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## DISCOUNTED CASH FLOW AND THE VALUE OF SECURITIES

Chapter 6 Time Value of Money

Chapter 7 The Valuation and  
Characteristics of Bonds

Chapter 8 The Valuation and  
Characteristics of Stock

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## CHAPTER OUTLINE



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The time value of money is based on the idea that a sum of money in your hand today is worth more than the same sum promised at some time in the future, even if you're absolutely certain to receive the future cash.

The idea is pretty easy to grasp if you think in terms of having a bank account and being promised an amount of money a year from now. Money in the bank earns interest, so it grows over time. The value today of the sum promised in one year is an amount that will grow into that sum if deposited in the bank now. In other words, a sum promised in a year is worth only as much as you'd have to put in the bank today to have that sum in a year.

That value obviously depends on the interest rate the bank is paying. The higher the interest rate, the faster money grows, so the less you'd have to deposit today to get a given amount next year.

Let's look at an example using a future amount of \$1,000. How much would a promise of \$1,000 in one year be worth today if the bank paid 5% interest? That question is equivalent to asking how much money will grow into \$1,000 in one year at 5% interest. The answer is \$952.38; we'll worry about how we got it a little later. The important thing to understand now is that if we deposit \$952.38 for one year, we'll earn interest of 5% of that amount,

$$\$952.38 \times .05 = \$47.62$$

which when added to the original deposit yields \$1,000.

$$\$952.38 + \$47.62 = \$1,000.00$$

The **present value** of a sum at a future time is the amount that must be deposited at interest today to have the sum at that time.

Therefore, a guaranteed promise of \$1,000 in a year is worth \$952.38 today if the interest rate is 5%.

We say that \$952.38 is the **present value** of \$1,000 in one year at 5%. Alternatively, we say that \$1,000 is the **future value** of \$952.38 after one year at 5%.

If the interest rate were 7%, the present value of \$1,000.00 in one year would be \$934.58, a smaller number:

$$\$934.58 \times .07 = \$65.42$$

and

$$\$934.58 + \$65.42 = \$1,000.00$$

In other words, a higher rate of interest makes the present value of a future amount smaller. This makes sense; the bank deposit is earning faster, so you don't have to put as much in to get to the desired amount at the end of the year.

The time value of money is one of the most important principles in finance and economics today. It's based on the simple ideas we've just stated, but the applications can get quite complicated as we'll see later in this chapter.

The subject can also be called **discounted cash flow**, abbreviated as DCF. Using that terminology in our first example, we would say the \$952.38 is the *discounted* value of the \$1,000.

Here's another way of looking at the same thing. Suppose you have a firm, written contract promising to pay you \$1,000 in one year's time, but you need as much cash as you can get today. You could take the contract, called a note, to a bank, which would **discount** it for you at whatever interest rate it charges. If the bank's interest rate was 5%, it would give you \$952.38 for the note. If the rate was 7%, the bank would be willing to give you only \$934.58.

The **discounted** value of a sum is its present value.



## OUTLINE OF APPROACH

Our study of time value will involve learning to deal with *amounts* and *annuities*. An amount problem is similar to what we've already been talking about, involving a single amount of money that grows at interest over time into a larger sum. An annuity problem deals with a stream of equal payments, each of which is placed at interest and grows over time.

We'll further divide each of these categories into two more. Within each we'll look at situations dealing with present values and those dealing with future values.

In all we'll be looking at four types of problems.

amount—present value

amount—future value

annuity—present value

annuity—future value

After we've mastered these, we'll put the techniques together and work with some relatively complicated compound problems.

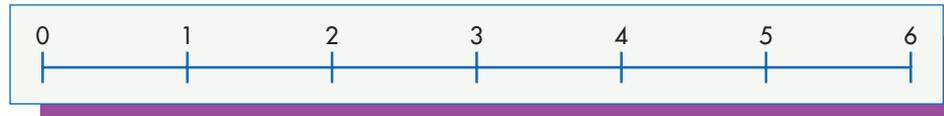
### Mathematics

As we approach each of the four categories, we'll develop a formula suited to doing that type of problem. The algebra needed to develop the formula may look a little intimidating to readers who aren't strong in math. Don't be alarmed. The math

required to do the financial problems once you accept the formula is quite simple. Developing the formulas is background that's good to know, but you don't have to remember it to do the practical work.

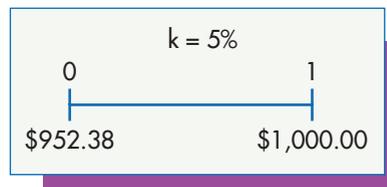
### Time Lines

Students sometimes find time value problems confusing. The time line is a graphic device that helps keep things straight. Time is divided into periods and portrayed along a horizontal line. Time zero is the present, and we count periods to the right.



A **time line** is a graphic portrayal of a time value problem.

Time 1 is the *end* of the first period, time 2 the end of the second, and so on. We can make notations above and below the time line to keep track of various pieces of the problem we're working on, such as interest rates and amounts. For example, a time line for the illustration we talked about before would look like this.



Most people don't need time lines for simple situations like this one, but the devices can help a lot in more complicated problems. We'll use them where appropriate as we go forward. We'll begin with yearly periods, but later we will introduce shorter spans.

### A Note about the Examples

Several of the examples in this chapter are used to teach important financial practices as well as to illustrate computational techniques. You should be sure to learn and understand the business situations described in each of these illustrations. The first one you'll encounter is in Example 6.2, which describes the equivalence of deferred payment terms and a cash discount.

## AMOUNT PROBLEMS

Amount problems involve a single sum of money that can be thought of as moving back and forth through time under the influence of interest. As it moves into the future, the sum gets larger as it earns interest. Conversely, as it moves back in time, the sum gets smaller. We'll begin with future value-oriented situations.

### THE FUTURE VALUE OF AN AMOUNT

To find the future value of an amount, we need a convenient way to calculate how much a sum of money placed at interest will grow into in some period of time. Let's start with a simple situation. Suppose we invest a sum of money in the bank at interest rate  $k$ . How much will it be worth at the end of one year?

Call the sum today  $PV$ , for present value, and the amount we'll have at the end of the year  $FV_1$ , for future value in one year. And call the decimal equivalent of the interest rate  $k$  ( $.05$  for  $5\%$ ).

At the end of a year we'll have the amount originally invested, PV, plus the interest on that amount,  $kPV$ . So we can write

$$FV_1 = PV + kPV$$

Factor PV out of the right side, and we have

$$(6.1) \quad FV_1 = PV(1 + k)$$

Now suppose we leave  $FV_1$  in the bank for another year, and we want to know how much we'll have at the end of that second year. We'll call that  $FV_2$ . The second year's calculation will be the same as the one we just did, but we'll use  $FV_1$  instead of PV.

$$FV_2 = FV_1 + kFV_1$$

Factor out  $FV_1$ .

$$FV_2 = FV_1(1 + k)$$

Now substitute for  $FV_1$  from equation 6.1 to get

$$(6.2) \quad \begin{aligned} FV_2 &= PV(1 + k)(1 + k) \\ FV_2 &= PV(1 + k)^2 \end{aligned}$$

Notice the similarity between equations 6.1 and 6.2.  $FV_1$  is equal to PV times  $(1 + k)$  to the *first* power, while  $FV_2$  is PV times  $(1 + k)$  to the *second* power. It's easy to see that if we performed the same calculation for a third year,  $FV_3$  would be equal to PV times  $(1 + k)$  to the *third* power, and so on, for as many years into the future as we'd care to go.

We can *generalize* the relationship and write

$$(6.3) \quad FV_n = PV(1 + k)^n$$

for any value of  $n$ . This expression gives us a very convenient way to calculate the future value of any present amount given that we know the interest rate,  $k$ , and the number of years the money is invested,  $n$ .

For example, if we deposited \$438 at 6% interest for five years, how much would we have? Using equation 6.3 gives

$$FV_5 = \$438(1.06)^5$$

Raising 1.06 to the fifth power on a calculator gives 1.3382, so

$$\begin{aligned} FV_5 &= \$438(1.3382) \\ &= \$586.13 \end{aligned}$$

The only messy part of the calculation is raising 1.06 to the fifth power. Looking at equation 6.3, we can see that calculating  $(1 + k)^n$  will always be tedious, especially for larger values of  $n$ .

However, notice that the value of  $(1 + k)^n$  depends only on the sizes of  $k$  and  $n$ , and that in business situations these variables take on a relatively limited number of values. Therefore, it's feasible to make up a table that contains the value of  $(1 + k)^n$  for common combinations of  $k$  and  $n$ . We'll call  $(1 + k)^n$  the *future value factor for  $k$  and  $n$* , and write it as  $FVF_{k,n}$ . Table 6.1 is a partial table of values for this factor. A more extensive version is given in Appendix A-1 for use in solving problems.

The **future value factor** for  $k$  and  $n$  is the calculated value of  $(1 + k)^n$ .

**Table 6.1**

The Future Value Factor for  $k$  and  $n$   
 $FVF_{k,n} = (1 + k)^n$

n	k						...
	1%	2%	3%	4%	5%	6%	
1	1.0100	1.0200	1.0300	1.0400	1.0500	1.0600	...
2	1.0201	1.0404	1.0609	1.0816	1.1025	1.1236	...
3	1.0303	1.0612	1.0927	1.1249	1.1576	1.1910	...
4	1.0406	1.0824	1.1255	1.1699	1.2155	1.2625	...
5	1.0510	1.1041	1.1593	1.2167	1.2763	1.3382	...
6	1.0615	1.1262	1.1941	1.2653	1.3401	1.4185	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

We can now rewrite equation 6.3 in a more convenient form by referring to the table.

$$(6.4) \quad FV_n = PV[FVF_{k,n}]$$

### Example 6.1

How much will \$850 be worth if deposited for three years at 5% interest?

**SOLUTION:** To solve the problem, write equation 6.4 and substitute the amounts given.

$$\begin{aligned} FV_n &= PV[FVF_{k,n}] \\ FV_3 &= \$850[FVF_{5,3}] \end{aligned}$$

Look up  $FVF_{5,3}$  in the three-year row under the 5% column of Table 6.1, getting 1.1576, and substitute.

$$\begin{aligned} FV_3 &= \$850[1.1576] \\ &= \$983.96 \end{aligned}$$

In time value problems, **three** of four variables are **given**, and **we solve for the fourth**.

### Problem-Solving Techniques

Equation 6.4 is the first of four formulas that you will use to solve a variety of time value problems. Each equation contains four variables. In this case the variables are  $PV$ ,  $FV_n$ ,  $k$ , and  $n$ . Every problem will give you three of the variables and ask you to find the fourth.

If you're asked to find  $PV$  or  $FV_n$ , the solution is very easy. Simply look up the factor for the given  $k,n$  combination in the table, and substitute in the equation along with the given  $PV$  or  $FV_n$ . The last problem gave us  $PV$  and asked for  $FV_n$ . Here's one that gives us the future value and requires us to find the present value.

### Example 6.2

Ed Johnson sold 10 acres of land to Harriet Smith for \$25,000. The terms of the agreement called for Harriet to pay \$15,000 down and \$5,000 a year for two years. What was the real purchase price if the interest rate available to Ed on invested money is 6%?

**SOLUTION:** What Ed is getting today is \$15,000 plus the present value of two \$5,000 payments, each to be received at different times in the future. The problem is to compute these PVs and add them to the \$15,000.

The present value of the payment due at the end of the first year is calculated by writing equation 6.4 and substituting the known elements.

$$\begin{aligned}FV_n &= PV[FVF_{k,n}] \\ \$5,000 &= PV[FVF_{6,1}]\end{aligned}$$

Table 6.1 or Appendix A-1 gives  $FVF_{6,1} = 1.0600$ . Substitute and solve for PV.

$$\begin{aligned}\$5,000 &= PV[1.0600] \\ PV &= \$4,716.98\end{aligned}$$

The second calculation is the same, but the payment is two years away, so we use  $FVF_{6,2} = 1.1236$ . That gives a present value of \$4,449.98.

In a present value sense, the actual sale amount is the sum of these two PVs and the down payment,

$$\$15,000.00 + \$4,716.98 + \$4,449.98 = \$24,166.96$$

That's \$833.04 less than the \$25,000 price quoted.

In real estate finance we would say that the *terms of sale* resulted in an effective price reduction of \$833.04, even though the real estate records would indicate a transaction price of \$25,000. Terms of sale state when and how the purchase price has to be paid. The seller's willingness to accept part of the price later is worth a specific amount of money. In other words, it is the equivalent of a cash discount.

## The Opportunity Cost Rate

Notice that in the last problem we calculated the present values using the interest rate available to the seller, 6%, even though nothing was actually invested at that or any other rate. We used the 6% rate because if the seller had received the full price at the time of sale, he would have been able to invest the deferred payments at that rate. Therefore, in a sense, he lost the income from that investment by giving the deferred payment terms.

We say that the lost interest income is the **opportunity cost** of giving the discount. In this case, because the seller's alternative is stated as a rate of interest at which he could have invested, we call it the *opportunity cost rate*.

The opportunity cost concept is a bit slippery. For example, you could argue that Ed Johnson might not have been able to sell the property to anyone without giving the deferred terms or an equivalent discount, and therefore there wasn't really any cost to the deferred terms at all. Nevertheless, we still say that the opportunity cost rate from Johnson's viewpoint is 6%.

