = [\lambda_t h + \Delta W_{t+h}^1] \quad (129)$$

We have just shown that the expectation of the right-hand side of this expression equals zero under $\tilde{P}^{t_{i+2}}$. So, under the $\tilde{P}^{t_{i+2}}$ we can write the new dynamics of the $F(t, t_i)$ as

$$F(t+h, t_i) = F(t, t_i) - \sigma^i F(t, t_i) \lambda_t h + \sigma^i F(t, t_i) \Delta W_{t+h}^2 \quad (130)$$

As can be seen from this expression, the new dynamics have a nonzero drift and the $F(t, t_i)$ is *not* a Martingale under the new measure. Yet, this process is arbitrage-free and easy to exploit in Monte Carlo type approaches. Since both dynamics are expressed under the same measure, the set of equations that describe the dynamics of the two forward rates can be used in pricing instruments that depend on these forward rates. The same pseudo-random numbers can be used in the two SDEs. Finally, the reader should remember that the discussion in this section depends on the discrete approximation of the SDEs.

5.2. Generalization

A generalization of the previous heuristic discussion leads to the Forward Libor Model. Suppose the setting involves n forward rates, $F(t_0, t_i), i = 0, \dots, n-1$, that apply to loans which begin at time t_i , and end at $t_{i+1} = t_i + \delta$. The $F(t_0, t_0)$ is the trivial forward rate and is the spot Libor with tenor δ . The terminal date is t_n .

Similar to the discussion in the previous section, assume that there is a single factor.²⁷ Using the t_{i+1} forward measure we obtain arbitrage-free Martingale dynamics for each forward rate $F(t, t_i)$:

$$dF(t, t_i) = \sigma^i F(t, t_i) W_t^{i+1} \quad t \in [0, \infty) \quad (131)$$

The superscript in W_t^{i+1} implies that²⁸

$$E_t^{\tilde{P}^{t_{i+1}}} [dW_t^{i+1}] = 0 \quad (132)$$

These arbitrage-free dynamics are very useful since they do not involve any interest rate modeling and are dependent only on the correct specification of the respective volatilities. However, when more than one forward rate determines a security's payoff in a nonlinear fashion, the process may have to be written under a unique working measure.

Suppose we chose \tilde{P}^{t_n} as the working measure.²⁹ The heuristic approach discussed in the previous section can be generalized to obtain the following arbitrage-free *system* of SDEs that involve recursive drift corrections in a one-factor case:

$$dF(t, t_i) = - \left[\sigma^i F(t, t_i) \sum_{j=i}^{n-1} \frac{\delta \sigma^j F(t, t_j)}{1 + F(t, t_j) \delta} \right] dt + \sigma^i F(t, t_i) dW_t^{t_n} \quad t \in [0, \infty) \quad (133)$$

²⁷ The multi-factor model and an extensive discussion of the Forward Libor Model can be found in many texts. Musiela and Rutkowski (1998), Brigo and Mercurio (2001), and Rebonato (2002) are some examples.

²⁸ The use of a stochastic differential here is heuristic.

²⁹ Sometimes this is called the *terminal measure*.

where the superscript in the $dW_t^{t_n}$ indicates that the working measure is \tilde{P}^{t_n} . The equations in this system are expressed under this forward measure for $i = 1, \dots, n$. Yet, only the last equation has Martingale dynamics

$$dF(t, t_{n-1}) = \sigma^{n-1} F(t, t_{n-1}) dW_t^{t_n} \quad t \in [0, \infty) \quad (134)$$

All other SDEs involve successive correction factors given by the first term on the right side. It is important to realize that all terms in these factors can be observed at time t . The dynamics do not need a modeling of actual drifts.

6. An Application

The forward measure and measure change technology are relevant for the pricing of many instruments. But there is one instrument class that has recently become quite popular with market participants and that can be priced with this technology. These are constant maturity swaps (CMS). They have properties that would illustrate some subtleties of the methods used thus far. In order to price them, forward rates need to be projected *jointly*.

First, we present a reading that illustrates some of the recent interest in this instrument class.

EXAMPLE:

Institutional investors, convinced that euro-zone interest rates are about to rise, have over the past month hoovered up over US\$4bn of notes paying coupons linked to constant maturity swap (CMS) rates. Swelling demand for these products could resuscitate the ailing market in step-up callable bonds and lead to a longer-term balance in European options markets. The CMS boom is being driven by European institutional investors keen to speculate on higher European interest rates.

The CMS deal structure is fairly generic and similar engineering was in evidence earlier in 1999. The Italy issue is typical, offering investors a 4% coupon in year one and 78% of the 10-year CMS rate for the remaining 19 years. Most deals include a floor limiting the investor's downside coupon rate.

CMS-based products appear very attractive in the current yield curve environment. They offer an above market first coupon and the chance to speculate on rising interest rates. They also guarantee a minimum coupon of at least 4%. (IFR, Issue 1281).

CMS swaps are instruments that build on the plain vanilla swaps in an interesting way. In a vanilla swap, a fixed swap rate is exchanged against a floating Libor that is an interest rate relevant for that particular settlement period *only*. In a CMS swap, this will be generalized. The fixed leg is exchanged against a floating leg, but the floating leg is *not* a “one-period” rate. It is itself a multi-period swap rate that will be determined in the future.

There are many versions of such exchanges, but as an example we consider the following. Suppose one party decides to pay 4% during the next three years against receiving a 2-year swap rate that will be determined at the beginning of each one of those years. The future swap rates are unknown at time t_0 and can be considered as floating payments, except they are not floating payments that depend on the perceived volatility for that particular year *only*. They are themselves averages of one-year rates. Clearly, such swaps have significant nonlinearities and we cannot do the same engineering as in the case of a plain vanilla swap.