The opportunity cost rate frequently isn't the same to different parties in the same transaction. In Example 6.2, Ed Johnson's opportunity cost rate is 6%, because that's the rate at which he can invest. But suppose Harriet Smith has to borrow to pay for the land and that she must do so at a rate of 10%. Her opportunity cost rate is then 10%, not 6%. To her the deferred payment terms are worth a discount of \$1,322.32, quite a bit more than what they implicitly cost Ed Johnson. (Verify this by calculating the effective price at 10% as we did in the example at 6%.)

In this example the deferred terms are a pretty good deal. They're worth more to the recipient than to the donor!

The **opportunity cost** of a resource is the benefit that would have been available from its next best use.

In general, the opportunity cost of using a resource in some way is the amount it could earn in the next best use.

## FINANCIAL CALCULATORS

Financial calculators take most of the drudgery out of time value problems. They work directly with mathematical relationships like equation 6.3 rather than with tables.

There's a temptation to skip the mathematical work we've been doing here and go directly to using a calculator without mastering the algebraic approach. That's a big mistake. If you go straight to the calculator, you'll never truly understand what's behind time value or be comfortable with it. Certainly in practice we use calculators almost exclusively, but it's very important to know what's behind the numbers that flash on the display.

In the rest of this chapter we'll concentrate on the approach we've been developing that uses financial tables, but we'll also show calculator solutions in the page margins.

**http://**

Some web sites have present/future value calculators, such as the one at

<http://www.ssfcu.org/tools.htm>

### How to Use a Typical Financial Calculator in Time Value

Recall that there are four variables in any time value problem. Values for three are given, and the fourth is unknown. Financial calculators have a key for each variable. To use a calculator, enter the three known variables, pressing the appropriate key after each input. Then press a compute key, followed by the key for the unknown variable. The calculator responds by displaying the answer.

There are actually five time value keys, because annuities require one that isn't used in the amount problems we've looked at so far. When we solve a problem, we use four keys and set to zero, or ignore, the fifth. The keys selected tell the calculator which kind of problem is being done. The time value keys and their meanings are as follows.

- n—Number of time periods
- I/Y—Interest rate (other labels: %i, I/YR, I% YR)
- PV—Present value
- FV—Future value
- PMT—Payment

The last key is the periodic payment associated with an annuity. We'll talk about it later when we get to annuities. For now it should be ignored (if you clear the time value registers before starting) or set to zero.

The compute key is usually labeled either CPT or 2nd. On some calculators there isn't a compute key; the calculator just knows the last key hit is the unknown.

Before trying a problem, take a look at your calculator's instruction manual. You may have to get into a particular mode of operation and clear the time value registers before starting. Advanced calculators also have a feature regarding the interest rate that needs to be set properly. They take the interest rate input and automatically divide it by a number of compounding periods per year. The default setting is usually 12, for 12 months a year. We'll get into non-annual compounding periods later. For now, set the calculator for one (1) period per year.

Now solve Example 6.1, using your calculator. Here's how.

1. The problem runs for three years. Enter 3 and then press n.
2. The interest rate is 5%. Enter 5 and then press I/Y.
3. The present value is \$850. Enter 850 and then press PV.
4. Press 2nd or CPT (if necessary) and then press FV.
5. The calculator displays 983.98 or -983.98.

Some calculators use a sign convention intended to reflect inflows as positive numbers and outflows as negatives. For example, if PV is entered as a positive, FV shows up as a negative. The idea is that PV is a deposit and FV is a withdrawal.

Notice that a calculator solution may be a little off a table solution because the table only carries four decimal places. The calculator carries 12 or more significant digits. Also notice that the interest rate is generally entered as a whole number even though the equations work with the decimal form.

In the remainder of this chapter we'll show abbreviated calculator solutions for examples in the margins. Here's an illustration showing the first \$5,000 payment in Example 6.2.

#### Calculator Solution

Key	Input
n	1
I/Y	6
FV	5,000
PMT	0
	<b>Answer</b>
PV	4,716.98

## THE EXPRESSION FOR THE PRESENT VALUE OF AN AMOUNT

Notice that we are able to use equation 6.4 to solve problems asking for *either* the present value or the future value. However, the expression is set up to make the future value calculation a little easier, because  $FV_n$  is isolated on the left side.

For convenience, we can develop another equation that's oriented toward solving the present value problem. We'll begin with equation 6.3,

$$FV_n = PV(1 + k)^n$$

Now simply solve for PV by dividing through by  $(1 + k)^n$  and switching the terms to opposite sides.

$$(6.5) \quad PV = FV_n \frac{1}{(1 + k)^n}$$

Slightly more sophisticated mathematical notation enables us to write the same thing with a negative exponent.

$$(6.6) \quad PV = FV_n(1 + k)^{-n}$$

The term  $(1 + k)^{-n}$  can be thought of as a factor depending only on  $k$  and  $n$  that can be tabulated. We'll call that factor the **present value factor for  $k$  and  $n$**  and write it as  $PVF_{k,n}$ . The values of  $PVF_{k,n}$  are given in Appendix A-2. We can now rewrite equation 6.6 by using this factor and reference to the table.

$$(6.7) \quad PV = FV_n[PVF_{k,n}]$$

We use this expression and the associated table just like we used equation 6.4. It too can be used to solve for either present or future values, but it is more conveniently formulated for present values. Do Example 6.2 on your own using equation 6.7.

The future and present value factors are **reciprocals**. **Either** amount equation can be used to solve any amount problem.

Solving for **k** or **n** involves searching a table.

### The Relation between the Future and Present Value Factors

It's important to notice that equations 6.4 and 6.7 really express the same relationship, since they both come from equation 6.3. It's also important to realize that the present and future value factors are reciprocals of one another. That is,

$$(6.8) \quad FVF_{k,n} = \frac{1}{PVF_{k,n}}$$

### More on Problem-Solving Techniques

So far we've looked at problems that ask us to solve for  $FV_n$  or  $PV$ . When the unknown element in the equation is  $k$  or  $n$ , the approach is a little different. Notice that in both equations 6.4 and 6.7,  $k$  and  $n$  appear as subscripts on the factors referring to table values. That means we can't use traditional algebraic methods to solve for an unknown  $k$  or  $n$ .

We'll change Example 6.1 a little to illustrate what we mean. In that problem we asked how much \$850 would grow into in three years at 5% interest and got an answer of \$983.96.

### Example 6.3

Suppose instead we were asked what interest rate would grow \$850 into \$983.96 in three years. In this case we have  $FV_3$ ,  $PV$ , and  $n$ , but we don't have  $k$ .

**SOLUTION:** We'll use equation 6.7 this time, just for variety. The general approach is to write the equation

$$PV = FV_n [PVF_{k,n}]$$

and substitute what's known,

$$\$850.00 = \$983.96 [PVF_{k,3}]$$

Notice that this equation can't be solved algebraically for  $k$ .

The approach we must take is to solve for the whole factor,  $PVF_{k,3}$ , and then find its value in the table. Once we've done that, we can read off the unknown value for  $k$  from the column heading. Solving for the factor gives

$$PVF_{k,3} = \$850.00 / \$983.96 = .8639$$

We have to find .8639 in Appendix A-2, but we don't have to search the entire table for it. We know that in this problem  $n = 3$ , so we can confine our search to the row for three years. Looking along that row we don't find .8639 exactly, but we do find .8638. That's close enough to assume that the difference is due to rounding error. Looking up to the top of the column, we read 5% as the solution to the problem.

#### Calculator Solution

Key	Input
n	3
PV	850.00
FV	983.96
PMT	0
	<b>Answer</b>
I/Y	5.0

### Solutions between Columns and Rows

Most of the time, solutions for  $k$  and  $n$  don't come out exactly on numbers in the table. That is, the calculated factor is somewhere between the columns or rows. The appropriate approach when that happens depends on the accuracy needed in the solution. For some purposes it's enough to round the answer to the closest tabulated row or column. If a more accurate answer is necessary and you're using tables like the ones provided here, you have to estimate between columns and rows.

In practice, financial calculators are used to solve time value problems. They give exact results without using tables. Before financial calculators were invented, people used enormously detailed tables that filled entire volumes. For illustrative purposes in what follows we'll just round to the nearest table value of  $n$  or  $k$ .

### Example 6.4

How long does it take money invested at 14% to double?

#### Calculator Solution

Key	Input
I/Y	14
FV	2
PV	1
PMT	0
	<b>Answer</b>
n	5.29

**SOLUTION:** Don't be confused by the fact that we're not given a present and future value in this case. What we are given is a relation between the two. If the money is to double in value, the future value must be twice the present value. Alternatively, we could ask how long it would take \$1 to double into \$2. We'll use equation 6.4 in this case,

$$FV_n = PV[FVF_{k,n}]$$

Solving for the factor and substituting yields

$$FVF_{14,n} = FV_n/PV = 2.000$$

Next we look for 2.0000 in Appendix A-1, confining our search to the column for  $k = 14\%$ . We find the table value is between five and six years.

n	14%
5	1.9254
6	2.1950

Clearly, 2.0000 is closer to 1.9254 than it is to 2.1950; therefore, the nearest whole integer number of years is 5. Notice that the calculator solution gives an exact answer of 5.29 years.

## ANNUITY PROBLEMS

The second major class of time value problems involves streams of payments called *annuities*. These are generally more complex than amount problems and harder to visualize, so using time lines can be important.

### ANNUITIES

An **annuity** is a stream of equal payments, made or received, separated by equal intervals of time. Hence, \$5 a month for a year is an annuity. A stream of monthly payments that alternates from \$5 to \$10 is not an annuity, nor is a stream of \$5 payments that skips an occasional month. Both the amount and the time interval must be constant to have an annuity.

When payments occur at the end of the time periods, we have what's called an *ordinary annuity*. This is the usual situation. If the payments occur at the beginning of each period, we call the stream an *annuity due*. Figures 6.1 and 6.2 show time lines for both cases for a stream of four \$1,000 payments.

An **annuity** is a finite series of equal payments separated by equal time intervals.

Figure 6.1

Ordinary Annuity

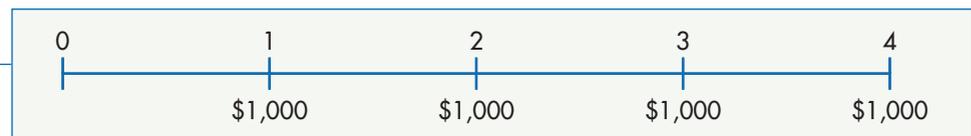
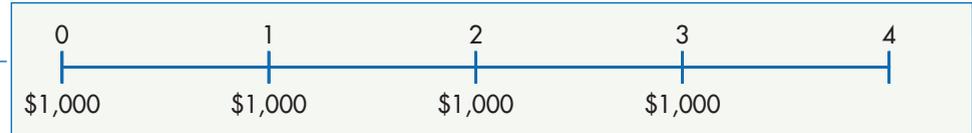


Figure 6.2

## Annuity Due



Annuities have definite beginning and end points in time; they don't go on forever. A stream of equal payments at regular time intervals that does go on forever is called a *perpetuity*. It has to be handled by its own rules, which we'll study later in the chapter.

### The Time Value of Annuities

Annuities are common in business and have important time value implications. For example, suppose a long-term contract calls for payments of \$5,000 a year for 10 years. A question that arises immediately concerns the value of the agreement today. That is, if the recipient wants to discount all the payments for immediate cash, how much will they be worth in total?

A similar question asks for the future value of the entire annuity if all 10 payments are put in the bank when received and left there until the end of the contract.

Either of these questions can be answered by taking the present or future value of each payment separately and adding the results. That is a tedious process, however, involving 10 separate calculations. It's much more convenient to develop expressions that enable us to calculate the present or future value of the entire annuity at once.

We'll begin with the future value problem.

## THE FUTURE VALUE OF AN ANNUITY—DEVELOPING A FORMULA

We can develop an expression for the future value of an annuity that's similar to the formulas we studied for amounts. We'll approach the task by examining the future value of a three-year ordinary annuity, using the tools we acquired in dealing with amounts.

We'll portray the annuity along a time line and represent the yearly cash payment as a variable called *PMT*, as shown in Figure 6.3.

Figure 6.3

### Time Line Portrayal of an Ordinary Annuity

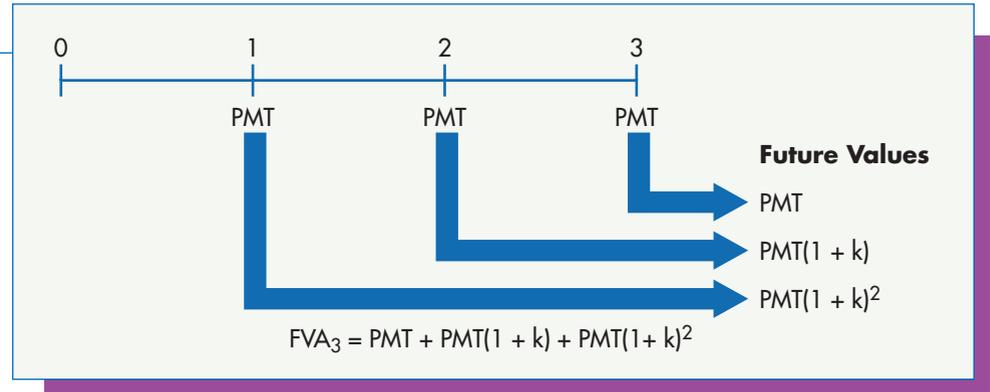


### The Future Value Problem

Precisely stated, the assumption behind the future value of an annuity is that each amount, *PMT*, earns interest at some rate, *k*, from the time it appears on the time line until the end of the last period. The **future value of the annuity** is simply the sum of all the payments and all the interest. This is the same as taking the future value of each *PMT* treated as an amount and adding them.

For example, imagine someone gives you \$100 a year for three years and that you put each payment in the bank as soon as you get it. The future value of an annuity

The **future value of an annuity** is the sum, at its end, of all payments and all interest if each payment is deposited when received.

**Figure 6.4****Future Value of a Three-Year Ordinary Annuity**

problem is to calculate how much you have at the end of the third year. It's clearly more than \$300 because of interest earned.

### The Future Values of the Individual Payments

We'll develop an expression for the future value of an annuity by projecting the future value of each payment to the end of the stream individually. The approach is illustrated in Figure 6.4. We'll call the end of the third year time 3, the end of the second time 2, and so on.

First consider the third payment. It occurs at the end of the annuity, so it spends no time earning interest at all. Therefore, its value at time 3 is simply PMT.

The second payment occurs at time 2, one year before the end of the annuity, and spends one year earning interest. Its value at time 3 is  $PMT(1+k)$ . Think of this as the future value of the present amount PMT for one year at interest rate  $k$ . This comes from equation 6.1 with PMT substituted for PV.

Now consider the first payment. It occurs at the end of the first year and spends two years earning interest. Its value by the end of the annuity is  $PMT(1+k)^2$ .

All this is portrayed graphically in Figure 6.4, along with the sum of the future values of the three payments, which we're calling  $FVA_3$  for future value of an annuity of three periods.

### The Three-Year Formula

Let's rewrite the expression for  $FVA_3$  from Figure 6.4 with two changes and then examine what we have. The first change will be to explicitly recognize the exponent of 1 on  $(1+k)$  in the middle term on the right. That is,  $(1+k) = (1+k)^1$ . It's simply common practice not to write an exponent of 1 even though it's there. The second change involves recognizing that anything raised to a zero exponent equals 1. That is,  $x^0 = 1$  for any value of  $x$ . In this case we're going to multiply the first term on the right by  $(1+k)^0$ . This gives

$$(6.9) \quad FVA_3 = PMT(1+k)^0 + PMT(1+k)^1 + PMT(1+k)^2$$

Notice the regular progression of the terms on the right side of equation 6.9. Each contains PMT multiplied by an increasing power of  $(1+k)$  starting from zero. Notice also for this three-year case that there are three terms and that the exponents start with zero and increase to two, one less than the number of years.

### Generalizing the Expression

Now imagine we have a four-year annuity, and we want to develop a similar expression for it. How would that expression differ from what we've written here for the three-year case?

In a four-year model the first payment would earn interest for three years, so its future value would be  $PMT(1 + k)^3$ . The second payment would earn interest for two years, and the third for one year; the fourth would earn no interest at all. These latter payments would be just like the ones in the three-year case, so the only thing different in the four-year model is the addition of the new term  $PMT(1 + k)^3$ . That addition fits our progression perfectly. It adds one more term with the next higher exponent.

You should be able to see that this could be done for any number of additional years. Each will add one more term with the next higher exponent of  $(1 + k)$ . Further, the highest exponent will always be one less than the number of years. Hence, we can generalize equation 6.9 for any number of years,  $n$ .

$$(6.10) \quad FVA_n = PMT(1 + k)^0 + PMT(1 + k)^1 + PMT(1 + k)^2 + \dots + PMT(1 + k)^{n-1}$$

Equation 6.10 can be written more conveniently by using the mathematical symbol  $\Sigma$ , which implies summation over the values of some index.

$$(6.11) \quad FVA_n = \sum_{i=1}^n PMT(1 + k)^{n-i}$$

As  $i$  ranges from 1 to  $n$ , each of the terms of equation 6.10 is formed in reverse order. For example, when  $i = 1$ ,  $n - i = n - 1$ , and we get the last term. When  $i = 2$  we get the next to last term, and so on, until  $i = n$  and  $n - i = 0$ , which gives us the first term.

Because  $PMT$  appears identically in every term, we can factor it outside of the summation.

$$(6.12) \quad FVA_n = PMT \sum_{i=1}^n (1 + k)^{n-i}$$

### The Future Value Factor for an Annuity

Now look at the entire summation term. It depends only on the values of  $n$  and  $k$ . For example, for  $n = 3$  years the summation is

$$(1 + k)^0 + (1 + k)^1 + (1 + k)^2$$

which is equivalent to

$$1 + (1 + k) + (1 + k)^2$$

In general, the summation term for  $n$  years is

$$1 + (1 + k) + (1 + k)^2 + \dots + (1 + k)^{n-1}$$

This expression can be calculated for pairs of values of  $n$  and  $k$  and placed in a table. The idea is identical to what we did in developing the future value factor for an amount [ $FVF_{k,n} = (1 + k)^n$ ], only this expression is more complex.

We'll call the summation in equation 6.12 the *future value factor for an annuity* and write it as  $FVFA_{k,n}$ . Values for ranges of  $k$  and  $n$  are given in Appendix A-3.

### The Final Formulation

The future value factor for an annuity can replace the summation in equation 6.12 like this.

$$FVA_n = PMT \sum_{i=1}^n (1 + k)^{n-i} = FVFA_{k,n}$$

Rewriting 6.12 using the factor, we get

$$(6.13) \quad FVA_n = PMT[FVFA_{k,n}]$$

## THE FUTURE VALUE OF AN ANNUITY—SOLVING PROBLEMS

We'll use equation 6.13 to solve future value problems where annuities are involved. Notice that there are four variables in this equation:  $FVA_n$  (the future value itself),  $PMT$  (the payment),  $k$  (the interest rate), and  $n$  (the number of periods). Problems will generally give three of them and ask for the fourth. The first step in problem solution is always writing down the equation and substituting the known elements. Once this is done, the solution procedure is very similar to that used for amount problems.

Annuity problems tend to be a bit more complex than amount problems, so it helps to draw a time line to keep the pieces straight.

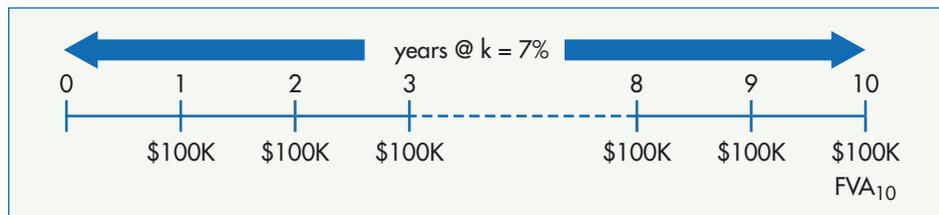
### Example 6.5

The Brock Corporation owns the patent to an industrial process and receives license fees of \$100,000 a year on a 10-year contract for its use. Management plans to invest each payment until the end of the contract to provide a fund for development of a new process at that time. If the invested money is expected to earn 7%, how much will Brock have after the last payment is received?

#### Calculator Solution

Key	Input
n	10
I/Y	7
PMT	100,000
PV	0
	<b>Answer</b>
FV	1,381,645

**SOLUTION:** The time line for this straightforward problem looks like this.<sup>1</sup>



First write equation 6.13,

$$FVA_n = PMT[FVFA_{k,n}]$$

1. A capital K is frequently used to denote thousands of dollars, replacing a comma and three zeros. M can be used to denote millions.

and substitute the given information,

$$FVA_{10} = \$100,000[FVFA_{7,10}]$$

Next look up  $FVFA_{7,10}$  in Appendix A-3, getting 13.8164. Substitute and solve for the future value.

$$\begin{aligned} FVA_{10} &= \$100,000[13.8164] \\ &= \$1,381,640 \end{aligned}$$

Notice that the actual money received is only \$1,000,000; the rest is interest.

## Calculator Solutions for Annuities

Annuity problems are similar to amount problems in that they have four variables of which three are given and one is unknown. However, the variables are somewhat different.

All amount problems involve both the present and future values of the amount. Annuity problems involve a payment (PMT) and *either* the future value or the present value of the annuity. Hence, in an annuity problem we use the PMT key, and we zero either PV or FV, depending on the nature of the problem.

Example 6.5 is a future value of an annuity problem, so we use the FV key, and we zero the PV key. That along with putting in a value for PMT tells the calculator what kind of a problem it's doing. Notice that although we write the future and present values of annuities as FVA and PVA, we just use the FV and PV buttons on the calculator.

## The Sinking Fund Problem

In Chapter 5 we learned that companies borrow money by issuing bonds for periods as long as 30 or 40 years. Bonds are non-amortizing debt, meaning borrowers make no repayment of principal during bonds' lives. Borrowers pay only interest until maturity, and then they must repay the entire principal in a lump sum. This means that on the maturity date, a bond-issuing company must either have a great deal of money on hand or must reborrow to pay off the old bonds coming due.

*Lenders can become quite concerned about this practice. They may feel that a borrowing company can generally earn enough to pay annual interest, but it won't have enough cash on hand at maturity to pay off principal. If the borrower's financial position deteriorates or if financial markets become tight, it may not be able to reborrow either. This can spell bankruptcy for the bond-issuing company and a big loss for the investor/lender.*

The solution to the problem can be a **sinking fund**. A sinking fund is a series of payments made into an account that's dedicated to paying off a bond's principal at maturity. Deposits are planned so that the amount in the bank on the date the bonds mature will just equal the principal due.

If lenders require a sinking fund for security, it's included as a provision in the bond agreement. The sinking fund problem is to determine the periodic deposit that must be made to ensure that the appropriate amount is available at the bond's maturity. This is a future value of an annuity problem in which the payment is unknown.

from the **CFO**

A **sinking fund** provides cash to pay off a bond's principal at maturity.

## Example 6.6

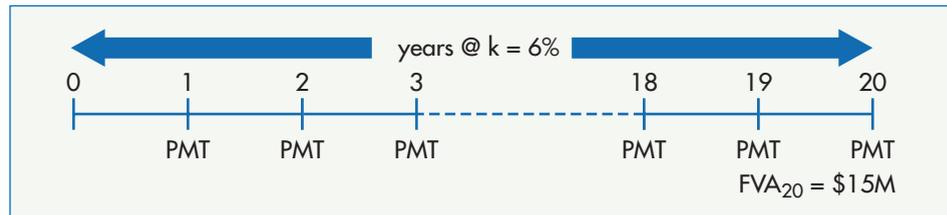
The Greenville Company issued bonds totaling \$15 million for 30 years. The bond agreement specifies that a sinking fund must be maintained after 10 years, which will retire the bonds at maturity. Although no one can accurately predict interest rates, Greenville's bank has estimated that a yield of 6% on deposited funds is realistic for long-term planning. How much should Greenville plan to deposit each year to be able to retire the bonds with the money put aside?

**SOLUTION:** First recognize that the time period of the annuity is the last 20 years of the bond issue's life, because the bond agreement states that the sinking fund must be maintained only after 10 years. In other words, time zero isn't today but the beginning of the eleventh year in the bond's life.

The problem's time line looks like this.

### Calculator Solution

Key	Input
n	20
I/Y	6
FV	15,000,000
PV	0
	<b>Answer</b>
PMT	407,768.35



First write the future value of an annuity formula, equation 6.13.

$$FVA_n = PMT[FVFA_{k,n}]$$

In this case, the future value itself is known. It's the principal amount of the bond issue that will have to be repaid, \$15 million. Also,  $k$  is 6% and  $n$  is 20 years, the duration of the sinking fund according to the contract. Substitute these values.

$$\$15,000,000 = PMT[FVFA_{6,20}]$$

Next look up  $FVFA_{6,20}$  in Appendix A-3, getting 36.7856, and substitute.

$$\$15,000,000 = PMT[36.7856]$$

Finally, solve for PMT.

$$PMT = \$407,768.26$$

Greenville will have to deposit just under \$408K per year starting in the eleventh year of the bond issue's life to ensure that the bonds will be retired on schedule without a problem.

At this point we're going to digress from time value problems themselves to study a little more detail about the workings of interest rates.

## COMPOUND INTEREST AND NON-ANNUAL COMPOUNDING

Until now we've been working with annually compounded interest. Although interest rates are always quoted in annual terms, they're usually not compounded annually, and that varies the actual amount of interest paid. Before going any further, let's be sure we know exactly what the term **compound interest** means.