An example of CMS swaps is shown in Figure 13-5. The reader can see that what is being exchanged at each settlement date against a fixed payment is a floating rate that is a function of *more than one* forward rate. Under these conditions it is impossible to project individual forward

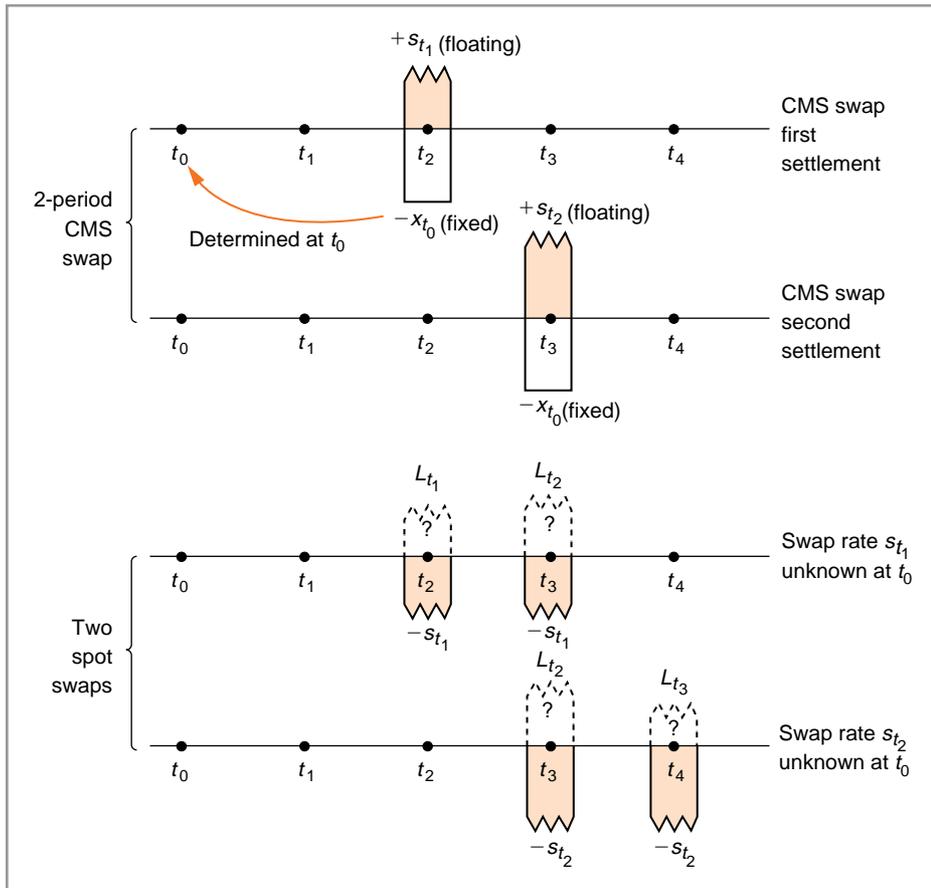


FIGURE 13-5

rates using individual zero-drift stochastic differential equations defined under different forward measures. Each leg of the CMS swap depends on more than one forward rate and these need to be projected *jointly*, under a single measure.

6.1. Another Example of Measure Change

This section provides another example to measure change technology from the FRA markets. Paid-in-arrears FRAs make time- t_{i+1} payoffs:

$$N\delta[F(t_0, t_i) - L_{t_i}] \tag{135}$$

The market-traded FRAs, on the other hand, settle at time t_i according to:

$$\frac{N\delta[F(t_0, t_i) - L_{t_i}]}{(1 + \delta L_{t_i})} \tag{136}$$

Finally, we have Libor-in-arrears FRAs that settle according to

$$N\delta[F(t_0, t_i) - L_{t_i}] \tag{137}$$

at time t_i . As we saw in Chapter 9, the Libor-in-arrear FRA payoffs settle in a “non-natural” way, since L_{t_i} -related payments would normally be received or paid at time t_{i+1} .

We now show that the paid-in-arrears FRA and market-traded FRAs lead to the same forward rate. First, remember that under the $\tilde{P}^{t_{i+1}}$ forward measure for paid-in-arrears FRAs, we have:

$$F(t_0, t_i) = E_{t_0}^{\tilde{P}^{t_{i+1}}} [L_{t_i}] \quad (138)$$

That is to say, the FRA rate $F(t_0, t_i)$ is the average of possible values the Libor rate might take:

$$F(t_0, t_i) = \sum_{j=1}^k L_{t_i}^j \tilde{p}_j^{t_{i+1}} \quad (139)$$

where j represents possible states of the world, which are assumed to be discrete and countable.

Now, consider the settlement amount of market-traded FRAs:

$$\frac{N\delta[F(t_0, t_i) - L_{t_i}]}{(1 + \delta L_{t_i})} \quad (140)$$

Would the forward rate implied by this contract be the same as the paid-in-arrears FRAs?

The answer is yes. Using the measure change technology, we discuss how this can be shown. The idea is to begin with the expectation of this settlement amount under the \tilde{P}^{t_i} measure, and show that it leads to the same forward rate. Thus, begin with

$$E_{t_0}^{\tilde{P}^{t_i}} \left[\frac{N\delta[F(t_0, t_i) - L_{t_i}]}{(1 + \delta L_{t_i})} \right] \quad (141)$$

Setting this equal to zero, and rearranging, leads to the pricing equation

$$F(t_0, t_i) = \frac{E_{t_0}^{\tilde{P}^{t_i}} \left[\frac{N\delta L_{t_i}}{(1 + \delta L_{t_i})} \right]}{E_{t_0}^{\tilde{P}^{t_i}} \left[\frac{N\delta}{(1 + \delta L_{t_i})} \right]} \quad (142)$$

Now we switch measures on the right-hand side of equation (142). We have two expectations and we will switch measures in both of them. But first, let $N = 1$ and, similarly, $\delta = 1$.