### Compound Interest

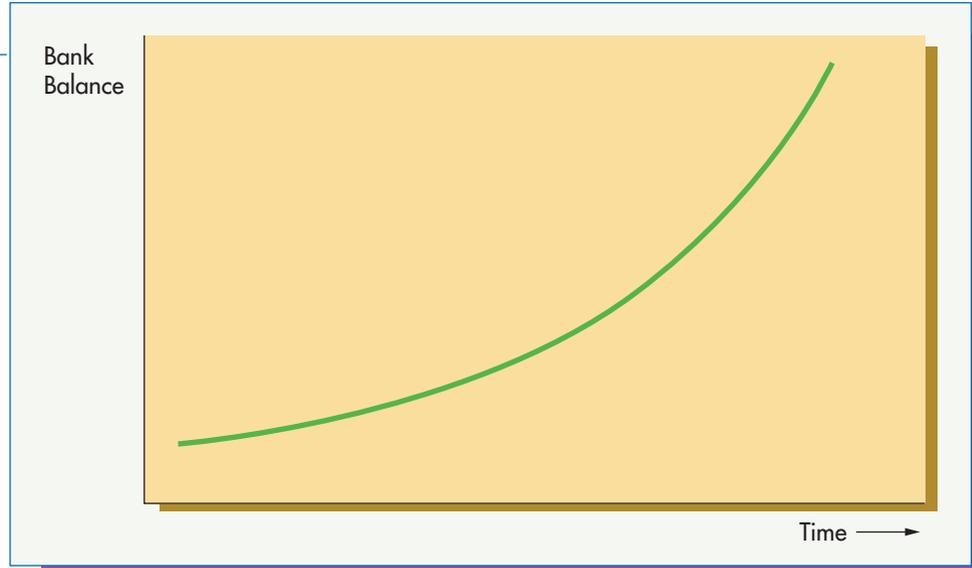
Compounding refers to the idea of earning interest on previously earned interest. Imagine putting \$100 in the bank at 10%. We'd earn \$10 in the first year and have a balance of \$110 at year end. In the second year we'll earn \$11 for a balance of \$121, in the third year \$12.10, and so on. The interest is larger each year because it's calculated on a balance that increases with the accumulation of all prior interest.

Under compound interest the balance in the bank grows at an exponential rate. Graphically, an amount placed at compound interest grows as shown in Figure 6.5. The increasing steepness of the curve as time progresses is characteristic of exponential growth.

**Compounding** refers to earning interest on interest.

**Figure 6.5**

**The Effect of Compound Interest**

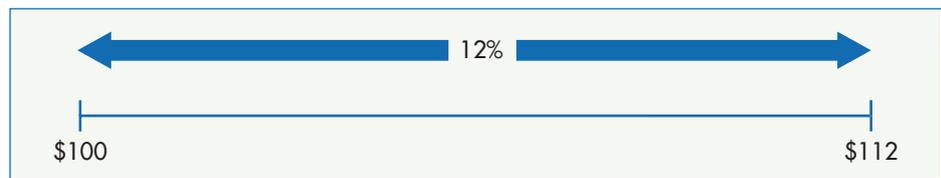


**Compounding Periods**

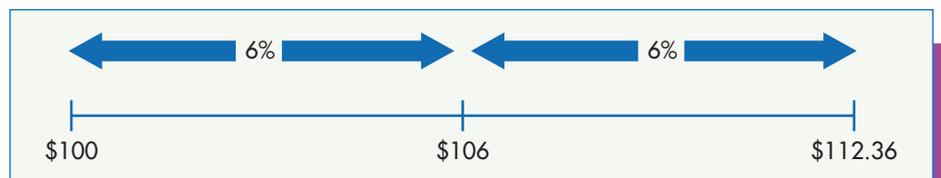
Every interest rate has an associated **compounding period**. Commonly used periods are annual, semiannual, quarterly, and monthly. When none is mentioned, an annual period is implied.

The compounding period associated with an interest rate refers to the frequency with which interest is credited into the recipient's account for the purpose of calculating future interest. The shorter the period, the more frequently interest is credited and the more interest is earned *on interest*.

An example will make the idea clear. If a bank pays 12% interest compounded annually, someone depositing \$100 is credited with \$12 at the end of a year, and the basis for the second year's interest calculation is \$112. A time line portrayal of the year would look like this.

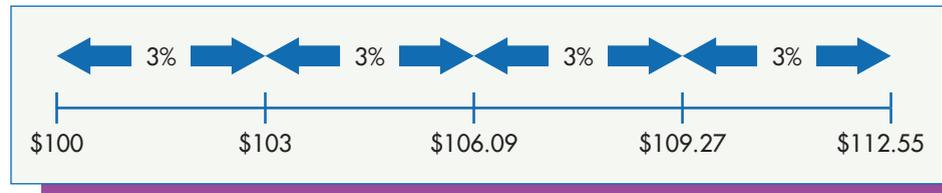


If the 12% is compounded semiannually, the year is divided into two halves and 6% interest is paid in each. However, the first half year's interest is credited to the depositor at midyear and earns additional interest in the second half. The additional interest is 6% of \$6 or \$0.36. The time line portrayal looks like this.



Interest is usually compounded **annually, semi-annually, quarterly, or monthly.**

Compounding 12% quarterly involves dividing the year into four quarters, each paying  $(12\%/4 =) 3\%$ . Each quarter's interest is credited at the end of the quarter. The time line looks like this.



It's easy to get each successive quarter's ending balance by multiplying the previous balance by 1 plus the *quarterly* interest rate in decimal form. That's 1.03 in this case. This is just the  $(1 + k)$  idea we've been working with, but  $k$  is stated for a quarterly compounding period.

Compounding 12% monthly involves dividing the year into 12 monthly periods, each bearing a 1% interest rate. If \$100.00 is initially deposited, the year-end balance will be \$112.68.

It's common practice to quote an annual rate and state the compounding period immediately afterward. The quarterly case in our 12% example would be quoted as "12% compounded quarterly." Those words literally mean 3% interest paid on quarterly periods.

The quoted rate, 12% in this case, is called the **nominal interest rate**. We'll write it as  $k_{\text{nom}}$ . The word "nominal" just means named.

It's possible to pay interest compounded on any time period; however, the periods we've mentioned are the most common in business. Daily compounding is encountered only rarely.

The theoretical limit as periods become shorter is continuous compounding in which interest is instantaneously credited as earned. Continuous compounding takes some special math that we'll discuss later.

### The Effective Annual Rate

Notice in the 12% example above that the final bank balance increases with more frequent compounding. Let's summarize those calculations. For an initial deposit of \$100 and a nominal rate of 12%, Table 6.2 shows the amounts in the bank at the end of one year.

**Table 6.2**

**Year-End Balances at Various Compounding Periods for \$100 Initial Deposit and  $k_{\text{nom}} = 12\%$**

Compounding	Final Balance
Annual	\$112.00
Semiannual	112.36
Quarterly	112.55
Monthly	112.68

These differences in a depositor's balance mean that although all four rates are quoted as 12%, different amounts of interest are actually being paid. As we've explained, the difference is due to the frequency of compounding.

It's important to quantify the effect of different compounding methods to avoid confusion in financial dealings. That is, people need to know just how much more

Interest rates are quoted by stating the **nominal rate** followed by the **compounding period**.

The **effective annual rate (EAR)** is the annually compounded rate that pays the same interest as a lower rate compounded more frequently.

monthly or quarterly compounding pays than annual compounding at any nominal rate. This need for clarification has led to the idea of an **effective annual rate**, referred to as EAR. It's the rate of annually compounded interest that is just equivalent to the nominal rate compounded more frequently. Stated another way, it's the annually compounded rate that gets the depositor the same account balance after one year that he or she would get under more frequent compounding.

Let's consider 12% compounded monthly as an example. What annually compounded interest rate will get a depositor the same interest? Table 6.2 shows that monthly compounding results in an ending balance of \$112.68 on an initial deposit of \$100; hence, the total interest paid is \$12.68.

The annually compounded rate that pays this much interest is calculated by dividing the interest paid by the principal invested.

$$\$12.68/\$100.00 = 12.68\%$$

Hence, 12.68% compounded annually is *effectively* equal to 12% compounded monthly. What are the EARs for semiannual and quarterly compounding at 12%?

Truth in lending legislation requires that lenders disclose the EAR on loans. Watch for it the next time you see an advertisement for a bank.

In general the EAR can be calculated for any compounding period by using the following formula.

$$(6.14) \quad \text{EAR} = \left(1 + \frac{k_{\text{nom}}}{m}\right)^m - 1$$

where *m* is the number of compounding periods per year (12 for monthly, 4 for quarterly, and 2 for semiannually).

The effect of more frequent compounding is greater at higher interest rates. Table 6.3 illustrates this point. At a nominal rate of 6%, the effective increase in interest due to monthly rather than annual compounding is only .17%, which represents a 2.8% increase in the rate actually paid (.17/6.00 = .028 = 2.8%). At 18%, however, the effective increase is 1.56%, which represents an 8.7% increase in what's actually paid.

**Table 6.3**

**Changes in the Effect of Compounding at Different Rates**

Nominal Rate	EAR for Monthly Compounding	Effective Increase	Increase as % of $k_{\text{nom}}$
6%	6.17%	.17%	2.8%
12	12.68	.68	5.7
18	19.56	1.56	8.7

**The APR and the EAR**

Some credit card companies charge monthly interest on unpaid balances at rates in the neighborhood of 1.5%. This represents a monthly compounding of interest on the cardholder's debt. They advertise that the annual percentage rate, known as the APR, is 18%, 12 times the monthly rate.

The **annual percentage rate (APR)** associated with credit cards is actually the nominal rate and is less than the **EAR**.

Don't confuse the **APR** with the **EAR**. The APR is actually the nominal rate. Table 6.3 shows that at a nominal rate of 18% the EAR for monthly compounding is 19.56%, somewhat more than 18%.

### Compounding Periods and the Time Value Formulas

Each of the time value formulas contains an interest rate,  $k$ , and a number time of periods,  $n$ . In using the formulas, the time periods must be compounding periods, and the interest rate must be the rate for a single compounding period.

The problems we've dealt with so far have all involved annual compounding. In that case, the compounding period is a year, and the appropriate interest rate is the nominal rate itself. Things are a little more complicated with non-annual compounding periods. Let's consider quarters as an example.

Suppose we have a time value problem that runs for five years and has an interest rate of 12%. If compounding is annual,  $k$  and  $n$  are simply 12 and 5, respectively. However, if compounding is quarterly, the appropriate period is one quarter and the rate for that period is  $(12\%/4 = )$  3%. Further, the time dimension of the problem needs to be stated as 20 quarters rather than five years ( $5 \text{ years} \times 4 \text{ quarters/year} = 20 \text{ quarters}$ ). Hence,  $k$  and  $n$  for the problem should be 3 and 20, respectively.

Whenever we run into a problem with non-annual compounding, we have to calculate the appropriate  $k$  and  $n$  for use in the formulas from the nominal rate and time given in the problem. Some simple rules make that relatively easy to do.

If a problem gives a nominal rate and states time in years, compute  $k$  and  $n$  for use in the formulas as follows.

Semiannual:	$k = k_{\text{nom}}/2$	$n = \text{years} \times 2$
Quarterly:	$k = k_{\text{nom}}/4$	$n = \text{years} \times 4$
Monthly:	$k = k_{\text{nom}}/12$	$n = \text{years} \times 12$

Recall that some calculators will automatically divide the interest input by a number of compounding periods for you. That feature is convenient if you're working with the same kind of compounding all the time. But because we're switching from one to another, it's better to leave the setting at one (1) and input the interest rate for the compounding period.

Let's try two problems involving the future value of an annuity to get used to these ideas (Examples 6.7 and 6.8).

### Example 6.7

You want to buy a car costing \$15,000 in  $2\frac{1}{2}$  years. You plan to save the money by making equal monthly deposits in your bank account, which pays 12% compounded monthly. How much must you deposit each month?

#### Calculator Solution

Key	Input
n	30
I/Y	1
FV	15,000
PV	0
	<b>Answer</b>
PMT	431.22

**SOLUTION:** In this situation the future value of a series of payments must accumulate to a known amount, indicating a future value of an annuity problem.

First calculate the correct  $k$  and  $n$ . Because compounding is monthly,

$$k = \frac{k_{\text{nom}}}{12} = \frac{12\%}{12} = 1\%$$

and

$$n = 2.5 \text{ years} \times 12 \text{ months/year} = 30 \text{ months.}$$

Next write the future value of an annuity expression and substitute.

$$FVA_n = PMT[FVFA_{k,n}]$$

$$\$15,000 = PMT[FVFA_{1,30}]$$

Use Appendix A-3 to find  $FVFA_{1,30} = 34.7849$  and substitute.

$$\$15,000 = PMT[34.7849]$$

Finally, solve for PMT.

$$PMT = \$431.22$$

### Example 6.8

Jeff and Susan Johnson have a daughter, Molly, just entering high school, and they've started to think about sending her to college. They expect to need about \$50,000 in cash when she starts. Although the Johnsons have a good income, they live extravagantly and have little or no savings. Susan analyzed the family budget and decided they could realistically put away \$750 a month or \$2,250 per quarter toward Molly's schooling. They're now searching for an investment vehicle that will provide a return sufficient to grow these savings into \$50,000 in four years. If quarterly compounding is assumed, how large a return (interest rate) do the Johnsons have to get to achieve their goal? Is it realistic?

**SOLUTION:** Once again we recognize this as a future value of an annuity problem because of the stream of payments involved and the fact that the Johnsons are saving for a known future amount. Because the problem runs for four years and the compounding is quarterly,  $n$  is calculated as

$$n = 4 \text{ years} \times 4 \text{ quarters/year} = 16 \text{ quarters.}$$

Equation 6.13 gives the future value of an annuity expression.

$$FVA_n = PMT[FVFA_{k,n}]$$

Substituting values from the problems, we have

$$\$50,000 = \$2,250[FVFA_{k,16}]$$

Solving for the factor yields

$$FVFA_{k,16} = 22.2222$$

In Appendix A-3 we search for this value along the row for 16 periods and find that it lies between 4% and 4.5%. In this case it's fairly easy to estimate that the factor is about half of the way between 4% and 4.5%.

Hence, the approximate solution is 4.2%; however, that's a quarterly rate. The appropriate nominal rate is

$$4.2\% \times 4 = 16.8\%.$$

This is a high rate of return to expect on invested money. Is it reasonable to expect such a rate to be sustained over four years?

There's no definite answer to that question. There have been times when that expectation would have been reasonable, but such a high rate can always be expected to involve substantial risk. Because they probably don't want to risk not being able to send Molly to college, the Johnsons should probably try to save a little more and opt for a more conservative investment.

#### Calculator Solution

Key	Input
n	16
PMT	2,250
FV	50,000
PV	0
	<b>Answer</b>
I/Y	4.2

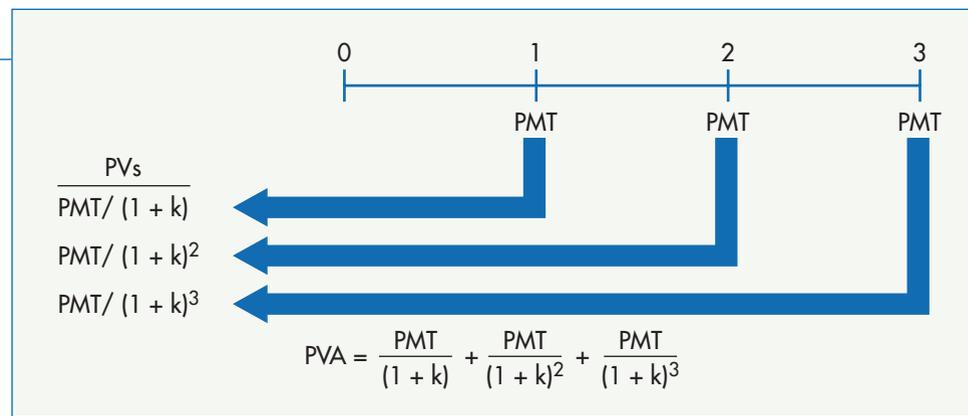
## THE PRESENT VALUE OF AN ANNUITY—DEVELOPING A FORMULA

The present value of an annuity is simply the sum of the present values of all of the annuity's payments. We could always calculate these individually, but it's much easier to develop a formula to do all the calculations in one step as we did with the future value. The method we'll use is similar to that used in developing the future value formula, but we'll proceed more quickly because we've used the approach before.

We begin with a time line portrayal of a three-period annuity and write down the present value of each payment in terms of the interest rate,  $k$ . In this case we divide by powers of  $(1 + k)$  instead of multiplying as in the future value case. Review equations 6.5 and 6.6 to see that this gives the present value of an amount. Figure 6.6 is the time line portrayal.

**Figure 6.6**

**Present Value of a Three-Period Ordinary Annuity**



[http://](http://www.stonestreet.com)

Stone Street Capital will buy your winning lottery ticket by paying you its present value. See its site at

<http://www.stonestreet.com>

The present value is formed for the first payment by dividing the payment amount by  $(1 + k)$ , for the second payment by dividing by  $(1 + k)^2$ , and so forth. Notice that this is equivalent to multiplying by present value factors, because  $1/(1 + k)$  is the present value factor for  $k$  and one period,  $PVF_{k,1}$ ;  $1/(1 + k)^2$  is the present value factor for  $k$  and two periods; and so on.

The present value of the three-period annuity is

$$(6.15) \quad PVA = \frac{PMT}{(1+k)} + \frac{PMT}{(1+k)^2} + \frac{PMT}{(1+k)^3}$$

which can also be written as

$$(6.15) \quad PVA = PMT(1+k)^{-1} + PMT(1+k)^{-2} + PMT(1+k)^{-3}$$

with negative exponents to indicate one over the powers of  $(1 + k)$ .

Notice how regular the expression is. Every payment produces a term involving  $PMT$  divided by  $(1 + k)$  to a successively larger power beginning with 1.

Examining Figure 6.6, we can easily see that adding more periods to the annuity would just add more terms to the equation. For example, a fourth payment would produce a term  $PMT(1 + k)^{-4}$ , and so on.

Thus we can generalize equation 6.15 for any number of periods,  $n$ , as follows.

$$(6.16) \quad PVA = PMT(1+k)^{-1} + PMT(1+k)^{-2} + \dots + PMT(1+k)^{-n}$$

Next we can factor PMT out of the right side of equation 6.16 and use summation notation to represent the terms involving negative powers of  $(1 + k)$ .

$$(6.17) \quad PVA = PMT \left[ \sum_{i=1}^n (1 + k)^{-i} \right]$$

Once again, we notice that the expression in the brackets is a function of only  $k$  and  $n$ , and can be tabulated for likely values of those variables. This is the present value factor for an annuity and is written as  $PVFA_{k,n}$ .

$$(6.18) \quad PVFA_{k,n} = \sum_{i=1}^n (1 + k)^{-i}$$

Values of the present value factor for an annuity are tabulated in Appendix A-4.

Finally, we can rewrite equation 6.17 by substituting from equation 6.18. The resulting expression is convenient for use in solving problems when used in conjunction with Appendix A-4.

$$(6.19) \quad PVA = PMT[PVFA_{k,n}]$$

### THE PRESENT VALUE OF AN ANNUITY—SOLVING PROBLEMS

Equation 6.19 for the present value of an annuity works just like equation 6.13 does for the future value of an annuity. There are four variables: PVA (the present value itself), PMT (the payment),  $k$  (the interest rate), and  $n$  (the number of periods). Problems will generally present three of them as known and ask you to find the fourth. The general approach is similar to what we've already been doing.

#### Example 6.9

The Shipson Company has just sold a large machine to Baltimore Inc. on an installment contract. The contract calls for Baltimore to make payments of \$5,000 every six months (semiannually) for 10 years. Shipson would like its cash now and asks its bank to discount the contract and pay it the present (discounted) value. Baltimore is a good credit risk, so the bank is willing to discount the contract at 14% compounded semiannually. How much will Shipson receive?

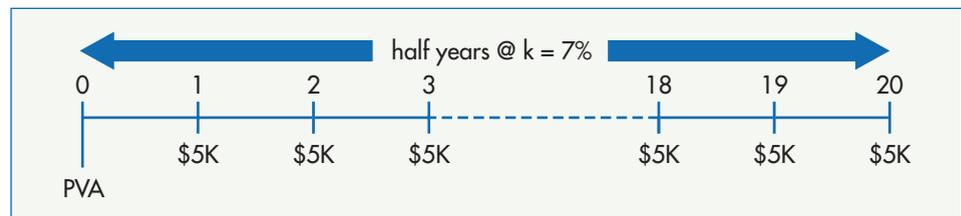
**SOLUTION:** The contract represents an annuity with payments of \$5,000. The bank is willing to buy it for its present value at a relatively high rate of interest. The higher the rate of interest, the lower the price the bank is willing to pay for the contract.

First calculate the appropriate  $k$  and  $n$  for semiannual compounding.

$$k = k_{\text{nom}}/2 = 14\%/2 = 7\%$$

$$n = 10 \text{ years} \times 2 = 20.$$

The time line looks like this.



#### Calculator Solution

Key	Input
n	20
I/Y	7
PMT	5,000
FV	0
	<b>Answer</b>
PV	52,970.07

Write equation 6.19 and substitute the known information.

$$PVA = PMT[PVFA_{k,n}]$$

$$PVA = \$5,000[PVFA_{7,20}]$$

Appendix A-4 gives  $PVFA_{7,20} = 10.5940$ . Substituting and solving for PVA yields

$$PVA = \$52,970$$

### Spreadsheet Solutions

Time value problems can be solved using spreadsheet programs like Lotus 123™ or Microsoft Excel™. The technique is similar to using a calculator. We'll explain the technique assuming that you're familiar with the basics of spreadsheet software.

Recall that there are four variables in every time value problem, but they're different depending on whether we're dealing with amounts or annuities. In amount problems we have PV, FV, k, and n, while in annuity problems we have *either* PVA or FVA, PMT, k, and n. Also recall that when using a calculator we use the PV and FV keys for both amounts and annuities. We can do this because the calculator is programmed to solve an annuity problem if we input a positive value for PMT and zero for either PV or FV. It solves an amount problem if we input zero for PMT. Hence, there is a total of five possible variables as follows.

k   n   PV   FV   PMT

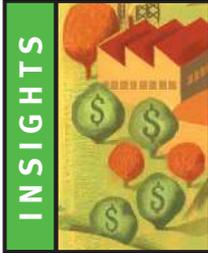
To solve a problem with a calculator we enter three numbers and zero a fourth. The calculator then gives us the unknown fifth variable.

A spreadsheet program uses similar logic. There are five spreadsheet time value functions. Each is used to calculate one of the five time value variables. Each function takes the *other* four variables as inputs. We'll use Microsoft Excel to illustrate. The five functions are as follows

To Solve For	Use Function
FV	FV(k, n, PMT, PV)
PV	PV(k, n, PMT, FV)
k	RATE(n, PMT, PV, FV)
n	NPER(k, PMT, PV, FV)
PMT	PMT(k, n, PV, FV)

To solve any time value problem, select the function for the unknown variable, put the problem values for the three known variables in the proper order within the parentheses, and input zero for the fourth variable. (Zero PMT for all amount problems, zero PV for future value of an annuity (FVA) problems, and zero FV for present value of an annuity (PVA) problems.)

There are two minor complications in the procedure. The first is simply that interest rates are entered in decimal form rather than as whole numbers, so use .07 for 7%. The second involves the signs of the cash figures. Notice that there are three cash variables—FV, PV, and PMT—one of which is always zero. Hence, in every problem there are two dollar variables. These must be of opposite signs.



## PRACTICAL FINANCE

### The Lottery: Congratulations, You're Rich—But Not as Rich as You Thought

State lottery jackpots are enormous sums of money, but they're not really as big as they're made out to be. That's because of the time value of money and the way the prizes are paid. Large lottery prizes are typically paid over 25 years, but the lottery authority states the winnings as the sum of the payments without consideration of time value. For example, a \$25 million prize is really \$1 million a year for 25 years, an annuity.

What the winner really has today is the present value of that annuity. If a lucky player wants her money immediately, she has to accept the discounted value of the stream of payments. Suppose the interest rate is 7%. A calculation using the present value of an annuity formula reveals that the winner's real prize is about \$11.7 million. That's nothing to sneeze at, but it is a far cry from \$25 million.

To make matters worse, winnings are taxable, largely in the top bracket. Let's be optimistic and assume the winner hires a good tax accountant and only winds up paying about 32% in taxes. That knocks down the immediately available, after-tax winnings to about \$8.4 million, less than a third of the amount advertised.

The easiest way to think of this issue is in terms of inflows and outflows. Imagine, for example, a simple problem in which we're depositing a sum of money in a bank today (PV) and withdrawing it with interest (FV) some years later. If we define flows into the bank as positives, then flows out must be negatives. Hence, if PV is positive, FV is negative, whether the figures are inputs or outputs of the calculation. The reverse definition is also OK, as long as the variables have opposite signs.

This convention can create a bit of a problem if you forget it and input only positive figures. In some applications the program simply gives you the correct answer with a negative sign, but in others the program doesn't work at all. Hence, the first thing to check when you get an error is the sign of your inputs.

Here's an example of the whole process. Suppose we want to calculate the amount we'll have in the bank after six years if we deposit \$4,000 today at 7% interest. We're looking for the future value of an amount, so we choose the first function,

$$\text{FV}(k, n, \text{PMT}, \text{PV})$$

and input as follows.

$$\text{FV}(.07, 6, 0, -4000)$$

Notice that we input 0 for PMT because we're not doing an annuity problem, we input the interest rate in decimal form, and we input the PV as a negative.

Now let's change things just a little to illustrate an annuity problem. Suppose we want to know the future value of a \$4,000 annual annuity for six years at 7%. We choose the future value (FV) function again and input as follows.

$$\text{FV}(.07, 6, -4000, 0)$$

Notice we're now telling the program that it's dealing with an annuity by inputting a nonzero number for PMT. And because there's no present value in a future value of an annuity calculation (see equation 6.13 on page 234), we input 0 for PV. The cell carrying this function will display the present value of our annuity.

Boot up your computer and verify that these examples yield answers of \$6,002.92 and \$28,613.16, respectively.

**Example 6.10** The bank in Example 6.9 discounts contracts for customers like Shipson frequently. Contracts can have payments of any constant amount for any number of periods, and the bank's interest rate changes frequently. Write a spreadsheet program for the bank that will calculate the discounted amount that should be paid on any contract like Shipson's after the interest rate, the term of the contract (in payment periods), and the amount of the payment are input into conveniently labeled cells.