Consider the numerator:

$$E_{t_0}^{\tilde{P}^{t_i}} \left[\frac{L_{t_i}}{(1 + L_{t_i})} \right] = \sum_{j=1}^k \frac{L_{t_i}^j}{(1 + L_{t_i}^j)} \tilde{p}_j^{t_i} \quad (143)$$

We know that for time t_i

$$\begin{aligned} \tilde{p}_j^{t_i} &= \frac{1}{B(t_0, t_i)} Q^j \\ \tilde{p}_j^{t_{i+1}} &= \frac{B(t_i, t_{i+1})^j}{B(t_0, t_{i+1})} Q^j \end{aligned} \quad (144)$$

Thus:

$$\tilde{p}_j^{t_i} = \frac{1}{B(t_0, t_i)} \frac{B(t_0, t_{i+1})}{B(t_i, t_{i+1})^j} \tilde{p}_j^{t_{i+1}} \quad (145)$$

Replacing the right-hand side in equation (143) we get

$$\sum_{j=1}^k \frac{L_{t_i}^j}{(1 + L_{t_i}^j)} \frac{1}{B(t_0, t_i)} \frac{B(t_0, t_{i+1})}{B(t_i, t_{i+1})^j} \tilde{p}_j^{t_{i+1}} \quad (146)$$

In this expression we recognize

$$\frac{B(t_0, t_{i+1})}{B(t_0, t_i)} = \frac{1}{1 + F(t_0, t_i)} \quad (147)$$

and

$$\frac{1}{B(t_i, t_{i+1})^j} = 1 + L_{t_i}^j \quad (148)$$

Using these we get the equivalence:

$$\sum_{j=1}^k \frac{L_{t_i}^j}{(1 + L_{t_i}^j)} \frac{1}{B(t_0, t_i)} \frac{B(t_0, t_{i+1})}{B(t_i, t_{i+1})^j} \tilde{p}_j^{t_{i+1}} = \sum_{j=1}^k \frac{L_{t_i}^j}{(1 + L_{t_i}^j)} \frac{1}{(1 + F(t_0, t_i))} (1 + L_{t_i}^j) \tilde{p}_j^{t_{i+1}} \quad (149)$$

Simplifying the common terms on the right-hand side reduces to

$$\sum_{j=1}^k \frac{L_{t_i}^j}{1 + F(t_0, t_i)} \tilde{p}_j^{t_{i+1}} \quad (150)$$

This we immediately recognize as the expectation:

$$E_{t_0}^{\tilde{P}^{t_{i+1}}} \left[\frac{L_{t_i}^j}{1 + F(t_0, t_i)} \right] \quad (151)$$

Now, consider the denominator in equation (142)

$$E_{t_0}^{\tilde{P}^{t_i}} \left[\frac{1}{(1 + L_{t_i}^j)} \right] = \sum_{j=1}^k \frac{1}{(1 + L_{t_i}^j)} \tilde{p}_j^{t_i} \quad (152)$$

Using equation (144) we switch to the $\tilde{P}^{t_{i+1}}$ measure:

$$\sum_{j=1}^k \frac{1}{(1 + L_{t_i}^j)} \tilde{p}_j^{t_i} = \sum_{j=1}^k \frac{1}{(1 + L_{t_i}^j)} \frac{1}{B(t_0, t_i)} \frac{B(t_0, t_{i+1})}{B(t_i, t_{i+1})^j} \tilde{p}_j^{t_{i+1}} \quad (153)$$

Use the equivalences in equation (144), substitute:

$$\sum_{j=1}^k \frac{1}{(1 + L_{t_i}^j)} \frac{1}{B(t_0, t_i)} \frac{B(t_0, t_{i+1})}{B(t_i, t_{i+1})^j} \tilde{p}_j^{t_{i+1}} = \sum_{j=1}^k \frac{1}{(1 + L_{t_i}^j)} \frac{1}{(1 + F(t_0, t_i))} (1 + L_{t_i}^j) \tilde{p}_j^{t_{i+1}} \quad (154)$$

Note that, again, the random $(1 + L_{t_i}^j)$ terms conveniently cancel, and on the right-hand side we obtain:

$$\begin{aligned} &= \sum_{j=1}^k \frac{1}{1 + F(t_0, t_i)} \tilde{p}_j^{t_{i+1}} \\ &= \frac{1}{(1 + F(t_0, t_i))} \end{aligned}$$

Putting the numerator and denominator together for general N and δ gives

$$F(t_0, t_i) = \frac{E_{t_0}^{\tilde{P}^{t_i}} \left[\frac{N\delta L_{t_i}}{(1+\delta L_{t_i})} \right]}{E_{t_0}^{\tilde{P}^{t_i}} \left[\frac{N\delta}{(1+\delta L_{t_i})} \right]} = \frac{E_{t_0}^{\tilde{P}^{t_{i+1}}} \left[N\delta \frac{L_{t_i}^j}{1+F(t_0, t_i)\delta} \right]}{\frac{N\delta}{1+F(t_0, t_i)\delta}} \quad (155)$$

We simplify the common terms to get

$$F(t_0, t_i) = E_{t_0}^{\tilde{P}^{t_{i+1}}} \left[L_{t_i}^j \right] \quad (156)$$

Hence, we obtained the desired result. The FRA rate of paid-in-arrears FRAs is identical to the FRA rate of market-traded FRAs and is an unbiased predictor of the Libor rate L_{t_i} , under the right forward measure.

We conclude this section with another simple example.

EXAMPLE:

We can apply the forward measure technology to mark-to-market practices as well. The paid-in-arrears FRA will settle at time t_{i+1} according to

$$[L_{t_i} - F(t_0, t_i)]N\delta \quad (157)$$

What is the value of this contract at time t_1 , with $t_0 < t_1 < t_i$?