**SOLUTION:** In this case the bank wants to calculate the present value of an annuity, so we'll use the PV function

$$PV(k, n, PMT, FV)$$

Since this program will only be used for one thing, we can customize the formula by zeroing FV and making the PMT variable negative.

$$PV(k, n, -PMT, 0)$$

Then the banker just has to input the appropriate interest rate, number of periods, and payment as a positive number. Here we'll input .07 for k, one-half of the annual 14% rate, 20 semi-annual periods, and payments of \$5,000. Notice that we do have to deal with non-annual compounding just as we did using calculators or tables.

Here is a spreadsheet that does the job, along with a note detailing the operative formula. Notice that the input cells are programmed directly into the PV function along with a negative sign for the payment and a zero for the future value variable. The banker inputs the appropriate values into the blue cells, and the discounted amount of the contract appears in the brown cell.

	A	B	C	D	E	F	G
1							
2			<b>DISCOUNTING A CONTRACT</b>				
3							
4							
5		<b>Inputs</b>					
6							
7		<b>Interest Rate</b>			0.07		
8							
9		<b>Payment Periods</b>			20		
10							
11		<b>Payment Amount</b>			5000		
12							
13							
14		<b>Discounted Value</b>			\$52,970.07		
15							
16							
17							
18		The formula in E14 is =PV(E7,E9,-E11,0)					
19							
20							

## Amortized Loans

The most common application of the present value of an annuity concept is in dealing with amortized loans. Debt is said to be amortized when the principal is paid off gradually during its life. Car loans, home mortgages, and many business loans are amortized.

An **amortized loan** is generally structured so that a *constant* payment is made periodically, usually monthly, over the loan's term. Each payment contains one month's interest and an amount to reduce principal. Interest is charged on the outstanding loan balance at the beginning of each month, so as the loan's principal is reduced, successive interest charges become smaller. Because the monthly payments are equal, successive payments contain larger proportions of principal repayment and smaller proportions of interest.

In applying the present value of an annuity formula to an amortized loan, the amount borrowed is always the present value of the annuity, PVA, and the loan payment is always PMT.

An **amortized loan's** principal is paid off regularly over its life.

**Example 6.11** Suppose you borrow \$10,000 over four years at 18% compounded monthly repayable in monthly installments. How much is your loan payment?

### Calculator Solution

Key	Input
n	48
I/Y	1.5
PV	10,000
FV	0
	<b>Answer</b>
PMT	293.75

**SOLUTION:** First notice that for monthly compounding  $k$  and  $n$  are

$$k = k_{\text{nom}}/12 = 18\%/12 = 1.5\%$$

$$n = 4 \text{ years} \times 12 \text{ months/year} = 48 \text{ months.}$$

Then write equation 6.19 and substitute.

$$PVA = PMT[PVFA_{k,n}]$$

$$\$10,000 = PMT[PVFA_{1.5,48}]$$

Appendix A-4 gives  $PVFA_{1.5,48} = 34.0426$ , and

$$PMT = \$293.75$$

**Example 6.12** Suppose you want to buy a car and can afford to make payments of \$500 a month. The bank makes three-year car loans at 12% compounded monthly. How much can you borrow toward a new car?

### Calculator Solution

Key	Input
n	36
I/Y	1
PMT	500
FV	0
	<b>Answer</b>
PV	15,053.75

**SOLUTION:** For monthly compounding,

$$k = k_{\text{nom}}/12 = 12\%/12 = 1\%$$

$$n = 3 \text{ years} \times 12 \text{ months/year} = 36 \text{ months.}$$

Write equation 6.19 and substitute.

$$PVA = PMT[PVFA_{k,n}]$$

$$PVA = \$500[PVFA_{1,36}]$$

Appendix A-4 gives  $PVFA_{1,36} = 30.1075$ , and

$$PVA = \$15,053.75$$

That is, the bank would lend you \$15,053.75.

**Loan amortization schedules** detail the interest and principal in each loan payment.

## Loan Amortization Schedules

A loan amortization schedule lists every payment and shows how much of it goes to pay interest and how much reduces principal. It also shows the beginning and ending balances of unpaid principal for each period.

To construct an amortization schedule we have to know the loan amount, the payment, and the periodic interest rate. That's PVA, PMT, and  $k$ . Let's use the loan in the last example as an illustration. Table 6.4 shows the completed computation for the first two lines. Follow the explanation in the next paragraph for the first line, verify the second line, and fill in the third and fourth lines yourself.

**Table 6.4**

### A Partial Amortization Schedule

Period	Beginning Balance	Payment	Interest @1%	Principal Reduction	Ending Balance
1	\$15,053.75	\$500.00	\$150.54	\$349.46	\$14,704.29
2	14,704.29	500.00	147.04	352.96	14,351.33
3	_____	500.00	_____	_____	_____
4	_____	500.00	_____	_____	_____
⋮	⋮	⋮	⋮	⋮	⋮

The loan amount is \$15,053.75. This is the beginning balance for the first monthly period. The payment is a constant \$500.00; that amount is entered on every row in the payment column. Although the nominal interest rate in this case is 12%, the monthly interest rate is 1% because compounding is monthly. Therefore, the monthly interest charge is calculated as 1% of the month's beginning loan balance.

$$\$15,053.75 \times .01 = \$150.54$$

As the payment is \$500 and \$150.54 goes to interest, the remaining ( $\$500.00 - \$150.54 =$ ) \$349.46 reduces principal. The ending loan balance is the beginning balance less the principal reduction, ( $\$15,053.75 - \$349.46 =$ ) \$14,704.29. This amount becomes the beginning balance for the next period, and the process is repeated.

This procedure carried out for 36 monthly periods will bring the ending balance to zero at the end of the last period. It's important to notice what happens to the composition of the payment as the loan is paid down. The interest charge declines, and the portion devoted to principal reduction increases, while the total payment remains constant.

### Calculator Solution

<b>Key</b>	<b>Input</b>
n	360
I/Y	.5
PV	100,000
FV	0
	<b>Answer</b>
PMT	599.55

### Mortgage Loans

Loans used to buy real estate are called **mortgage loans** or just *mortgages*. A home mortgage is often the largest single financial transaction in an average person's life. A typical mortgage is an amortized loan with monthly compounding and payments that run for 30 years; that's 360 payments.

In the beginning of a mortgage's life, most of the payment goes to pay interest. For example, consider that a 30-year mortgage at 6% (compounded monthly) for \$100,000 has a monthly payment of \$599.55 (verify this by using equation 6.19 and Appendix A-4). The first month's interest on such a loan is  $\frac{1}{2}\%$  of \$100,000 or \$500. Hence, \$99.55 is applied to principal. The first payment is 83.4% interest!

This situation reverses toward the end of the mortgage when most of the payment is principal. In other words, during the early years of a mortgage, the principal is paid down slowly, but near the end it's amortized quickly.

This payment pattern has two important implications for homeowners. The most important is related to the fact that mortgage interest is tax deductible. Early mortgage payments provide homeowners with a big tax deduction, while later payments don't.

Consider the first payment on the loan we just used as an example. If the homeowner is in the 25% tax bracket, he or she will save \$125 in taxes by making that payment because it contains deductible interest of \$500 ( $\$500 \times .25 = \$125$ ).

Hence, the *effective* cost of the loan payment is found as follows.

Payment	\$599.55
Tax savings	<u>125.00</u>
Net	\$474.55

In effect, the government shares the cost of home ownership, especially in the early years. Later on, although equity builds up faster, the tax benefit isn't nearly as great. (In this context equity means the portion of the home's value that belongs to the homeowner as opposed to being supported by a bank loan.)

The second implication of the mortgage payment pattern is that halfway through a mortgage's life, the homeowner hasn't paid off half the loan. To see that in the loan we've been talking about, let's calculate the present value of the second half of the payment stream as of the end of year 15. That will be the amount one could borrow making 180 payments of \$599.55. Because this is what's left after 15 years, it must represent the remaining loan balance. We'll use the same expression for the present value of an annuity.

**Calculator Solution**

<b>Key</b>	<b>Input</b>
n	180
I/Y	.5
PMT	599.55
FV	0
	<b>Answer</b>
PV	71,048.78

$$\begin{aligned}
 PVA &= PMT[PVFA_{k,n}] \\
 &= \$599.55[PVFA_{.5,180}] \\
 &= \$599.55[118.504] \\
 &= \$71,049.07
 \end{aligned}$$

Thus, halfway through the life of this \$100,000 mortgage, roughly \$71,000 is still outstanding. In other words, only about 29% of the original loan has been paid off.

Another interesting feature of a long-term amortized loan like a mortgage is the total amount of interest paid over the entire term. At 6% the homeowner pays approximately 87% of the amount of the loan in interest even after considering the tax savings.

Total payments ( $\$599.55 \times 360$ )	\$215,838.00
Less original loan	<u>100,000.00</u>
Total interest	\$115,838.00
Tax savings @ 25%	<u>28,959.50</u>
Net interest cost	\$ 86,878.50

Of course, this effect varies dramatically with the interest rate. Over the last 30 years, rates have varied between 5% and 16%. Recently, in the early 2000s, they've

Early mortgage payments are **almost all interest** and provide a big tax savings.

been on the low end of that range. Verify that the net after-tax interest cost of a \$100,000 mortgage loan at 8% is \$123,116.28.<sup>2</sup>

### Amortized Loans and Tax Planning

Because interest payments on business loans and home mortgages are tax deductible, we're sometimes interested in projecting the total interest and principal payments to be made during a particular future year of a loan's life. If we don't want to write out the entire amortization schedule, we can solve the problem by calculating the loan balance at the beginning and end of the year.

To illustrate, let's calculate the principal and interest payments in the third year of the \$100,000 loan with which we've been working. The loan balance at the beginning of the third year will be the amount left after 24 payments have been made and there are 336 left to go. We find it by using equation 6.19 ( $PVFA_{.5,336} = 162.569$ ;  $PVFA_{.5,324} = 160.260$ ).

#### Calculator Solution

Key	Input
n	336
I/Y	.5
PMT	599.55
FV	0
	<b>Answer</b>
PV	97,468.15

$$\begin{aligned} PVA &= PMT[PVFA_{k,n}] \\ &= \$599.55[PVFA_{.5,336}] \\ &= \$599.55[162.569] \\ &= \$97,468.24 \end{aligned}$$

Similarly, 324 payments will be left after three years.

#### Calculator Solution

Key	Input
n	324
I/Y	.5
PMT	599.55
FV	0
	<b>Answer</b>
PV	96,083.99

$$\begin{aligned} PVA &= \$599.55[PFVA_{.5,324}] \\ &= \$599.55[160.260] \\ &= \$96,083.88 \end{aligned}$$

The difference between these balances, \$1,384.36, is the amount paid into principal during the year. There are 12 payments totaling ( $\$599.55 \times 12 =$ ) \$7,194.60, so the interest portion is this amount less the contribution to equity.

Total payments	\$7,194.60
Principal reduction	<u>(1,384.36)</u>
Deductible interest	\$5,810.24

## THE ANNUITY DUE

So far we've dealt only with ordinary annuities in which payments occur at the ends of time periods. When payments occur at the beginnings of time periods, we have an annuity due, and our formulas need to be modified somewhat.

### The Future Value of an Annuity Due

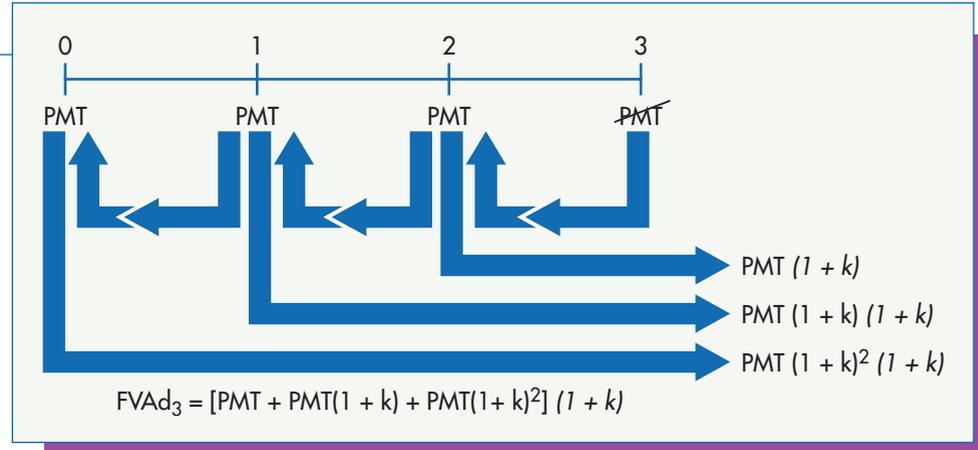
First consider the future value of an annuity formula as developed in Figure 6.4. Review that figure on page 232 now. Because the end of one period is the beginning of the next, we can create the annuity due by simply shifting each payment back one period in time. This is shown schematically in Figure 6.7. There is now a payment at time 0, but none at time 3.

In an **annuity due** payments occur at the **beginning** of each period.

2. At 12%, a typical mortgage rate in the 1980s, the first mortgage payment is 97% interest, and the net after-tax interest cost over 30 years is almost twice the amount borrowed.

Figure 6.7

## Future Value of a Three-Period Annuity Due



Because each payment is received one period earlier, it spends one period longer in the bank earning interest. Therefore, each payment's future value at the end of the annuity will be whatever it was before times  $(1+k)$ . The additional  $(1+k)$  is shown in italics in the diagram.