It is market convention to replace the random variable L_{t_i} with the corresponding forward rate of time t_1 . We get

$$[F(t_1, t_i) - F(t_0, t_i)]N\delta \quad (158)$$

which, in general, will be nonzero. How do we know that this is the correct way to mark the contract to market? We simply take the time- t_1 expectation of:

$$[L_{t_i} - F(t_0, t_i)]N\delta \quad (159)$$

with respect to the natural forward measure of t_{i+1}

$$E_{t_1}^{\tilde{P}^{t_{i+1}}} [L_{t_i} - F(t_0, t_i)]N\delta = [F(t_1, t_i) - F(t_0, t_i)]N\delta \quad (160)$$

where we use the fact that under the $\tilde{P}^{t_{i+1}}$, the $F(t_1, t_i)$ is an unbiased estimate of L_{t_i} .

6.2. Pricing CMS Swaps

Pricing CMS swaps is known to involve convexity adjustments. Staying within the context of the simple framework used in this chapter, the industry first obtains the $t_1 \times t_2$ and $t_2 \times t_3$ swaption volatilities. Then, knowing that the swap is a Martingale under the “annuity” measure treated in Chapter 21, various transformations under specific assumptions are performed and then the convexity correction to the forward swap rate is estimated. In other words, the industry calculates the ϵ_t in the equation

$$cms_t = s_t^f + \epsilon_t \quad (161)$$

where cms_t is the CMS rate, s_t^f is the relevant forward swap rate, and ϵ_t is the convexity correction.

It is straightforward to price CMS swaps using the forward Libor dynamics discussed earlier and then use successive measure changes for the required mean corrections. Because CMS swaps offer a good example for such an application, we show a simple case.

Consider a two-period forward CMS swap where a fixed CMS rate x_{t_0} is paid at times t_2 and t_3 against the floating two-period cash swap rate at these times. The present value of the cash flows under the \tilde{P}^{t_3} forward probability is given by

$$0 = E_{t_0}^{\tilde{P}^{t_3}} \left[(x_{t_0} - s_{t_1}) \frac{1}{(1 + L_{t_0}\delta)(1 + L_{t_1}\delta)} + (x_{t_0} - s_{t_2}) \frac{1}{(1 + L_{t_0}\delta)(1 + L_{t_1}\delta)(1 + L_{t_2}\delta)} \right] N \quad (162)$$

where the settlement interval is assumed to be one, and N is the notional swap amount. The s_{t_1} and s_{t_2} are the two 2-period swap rates unknown at time t_0 . They are given by the usual spot swap formula shown in equation (49).

Setting $\delta = 1$, and rearranging this equation, we obtain

$$x_{t_0} = \frac{E_{t_0}^{\tilde{P}^{t_3}} \left[s_{t_1} \frac{1}{(1+L_{t_0})(1+L_{t_1})} + s_{t_2} \frac{1}{(1+L_{t_0})(1+L_{t_1})(1+L_{t_2})} \right]}{E_{t_0}^{\tilde{P}^{t_3}} \left[\frac{1}{(1+L_{t_0})(1+L_{t_1})} + \frac{1}{(1+L_{t_0})(1+L_{t_1})(1+L_{t_2})} \right]} \quad (163)$$

Hence, to find the value of the CMS rate x_{t_0} , all we need to do is write down the dynamics of the forward Libor processes, $F(t_0, t_1)$ and $F(t_0, t_2)$, under the *same* forward measure \tilde{P}^{t_3} as done earlier, and then select Monte Carlo paths.

It is clear that proceeding in this way and obtaining Monte Carlo paths from the arbitrage-free forward Libor dynamics requires calibrating the respective volatilities σ^i . But once this is done, and once the correction factors are included in the proper equations, the Monte Carlo paths can be selected in a straightforward manner. The CMS rate can then be calculated from

$$x_{t_0} = \frac{\sum_{j=1}^M \left[s_{t_1}^j \frac{1}{(1+L_{t_0}^j)(1+L_{t_1}^j)} + s_{t_2}^j \frac{1}{(1+L_{t_0}^j)(1+L_{t_1}^j)(1+L_{t_2}^j)} \right]}{\sum_{j=1}^M \left[\frac{1}{(1+L_{t_0}^j)(1+L_{t_1}^j)} + \frac{1}{(1+L_{t_0}^j)(1+L_{t_1}^j)(1+L_{t_2}^j)} \right]} \quad (164)$$

where the swap rates $s_{t_i}^j$ themselves depend on the same forward rate trajectories and, hence, can be calculated from the selected paths.

The same exercise can be repeated by starting from perturbed values of volatilities and initial forward rates to obtain the relevant Greeks for risk-management purposes as well.

7. In-Arrears Swaps and Convexity

Although an overwhelming proportion of swap transactions involve the vanilla swap, in some cases parties transact the so-called *Libor in-arrears swap*. In this section we study this instrument because it is a good example of how Forward Libor volatilities enter pricing *directly* through convexity adjustments.

But first we need to clarify the terminology. In a vanilla swap, the Libor rates L_{t_i} are assumed to “set” at time t_i whereas the floating *payments* are made in arrears at times t_{i+1} .³⁰ In the case

³⁰ It is important to remember that, in reality, there is another complication. The Libor is set, by convention, 2 business days before time t_i and the payment is made at t_{i+1} . Here we are ignoring this convention because it really does not affect the understanding of the instruments and pricing, while making the formulas easier to understand. The reader can incorporate such real life modifications in the formulas given below.

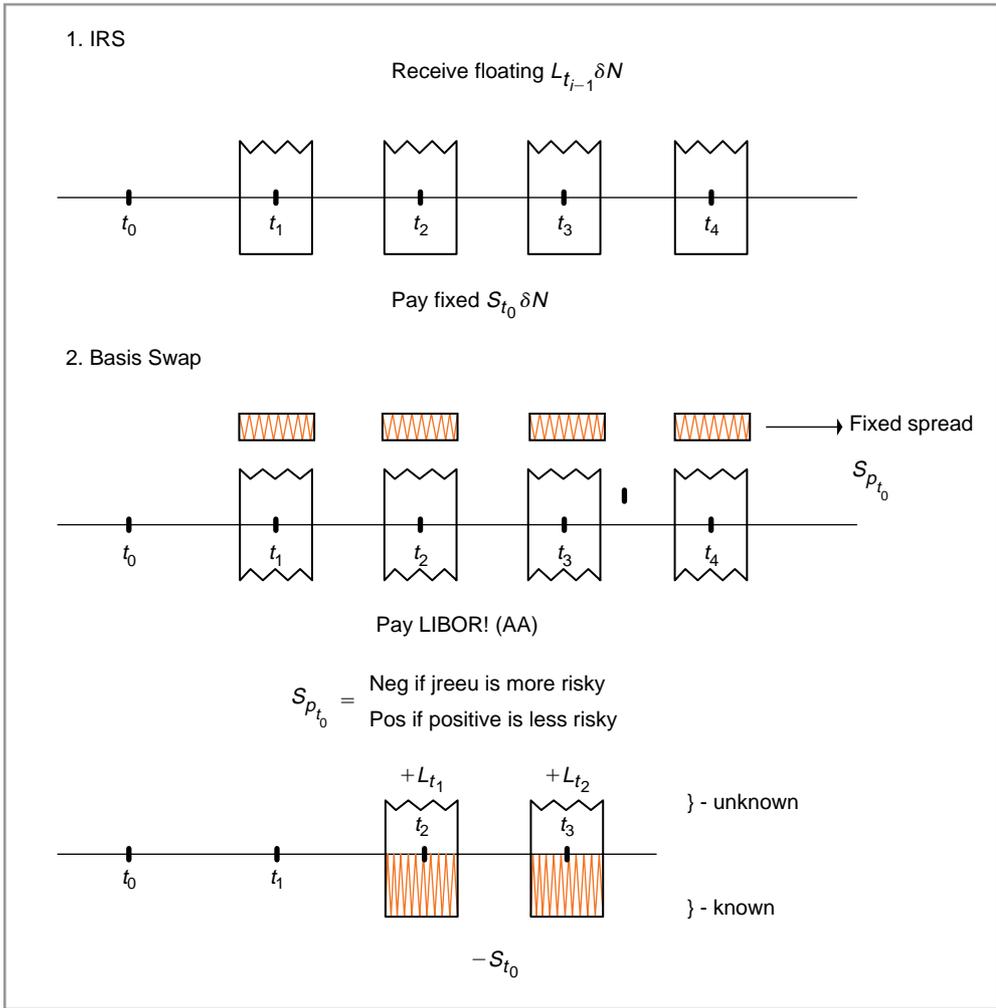


FIGURE 13-6

of an in-arrears swap, the payment days are kept the same, but the time- t_{i+1} settlement will use the Libor rate $L_{t_{i+1}}$ that has *just* been observed at time t_{i+1} ³¹ to determine the floating payment. Thus, in a sense the *setting* of the Libor rate is in arrears, hence the name of the in-arrears swap.³² The difference between Libor resets is shown in Figure 13-6. Libor in-arrears swaps *set* the libor rates in arrears.

The simple modification of paying the $L_{t_{i+1}}$ observed at time t_{i+1} rather than the previously observed L_{t_i} makes a significant difference in pricing. We will work with a simple case of a two-period (forward) swap first, and then give the generalized formulas.

7.1. Valuation

The valuation of the *fixed-leg* of the in-arrears swap is the same as that of the vanilla swap, except of course the swap coupons are different. Let the s_{t_0} be the *vanilla* swap rate fixed at

³¹ See the previous footnote concerning the two-business-days convention.

³² Although in case of vanilla swaps the payments are also in arrears.

time t_0 , N and δ be the notional amount and accrual parameters respectively. Then the fixed leg payments are easy to value:

$$\begin{aligned}\text{Fixed-Leg}_{t_0} &= B(t_0, t_2)s_{t_0}\delta N + B(t_0, t_3)s_{t_0}\delta N \\ &= [B(t_0, t_2) + B(t_0, t_3)]s_{t_0}\delta N\end{aligned}\quad (165)$$

It is clear that in case of the in-arrears swap we will have a similar expression:

$$\text{Fixed-Leg}_{t_0} = [B(t_0, t_2) + B(t_0, t_3)]is_{t_0}\delta N \quad (166)$$

where the is_{t_0} is the swap rate of the in-arrears swap.

The difference in valuations emerge in the *floating-leg*. Note that in the case of the in-arrears swap, the expected value under the P^{t_2} forward measure of the floating rate payments would be

$$\text{Floating-Leg}_{t_0} = E_{t_0}^{P^{t_2}} [B(t_0, t_2)L_{t_2}]\delta N + E_{t_0}^{P^{t_2}} [B(t_0, t_3)]\delta N \quad (167)$$

Multiply and divide by $(1 + L_{t_2}\delta)$ and $(1 + L_{t_3}\delta)$ respectively, we obtain

$$\text{Floating-Leg} = E_{t_0}^{P^{t_2}} \left[B(t_0, t_2)L_{t_2} \frac{(1 + L_{t_2}\delta)}{(1 + L_{t_2}\delta)} \right] \delta N + E_{t_0}^{P^{t_2}} \left[B(t_0, t_3)L_{t_3} \frac{(1 + L_{t_3}\delta)}{(1 + L_{t_3}\delta)} \right] \delta N \quad (168)$$

But in the case of finite-state random quantities we have the usual correspondence between the t_3 and t_2 forward measures:

$$p_i^{t_2} \frac{B(t_2, t_3)^i}{B(t_0, t_3)} B(t_0, t_2) = p_i^{t_3} \quad (169)$$

Also, by definition

$$B(t_2, t_3)^i = \frac{1}{(1 + L_{t_2}^i \delta)} \quad (170)$$

This means that after regrouping, changing measures from P^{t_2} to P^{t_3} :

$$E_{t_0}^{P^{t_2}} \left[B(t_0, t_2)L_{t_2} \frac{(1 + L_{t_2}\delta)}{(1 + L_{t_2}\delta)} \right] \delta N = E_{t_0}^{P^{t_3}} [B(t_0, t_3)L_{t_2}(1 + L_{t_2}\delta)]\delta N \quad (171)$$