The future value of the annuity due, which we'll call  $FVAd_3$ , is then

$$FVAd_3 = PMT(1+k) + PMT(1+k)(1+k) + PMT(1+k)^2(1+k)$$

which can be rewritten by factoring out the additional  $(1+k)$  as

$$(6.20) \quad FVAd_3 = [PMT + PMT(1+k) + PMT(1+k)^2](1+k)$$

It's easy to see that no matter how many periods we choose to add, every term in an annuity due will be the same as it is in an ordinary annuity multiplied by an extra  $(1+k)$ . Therefore, we can generalize equation 6.20 to  $n$  periods.

$$(6.21) \quad FVAd_n = [PMT + PMT(1+k) + \dots + PMT(1+k)^{n-1}](1+k)$$

Once we've done that, the term inside the brackets can be developed into the ordinary annuity formula just as before. The only thing changed is the addition of the  $(1+k)$  factor on the right. Hence, the final formula for an annuity due is just our old formula for an ordinary annuity multiplied by  $(1+k)$ .

$$(6.22) \quad FVAd_n = PMT[FVFA_{k,n}](1+k)$$

Situations in which an annuity due is appropriate can be recognized when words such as "starting now," "starting today," or "starting immediately" are used to describe a payment stream.

**Example 6.13** The Baxter Corporation started making sinking fund deposits of \$50,000 per quarter today. Baxter's bank pays 8% compounded quarterly, and the payments will be made for 10 years. What will the fund be worth at the end of that time?

**Calculator Solution** **SOLUTION:** First calculate  $k$  and  $n$ .

Key	Input
n	40
I/Y	2
PMT	50,000
PV	0
<b>Answer</b>	
FV	3,020,099
	× 1.02
	<u>3,080,501</u>

$$k = 8\%/4 = 2\%$$

$$n = 10 \text{ years} \times 4 \text{ quarters/year} = 40 \text{ quarters}$$

Next write equation 6.22 and substitute known values from the problem.

$$FVAd_n = PMT[FVFA_{k,n}](1 + k)$$

$$FVAd_{40} = \$50,000[FVFA_{2,40}](1.02)$$

Get  $FVFA_{2,40} = 60.4020$  from Appendix A-3 and substitute.

$$\begin{aligned} FVAd_{40} &= \$50,000[60.4020](1.02) \\ &= \$3,080,502 \end{aligned}$$

Advanced calculators let you set annuity payments at either the beginning or end of periods. If you set the beginning, the calculator takes care of the  $(1 + k)$  multiplication automatically. However, if you're only doing an occasional annuity due problem, it's just as easy to multiply manually.

### The Present Value of an Annuity Due

Applying very similar logic to the derivation of the present value of an annuity expression developed in Figure 6.7 yields a formula for the present value of an annuity due, which we'll call PVAd.

$$(6.23) \quad PVAd = PMT[PVFA_{k,n}](1 + k)$$

This expression is used in the same way as equation 6.22.

As an exercise, work your way through the development of equation 6.23 by using Figure 6.7 and the approach we've just gone through in developing equation 6.22.

### Recognizing Types of Annuity Problems

The most common errors in working annuity problems result from confusion over whether to use the present or future value technique. Here's a little guidance on how to keep the two straight.

First, an annuity problem is always recognized by the presence of a *stream of equal* payments. Whether the value of the payments is known or unknown, a series of them means an annuity.

Annuity problems always involve some kind of a transaction at one end of the stream of payments or the other. If the transaction is at the end of the stream, you have a future value problem. If the transaction is in the beginning, you have a present value problem. Here's a graphic representation of this idea.



A loan is always a present value of an annuity problem. The annuity itself is the stream of loan payments. The transaction is the transfer of the amount borrowed from the lender to the borrower. That always occurs at the beginning of the payment stream.

Putting aside money to pay for something in the future (saving up) is always a future value of an annuity problem. For example, suppose we're saving up to buy a car by depositing equal sums in the bank each month. The deposits are the payments, and the car purchase is the transaction at the end of the payment stream.

## PERPETUITIES

A series of equal payments that occur at equal intervals and go on forever is called a **perpetuity**. You can think of a perpetuity as an infinite annuity although it's not really an annuity.

The concept of future value clearly doesn't make sense for perpetuities, because there's no end point in time to which future values can be projected. The present value of a perpetuity, however, does make sense.

The present value of a perpetuity, like that of an annuity, is the sum of the present values of all the individual payments. At first that doesn't seem to make sense either, because you'd think the sum of the present values of an infinite number of payments would be an infinite number itself.

However, the *present value* of each payment in an infinite stream is a diminishing series of numbers. Each payment's PV contribution to the sum is smaller than that of the one before because of the fact that it's farther out into the future. Mathematically, the sum of such a diminishing series of numbers turns out to be finite. Further, the computation of that finite value is rather simple.

The present value of a perpetuity of payments of amount PMT, at interest rate  $k$ , which we'll call  $PV_P$ , is just

$$(6.24) \quad PV_P = \frac{PMT}{k}$$

where  $k$  is the interest rate for the period on which the payment is made. For example, if the payment is made quarterly and interest is compounded quarterly,  $k$  is the interest rate for quarterly compounding,  $k_{\text{nom}}/4$ .

Notice that the present value of a perpetuity at a given interest rate is a sum that, if deposited at that rate, will just earn the amount of the payment each period without compounding. To see that, just solve equation 6.24 for PMT.

A **perpetuity** is a stream of regular payments that goes on **forever**.

**Example 6.14** The Longhorn Corporation issues a security that promises to pay its holder \$5 per quarter indefinitely. Money markets are such that investors can earn about 8% compounded quarterly on their money. How much can Longhorn sell this special security for?

### Preferred Stock

**SOLUTION:** Longhorn's security represents a perpetuity paid on a quarterly basis. The security is worth the present value of the payments promised at the going interest rate.

$$PV_P = \frac{PMT}{k} = \frac{\$5.00}{.02} = \$250$$

**Preferred stock** dividends are a perpetuity.

Securities that offer a deal like this are called *preferred stocks*. We'll study preferred stocks in some detail in Chapter 8.

**Example 6.15****Capitalization of Earnings**

Ebertek is a privately held corporation that is currently being offered for sale. Big Corp. is considering buying the firm. Ebertek's revenues and earnings after tax have averaged \$40 million and \$2.5 million, respectively, for the last five years without much variation around those averages. Interest rates are about 10%. What is a realistic starting point for price negotiations?

**SOLUTION:** If the parties agree that Ebertek's earnings stream is stable, a fair price for the company is the present value of those earnings in perpetuity. In other words, the fair price is the present value of a perpetuity of annual payments equal in size to the annual earnings. In this case the company should be worth approximately

$$PV_p = \frac{PMT}{k} = \frac{\$2,500,000}{.10} = \$25,000,000$$

A steady stream of **earnings** is **capitalized** at its present value as a perpetuity.

This valuation process is called the *capitalization of earnings*, at the relevant interest rate, which is 10% in this case. In essence, we equate the stream of payments to an amount of capital (money) that would earn an equivalent series of payments at the current interest rate.

In this situation, negotiations would move up or down from this starting point depending on whether future earnings prospects look better or worse than the earnings record of the recent past.

**Continuous Compounding**

In the section on compound interest earlier in this chapter, we discussed compounding periods of less than a year. We specifically addressed annual, semiannual, quarterly, and monthly periods.

Compounding periods can theoretically be even shorter than a day. Hours, minutes, or seconds are indeed possible. In the limit, as time periods become infinitesimally short, we have the idea of *continuous compounding* in which interest is instantaneously credited to the recipient's account as it is earned.

The development of formulas for continuous compounding is more mathematically advanced than we want to deal with in this text. Therefore, we'll just present an expression for amount problems without derivation.

$$(6.25) \quad FV_n = PV(e^{kn})$$

where  $k$  is the nominal rate *in decimal form*, and  $n$  is the number of years in the problem.

The letter  $e$  represents a special number in advanced mathematics whose decimal value is 2.71828. . . . All financial and engineering calculators have an  $e^x$  key for calculating exponential values of  $e$ . Notice that you can use equation 6.25 to solve for either the present or future value of an amount. Fractional values for  $k$  and/or  $n$  can be used directly in this equation.

**Example 6.16**

The First National Bank of Charleston is offering continuously compounded interest on savings deposits. Such an offering is generally more of a promotional feature than anything else.

- If you deposit \$5,000 at  $6\frac{1}{2}\%$  compounded continuously and leave it in the bank for  $3\frac{1}{2}$  years, how much will you have?
- What is the equivalent annual rate (EAR) of 12% compounded continuously?

**SOLUTION:** To solve part (a), write equation 6.25 and substitute from the problem.

$$\begin{aligned} FV_n &= PV(e^{kn}) \\ FV_{3.5} &= \$5,000(e^{(.065)(3.5)}) \\ &= \$5,000(e^{.2275}) \end{aligned}$$

Use a calculator to calculate  $e^{.2275} = 1.255457$ , then multiply.

$$FV_{3.5} = \$6,277.29$$

For part (b), calculate the interest earned on a \$100 deposit at 12% compounded continuously in one year.

$$\begin{aligned} FV_n &= PV(e^{kn}) \\ FV_1 &= \$100(e^{(.12)(1)}) \\ &= \$100(e^{.12}) \\ &= \$100(1.1275) \\ &= \$112.75 \end{aligned}$$

Because the initial deposit was \$100, the interest earned is \$12.75, and the EAR is  $\$12.75/\$100 = 12.75\%$ .

Compare this result to the year-end balances and resulting EARs for other compounding periods at 12% shown in Table 6.2 on page 238 and the related discussion.

Table 6.5 summarizes all of the time value formulas we've developed.

**Table 6.5**

**Time Value Formulas**

Equation Number	Formula	Table
<b>Amounts</b>		
6.4	$FV_n = PV[FVF_{k,n}]$	A-1
6.7	$PV = FV_n[PVF_{k,n}]$	A-2
<b>Ordinary Annuities</b>		
6.13	$FVA_n = PMT[FVFA_{k,n}]$	A-3
6.19	$PVA = PMT[PVFA_{k,n}]$	A-4
<b>Annuities Due</b>		
6.22	$FVAd_n = PMT[FVFA_{k,n}](1 + k)$	A-3
6.23	$PVAd = PMT[PVFA_{k,n}](1 + k)$	A-4
<b>Perpetuity</b>		
6.24	$PV_p = PMT/k$	
<b>Continuous Compounding</b>		
6.25	$FV_n = PV(e^{kn})$	

**A Note on the Similarity of the Equations**

Either of the two amount equations can be used to solve any amount problem, because both come from equation 6.3. The four variables are the same, and the time value factors are reciprocals of one another.

The two annuity expressions appear to have the same symmetry, but they don't. The annuity equations are not interchangeable, and each is suited only to its own type of problem. Further, there isn't a reciprocal relationship between the factors. You therefore must choose the correct annuity formula before starting a problem.

## MULTIPART PROBLEMS

Real situations often demand putting two or more time value problems together to get a final solution. In such cases, a time line portrayal can be critical to keeping things straight. Here are two examples.

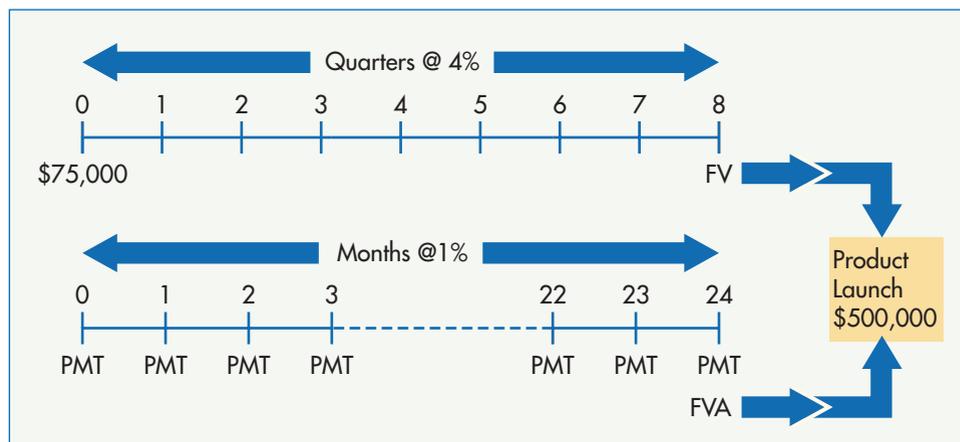
**Example 6.17** Exeter Inc. has \$75,000 invested in securities that earn a return of 16% compounded quarterly. The company is developing a new product that it plans to launch in two years at a cost of \$500,000. Exeter's cash flow is good now but may not be later, so management would like to bank money from now until the launch to be sure of having the \$500,000 in hand at that time. The money currently invested in securities can be used to provide part of the launch fund. Exeter's bank has offered an account that will pay 12% compounded monthly. How much should Exeter deposit with the bank each month to have enough reserved for the product launch?

**SOLUTION:** Two things are happening at once in this problem. Exeter is saving up money by making monthly deposits (an annuity), and the money invested in securities (an amount) is growing independently at interest.