Changing the measure from P^{t_3} to P^{t_4} , a similar set of equations gives

$$E_{t_0}^{P^{t_3}} \left[B(t_0, t_3)L_{t_3} \frac{(1 + L_{t_3}\delta)}{(1 + L_{t_3}\delta)} \right] = E_{t_0}^{P^{t_4}} [B(t_0, t_4)L_{t_3}(1 + L_{t_3}\delta)] \quad (172)$$

Thus the valuation of the floating-leg becomes

$$\text{Floating-Leg}_{t_0} = [E_{t_0}^{P^{t_3}} [B(t_0, t_3)L_{t_2}(1 + L_{t_2}\delta)] + E_{t_0}^{P^{t_2}} [B(t_0, t_4)L_{t_3}(1 + L_{t_3}\delta)]]\delta N \quad (173)$$

The right hand side can be expanded to

$$\begin{aligned}\text{Floating-Leg}_{t_0} &= B(t_0, t_3)[E_{t_0}^{P^{t_3}} [L_{t_2}] + E_{t_0}^{P^{t_3}} [(L_{t_2}\delta)^2]]\delta N + B(t_0, t_4)[E_{t_0}^{P^{t_4}} [L_{t_3}] \\ &\quad + E_{t_0}^{P^{t_4}} [(L_{t_3}\delta)^2]]\delta N\end{aligned}\quad (174)$$

But the forward rates are Martingales with respect to their own measures. So,

$$F_{t_0}^{t_2} = E_{t_0}^{P^{t_3}} [L_{t_2}] \quad (175)$$

$$F_{t_0}^{t_3} = E_{t_0}^{P^{t_4}} [L_{t_3}] \quad (176)$$

And

$$E_{t_0}^{P^{t_3}} [L_{t_2}^2] = (F_{t_0}^{t_2})^2 e^{\int_{t_0}^{t_2} \sigma(u)_{t_2}^2 du} \quad (177)$$

$$E_{t_0}^{P^{t_4}} [L_{t_3}^2] = (F_{t_0}^{t_3})^2 e^{\int_{t_0}^{t_3} \sigma(u)_{t_3}^2 du} \quad (178)$$

So we get the final result as:

$$\begin{aligned} \text{Floating-Leg}_{t_0} &= B(t_0, t_3) [F_{t_0}^{t_2} + \delta (F_{t_0}^{t_2})^2 e^{\int_{t_0}^{t_2} \sigma(u)_{t_2}^2 du}] \\ &\quad + B(t_0, t_4) [F_{t_0}^{t_3} + \delta (F_{t_0}^{t_3})^2 e^{\int_{t_0}^{t_3} \sigma(u)_{t_3}^2 du}] \delta N \end{aligned} \quad (179)$$

This can be expressed as the floating leg of a vanilla swap plus an adjustment, called the *convexity adjustment*:

$$\begin{aligned} \text{Floating-Leg}_{t_0} &= [B(t_0, t_3) F_{t_0}^{t_2} + B(t_0, t_4) F_{t_0}^{t_3}] \delta N + [B(t_0, t_3) \delta (F_{t_0}^{t_2})^2 e^{\int_{t_0}^{t_2} \sigma(u)_{t_2}^2 du} \\ &\quad + B(t_0, t_4) \delta (F_{t_0}^{t_3})^2 e^{\int_{t_0}^{t_3} \sigma(u)_{t_3}^2 du}] \delta N \end{aligned} \quad (180)$$

For small settlement intervals, Δ and constant volatilities the approximation becomes

$$[B(t_0, t_3) \delta (F_{t_0}^{t_2})^2 e^{\Delta \sigma_{t_2}^2} + B(t_0, t_4) \delta (F_{t_0}^{t_3})^2 e^{\Delta \sigma_{t_3}^2}] \delta N \quad (181)$$

Note that the second bracketed term in the convexity adjustment is positive. This makes the value of the floating rate payments in the in-arrears swap be *greater* than the value of the floating payments in the vanilla swap. The consequence of this is that the in-arrears fixed swap rate denoted by \tilde{s}_t is bigger than the vanilla fixed rate

$$s_{t_0} < \tilde{s}_{t_0} \quad (182)$$

A number of comments can be made. First note that the volatilities can be obtained from the corresponding caplet volatilities. Second, note that the value of the in-arrears swap does not depend on the correlation between various forward rates. Third, the volatilities are likely to be different than the swaption volatilities.

7.2. Special Case

A special case of this is if we look at a single period in-arrears swap. Then we get a relation between forward rates and futures rates.

A Eurodollar contract leads to an exchange of f_{t_0} for L_{t_i} at time t_i . The forward contract leads to an exchange of F_{t_0} for $L_{t_{i-1}}$ at t_i . So this is the one period replica of the comparison we just made.

This means

$$f_{t_0}^{t-i} - F_{t_0}^{t-i} = \delta (F_{t_0})^2 e^{\int_{t_0}^{t_i} \sigma(u)_{t_i}^2 du} \quad (183)$$

Directly from equation (177) of the previous section. The right-hand side is known as the convexity adjustment that needs to be applied when going from futures to forward rates. Note that the futures *price* of the contract will then be smaller than the forward price.³³

³³ Because we subtract a bigger term.

8. Cross-Currency Swaps

A cross-currency swap has two principal amounts, one for each currency. The initial principals can be exchanged or not. The non-exchange of the initial amounts is a minor issue. But eliminating the exchange of the final amounts changes the pricing structure significantly. The exchange rate used to determine the principals is the prevailing spot rate.

With an interest rate swap there is no exchange of principal at either the start or end of the transaction, as both principal amounts are the same and therefore net out. For a cross-currency swap it is essential that the parties agree to exchange principal amounts at maturity. The exchange of principal at the start is optional.

Like all swaps, a cross-currency swap can be replicated using on-balance sheet instruments, in this case with money market deposits or FRNs denominated in different currencies. This explains the necessity for principal exchanges at maturity as all loans and deposits also require repayment at maturity.³⁴

The initial exchange can be replicated by the bank by entering into a spot exchange transaction at the same rate quoted in the cross-currency swap. Actually, all foreign exchange forwards can be described as cross-currency swaps as they are agreements to exchange two streams of cash flows in different currencies. Many banks manage long-term foreign exchange forwards as part of the cross-currency swap business, given the similarities.

Like FX forwards, the cross-currency swap exposes the user to foreign exchange risk. The swap leg the party agrees to pay is a liability in one currency, and the swap leg they have agreed to receive is an asset in the other currency. One of the users of cross-currency swaps are debt issuers. In the Eurobond markets, issuers sell bonds in the currency with the lowest cost and swap their exposure to the desired currency using a cross-currency swap.

8.1. Pricing

At the inception of the swap, the present value of one leg must be equal to the present value of the other leg at the then-prevailing spot rate. Using this simple logic, it would seem natural that cash flows of Libor (flat) payments in one currency could be exchanged for cash flows of Libor (flat) payments in another currency.

In reality this is not true, and there is a constant spread for two reasons. First is the daily demand-supply imbalances that are always possible. There may be more demand for paying a Libor in a particular currency and this will lead to a positive spread to be paid. For major currencies such spreads are less than 15 bps. The second effect that leads to positive spreads is credit risk. Counterparties may not have the same credit risk and the currency swap spread may then reflect this. For example, if one party is paying the Philippine peso equivalent of the Libor rate against USD Libor, then this party is likely to have a higher credit risk. So the party will pay a higher spread.

8.2. Conventions

The usual convention for quoting the currency swap spread, also called the *basis*, is to quote it relative to the USD-Libor.

³⁴ While the corporate can elect not to exchange principal at the start, the bank needs to.

9. Differential (Quanto) Swaps

Financial Accounting Standard (FAS) 133 and the International Accounting Standard (IAS) 39 set the accounting rules for derivatives for the United States and for European companies respectively. According to these rules, a derivative position will have to be marked-to-market and included in income statements unless it qualifies for *hedge accounting*.³⁵

A quanto (differential) swap is a special type of cross-currency swap. It is an agreement where one party makes payments in, say, USD-Libor and receives payments, say, in Euro-Libor. However, the *important* point is that both parties make payments in the *same* currency. In other words, the quanto swap value is an exposure on pure play of international interest rate differentials, and has no foreign exchange risk.

Quanto swap popularity depends on the relative shapes of the forward curves in the two underlying money markets. Quanto swaps become “cheap” if one of the two forward curves is lower at the short end and higher at the long end.

9.1. Basis Swaps

This discussion is limited to U.S. dollar markets. The particular interest rates discussed below will change if other currencies are considered, since basis swaps are directly related to the business environment in an economy.

In a *basis swap*, one party will pay USD-Libor and will receive *another money-market rate*. Most bank liabilities are in fact Libor based, but assets are not. A corporation that deals mainly in the *domestic* U.S. economy may be exposed to commercial paper (CP) rates; a bank may be exposed to T-bill rates, and another to the *prime rate*. Basis swaps could then be used to protect the party with respect to changes in these different money market rates.

The most common types of basis swaps are Fed Funds against Libor,³⁶ T-bill rates against Libor, CP rates against Libor, and the prime rate against Libor.³⁷ Which basis swap a client picks depends on his or her business. For example, a party that has concerns about credit squeezes can use Fed Funds-Libor basis swaps or the T-bill-Libor basis swaps. During a credit squeeze, a flight to safety will make the basis swap spread increase. On the other hand, the Prime-Libor basis swap can hedge the exposures of those players involved in credit card, auto, or consumer loans.

10. Conclusions

This chapter was devoted to the connections between the swap, FRA, and bond markets. Our discussion led us to the issue of constructing a satisfactory yield curve, which is the fundamental task of a financial engineer. Two main tools were introduced in the chapter. The first was the *T*-forward measures and the second was the related measure change technology. This permitted setting up convenient arbitrage-free dynamics for a sequence of forward rates. These dynamics were then used as a tool for calculating the desired quantities using the formulas that connect swap rates, forward rates, and their derivatives.

³⁵ Qualifying for hedge accounting is a lengthy and costly process. At the end the qualification is still random. However, some vanilla instruments such as vanilla swaps can qualify relatively easily.

³⁶ The Fed Funds market consists of overnight lending of free reserves kept at the Federal Reserve between high quality banks. The quality of the banks and the sort tenor implies that Fed Funds rate will be lower than, say, Libor.

³⁷ The prime rate is not an interbank rate. It applies to the best retail clients.

The next topic was the Forward Libor Model. Here, the essential idea was to obtain sequential correction factors to express the dynamics of various forward rates under a single forward measure.

Suggested Reading

*The standard readings for this chapter will make interesting reading for a financial engineer. **Brace, Gatarek, Musiela** (1997), and **Jamshidian** (1997) are the fundamental readings. **Miltersen, Sandmann, and Sondermann** (1997) is another important reference. **Glasserman and Zhao** (2000) is a good source on the discretization of BGM models. Finally, the text by **Brigo and Mercurio** (2001) provides a comprehensive treatment of all this material. The recently published **Rebonato** (2002) is a good introduction. See also **Fries** (2007) and **Joshi** (2004).*

APPENDIX 13-1: Practical Yield Curve Calculations

The traditional way of calculating yield curves starts with liquid bond prices and then obtains the discounts and related yields. Thus, the method first calculates the implied zero-coupon prices, and then the corresponding yields and forward rates from observed coupon bond prices.

We will briefly review this approach to yield curve calculation. It may still be useful in markets where liquid interest rate derivatives do not trade. First, we need to summarize the concepts.

1. Par Yield Curve

Consider a straight coupon bond with coupon rate c exactly equaling the yield at time t for that maturity. The current price of this “par” bond will be exactly 100, the par value. Such a bond will have a yield to maturity, the *par yield*. The current price of these bonds is equal to 100 and their coupon would be indicative of the correct yield for that maturity and credit at that particular time.

We can write the present value of a three-period par bond, paying interest annually, as

$$100 = \frac{100c}{(1+y)} + \frac{100c}{(1+y)^2} + \frac{100(1+c)}{(1+y)^3} \quad (184)$$

where the par yield implies that $c = y$.

This property is desirable because with coupon bonds, the maturity does not give the correct timing for the average cash receipt, and if we consider bonds with coupons different than the par yield, the durations of the bonds would be different and the implied yields would also differ.

2. Zero-Coupon Yield Curve

We can also calculate a yield curve using zero-coupon bonds with par value 100 by exploiting the equality,

$$B(t, T) = \frac{100}{(1 + y_t^T)^{T-t}} \quad (185)$$

The y_t^T will correspond to the $(T - t)$ -maturity zero-coupon yield.

It turns out that the par yield curve and the zero coupon yield curve are different in general. We now show the calculations in an example.

EXAMPLE:

We would like to show the relationship between par yields and zero-coupon yields. Suppose we are given the following zero-coupon bond prices:

$$\begin{aligned} B(0, 1) &= 96.00 \\ B(0, 2) &= 91.00 \\ B(0, 3) &= 87.00 \end{aligned} \quad (186)$$

1. *What are the zero-coupon yields?*
2. *What are the par yields for the same maturities?*

To calculate the zero-coupon yields, we use the following formula:

$$B(t, T) = \frac{100}{(1 + y_t^T)^{T-t}} \quad (187)$$

and obtain:

$$\begin{aligned} 96.00 &= \frac{100}{(1 + y_0^1)^1} \\ 91.00 &= \frac{100}{(1 + y_0^2)^2} \\ 87.00 &= \frac{100}{(1 + y_0^3)^3} \end{aligned} \quad (188)$$

Solving for the unknown zero-coupon yields:

$$y_0^1 = 0.04167, \quad y_0^2 = 0.04828, \quad y_0^3 = 0.04752 \quad (189)$$

We now calculate par yields using the relationship with $t_n = T$

$$P(t_0, T) = \sum_{i=0}^n \tilde{y} B(t_0, t_i) + B(t_0, T) = 100 \quad (190)$$

The \tilde{y} that satisfies this equation will be the par yield for maturity T . The idea here is that, when discounted by the correct discount rate, the sum of the cash flows generated by a par bond should equal 100; i.e., we must have $P(t_0, T) = 100$. Only one \tilde{y} will make this possible for every T .

Calculating the par yields, we obtain

$$\tilde{y}_0^1 = y_0^1 = 0.04167, \quad \tilde{y}_0^2 = 0.04813, \quad \tilde{y}_0^3 = 0.04745 \quad (191)$$

As these numbers show, the par yields and the zero-coupon yields are slightly different.

3. Zero-Coupon Curve from Coupon Bonds

Traditional methods of calculating the yield curve involve obtaining a zero-coupon yield curve from arbitrary coupon bond prices. This procedure is somewhat outdated now, but it may still be useful in economies with newly developing financial markets. Also, the method is a good illustration of how synthetic asset creation can be used in yield curve construction. It is important to remember that all these calculations refer to default-free bonds.

Consider a two-year coupon bond. The *default-free* bond carries an annual coupon of c percent and has a current price of $P(t, t + 2)$. The value at maturity is 100. Suppose we know the level of the current annual interest rate r_t .³⁸ Then the portfolio

$$\left\{ \frac{100c}{(1 + r_t)} \text{ units of time } t \text{ borrowing, and buying two-period coupon bond, } P(t, t + 2) \right\} \quad (192)$$

³⁸ Alternatively, we can assume that the price of the one-period coupon bond $P(t, t + 1)$ is known.

will yield the cash flow equivalent to $1 + c$ units of a two-period discount bond. Thus, we have

$$P(t, t + 2) - \frac{100c}{1 + r_t} = (100(1 + c))/(1 + y_t^2)^2 \quad (193)$$

If the coupon bond price $P(t, t + 2)$ and the 1-year interest rate r_t are known, then the two-year zero-coupon yield y_t^2 can be calculated from this expression. Zero-coupon yields for other maturities can be calculated by forming similar synthetics for longer maturity zero-coupon bonds, recursively.

Exercises

1. You are given the following quotes for liquid FRAs paid in arrears. Assume that all time intervals are measured in months of 30 days.

Term	Bid/Ask
3×6	4.5–4.6
6×9	4.7–4.8
9×12	5.0–5.1
12×15	5.5–5.7
15×18	6.1–6.3

You also know that the current 3-month Libor rate is 4%.

- Calculate the discount bond prices $B(t_0, t_i)$, where $t_i = 6, 9, 12, 15,$ and 18 months.
 - Calculate the yield curve for maturities 0 to 18 months.
 - Calculate the swap curve for the same maturities.
 - Are the two curves different?
 - Calculate the par yield curve.
 - Calculate the zero-coupon yield curve.
2. You are given the following quotes for liquid swap rates. Assume that all time intervals are measured in years.

Term	Bid/Ask
2	6.2–6.5
3	6.4–6.7
4	7.0–7.3
5	7.5–7.8
6	8.1–8.4

You know that the current 12-month Libor rate is 5%.

- Calculate the FRA rates for the next five years, starting with a 1×2 FRA.
 - Calculate the discount bond prices $B(t_0, t_i)$, where $t_i = 1, \dots, 6$ years.
 - Calculate the yield curve for maturities of 0 to 18 months.
 - Calculate the par yield curve.
3. Going back to the data given in Exercise 2, calculate the following:
- The bid-ask on a forward swap that starts in two years with maturity in three years. The swap is against 12-month Libor.
 - The forward price of a coupon bond that will be delivered at time 2. The bond pays coupon 7% and matures in two years.