To figure out how much the firm has to deposit each month, we need to know how much those deposits have to accumulate into by the end of two years. That's not given, but it can be calculated. The stream of deposits must provide an amount equal to \$500,000 less whatever the securities investment will grow into.

Thus, we have two problems that must be handled sequentially. First we have an amount problem to find the future value of \$75,000. Once we have that figure, we'll subtract it from \$500,000 to get the contribution required from the annuity. Then we'll solve a future value of an annuity problem for the payment required to get that amount.

It's important to notice that  $k$  and  $n$  aren't the same for the two parts of the problem. For the amount problem, we have quarterly compounding over two years at 16%, so  $k = 4\%$  and  $n = 8$  quarters. In the annuity problem, we have monthly compounding of 12% for two years, so  $k = 1\%$  and  $n = 24$  months. The two-part time line follows.



**Calculator Solution**

Find the future value of \$75,000 by using equation 6.4.

Key	Input
n	8
I/Y	4
PV	75,000
PMT	0
	<b>Answer</b>
FV	102,643

$$\begin{aligned}
 FV_n &= PV[FVF_{k,n}] \\
 FV_8 &= \$75,000[FVF_{4,8}] \\
 &= \$75,000[1.36886] \\
 &= \$102,645
 \end{aligned}$$

**Calculator Solution**

Key	Input
n	24
I/Y	1
FV	397,355
PV	0
	<b>Answer</b>
PMT	14,731

Then the savings annuity must provide

$$\$500,000 - \$102,645 = \$397,355$$

In other words, the future value of the annuity of the savings deposits is \$397,355. Use equation 6.13 to solve for the required payment.

$$\begin{aligned}
 FVA_n &= PMT[FVFA_{k,n}] \\
 \$397,355 &= PMT[FVFA_{1,24}] \\
 \$397,355 &= PMT[26.9735] \\
 PMT &= \$14,731
 \end{aligned}$$

**Example 6.18**

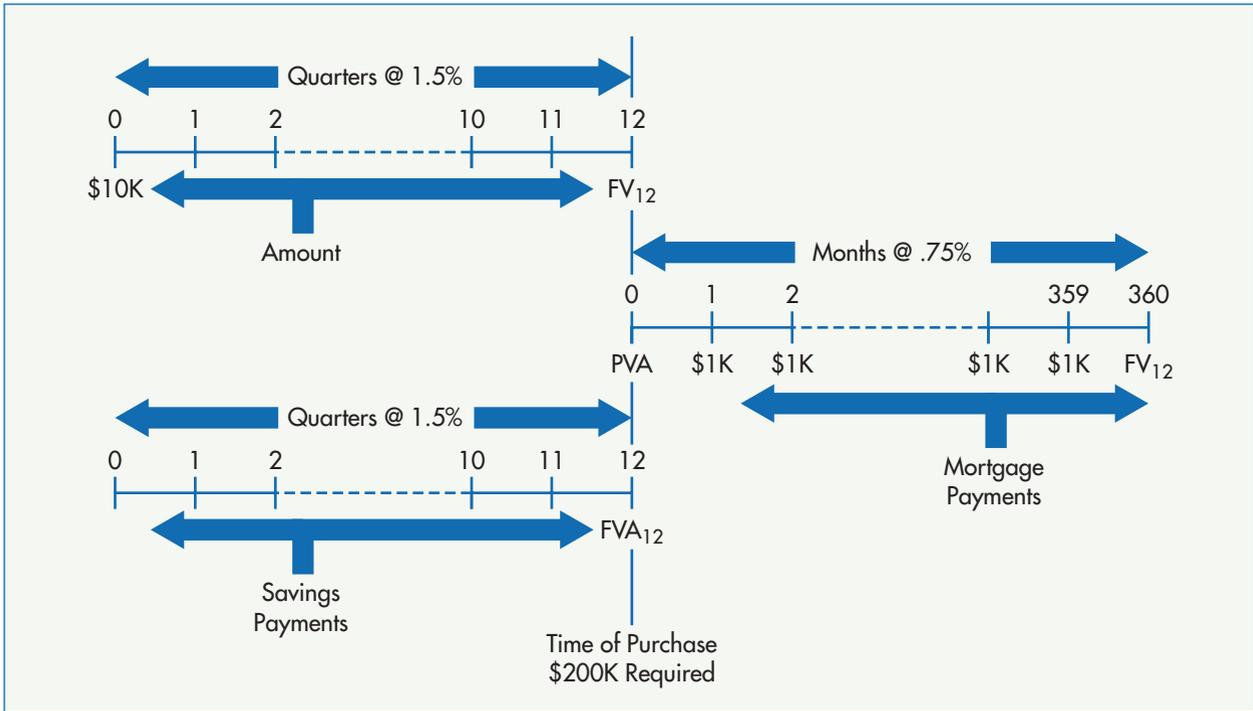
The Smith family plans to buy a new house three years from now for \$200,000. They'll take out a traditional 30-year mortgage at the time of purchase. Mortgage lenders generally base the amount they'll lend on the borrower's gross family income, allowing roughly 25% of income to be applied to the mortgage payment. The Smiths anticipate that their family income will be about \$48,000 at the time they'll purchase the house. The mortgage interest rate is expected to be about 9% at that time.

The mortgage alone won't provide enough cash to buy the house, and the family will need to have a down payment saved to make up the difference. They have a bank account that pays 6% compounded quarterly in which they've already saved \$10,000. They plan to make quarterly deposits from now until the time of purchase to save the rest. How much must each deposit be?

**SOLUTION:** We need three time lines to visualize this problem: one for the \$10,000 already in the bank, one for the loan, and one for the savings to be made over the next three years. A time line diagram is usually necessary in problems like this one at the top of page 258.

Notice that the problem is focused around the date of purchase of the house. The amount problem and the annuity of the savings end at that time, but that's when the loan begins. That is, time 0 for the loan isn't the present but a time three years in the future. Nevertheless, we'll refer to the loan amount as the present value of the annuity of the payments.

The problem asks us to calculate how much the Smiths need to save each quarter. To do that we have to know how much they need to save up in total. That's the future value of the annuity of their savings,  $FVA_{12}$  in the diagram. That sum is going to be \$200,000 less the amount that can be borrowed, less the amount that the money already in savings will have grown into. Those amounts are  $PVA$  and  $FV_{12}$ , respectively, in the diagram.



**Calculator Solution**

Key	Input
n	360
I/Y	.75
PMT	1,000
FV	0
	<b>Answer</b>
PV	124,282

**Calculator Solution**

Key	Input
n	12
I/Y	1.5
PV	10,000
PMT	0
	<b>Answer</b>
FV	11,956

**Calculator Solution**

Key	Input
n	12
I/Y	1.5
FV	63,762
PV	0
	<b>Answer</b>
PMT	4,889

First calculate the amount that can be borrowed by using the present value of an annuity formula (equation 6.19). A 30-year mortgage at 9% implies  $k = .75\%$  and  $n = 360$ . The Smith's annual income is \$48,000 or \$4,000 a month. Generally about 25% of that amount, \$1,000, can be used for a mortgage payment.

$$\begin{aligned}
 PVA &= PMT[PVFA_{k,n}] \\
 &= \$1,000[PVFA_{.75,360}] \\
 &= \$1,000[124.282] \\
 &= \$124,282
 \end{aligned}$$

Next calculate the future value of the \$10,000 already in the bank using equation 6.4. Six percent compounded quarterly for three years implies  $k = 1.5\%$  and  $n = 12$ .

$$\begin{aligned}
 FV_{12} &= \$10,000[FV_{1.5,12}] \\
 &= \$10,000[1.1956] \\
 &= \$11,956
 \end{aligned}$$

The savings requirement is \$200,000 less these amounts; that's \$63,762. This sum is the future value of the annuity of the savings deposits. We can solve for the required deposit by using the future value of an annuity formula (equation 6.13). Because this money is going into the same bank account as the previous \$10,000,  $k$  and  $n$  are the same.

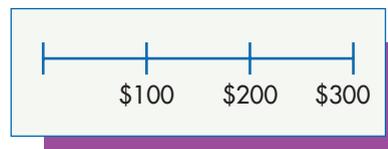
$$\begin{aligned}
 FVA_n &= PMT[FVFA_{k,n}] \\
 \$63,762 &= PMT[FVFA_{1.5,12}] \\
 \$63,762 &= PMT[13.0412] \\
 PMT &= \$4,889
 \end{aligned}$$

Don't be confused by the fact that the savings deposits and the \$10,000 already saved are in the same account. For purposes of calculation, they can be treated as though they're in identical but separate accounts.

Our figures show that the Smiths would have to deposit almost \$4,900 a quarter, which is about \$1,630 a month. That's probably a bit too much to be realistic at their income level.

## UNEVEN STREAMS AND IMBEDDED ANNUITIES

Many real-world problems involve streams of payments that aren't even. When that occurs, we can't use the annuity formulas to calculate present and future values, and generally we must treat each payment as an independent amount problem. For example, consider the payment stream represented by the following time line.<sup>3</sup>



The only way to deal with this stream is to handle each payment as an individual amount. That's not too hard if we're looking for the present or future value, but it's quite difficult if we're looking for an interest rate that yields a particular present or future value.

For example, we might be asked to find the interest rate at which the PVs of the individual amounts just add up to \$500.

The correct approach to that question is iterative. That means we guess at an interest rate and calculate the PV of the stream. If the calculated PV isn't \$500, we make another, better guess and recalculate. As we'll see shortly, there's a way of making sure the second guess moves us closer to the solution. We'll do this problem as an illustration.

**Example 6.19** Calculate the interest rate at which the present value of the stream of payments shown above is \$500.

**SOLUTION:** We'll use the present value of an amount formula for each successive payment and start off by guessing at an interest rate of 12%. The present value of the entire stream is then

$$\begin{aligned}
 PV &= FV_1[PVF_{k,1}] + FV_2[PVF_{k,2}] + FV_3[PVF_{k,3}] \\
 &= \$100[PVF_{12,1}] + \$200[PVF_{12,2}] + \$300[PVF_{12,3}] \\
 &= \$100(.8929) + \$200(.7972) + \$300(.7118) \\
 &= \$462.27
 \end{aligned}$$

3. Although we haven't shown one here, you should recognize that one or more payments in a stream like this can be negative. A negative payment simply means that money is going the other way. For example, if a series of payments represents projected profits from a business, a negative number would just reflect a loss in some period. It would make a negative contribution to the present or future value calculation.

This figure is lower than the \$500 we're looking for, so our guess was wrong. Because our guess discounted the figures by too much, and higher interest rates discount amounts more, we conclude that the next guess should be lower.

Using 11% gives \$471.77, which is closer but still not high enough. Try a few more iterations to show that the answer is between 8% and 9%.

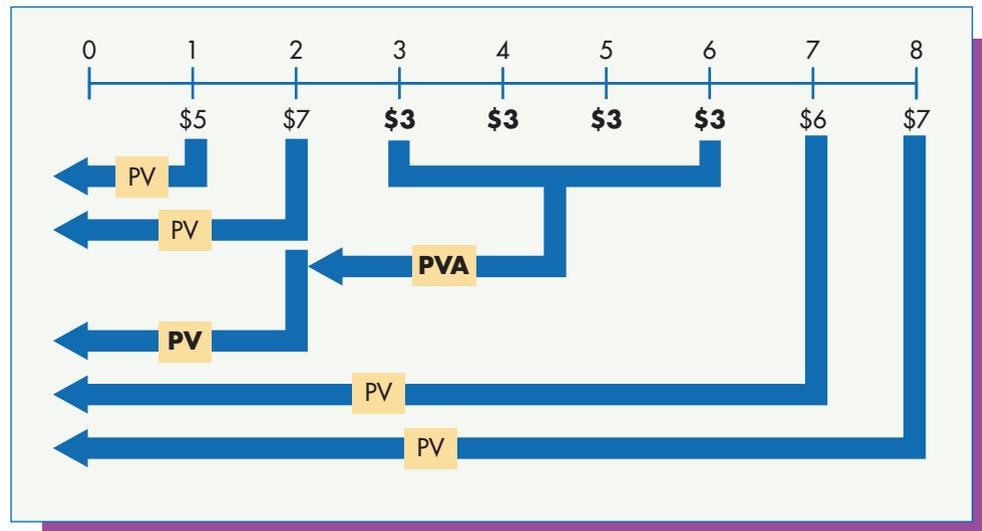
### Imbedded Annuities

Sometimes uneven streams have regular sections, and we can use the annuity formula to reduce the number of calculations required to compute the present or future values.

Consider calculating the present value of the following uneven stream in which the third through sixth payments represent a \$3 annuity of four periods.

Instead of calculating the present value of each term, we can recognize the annuity and use the PVA formula for that part. However, we have to remember that the annuity formula gives the present value at the beginning of the annuity. In this case that's at time 2, not at time 0. Hence, we have to bring the "present" value of the annuity back another two periods as an amount to get its "present" value as of time 0 as indicated schematically in the diagram.

The "present value" of an imbedded annuity is moved back in time as an amount.



### Example 6.20 Calculate the present value of the uneven stream above at 12%.

**SOLUTION:** First handle the first two and the last two payments as simple amount problems.

$$\text{Payment 1: } PV = FV_1[PVF_{12,1}] = \$5(.8929) = \$4.46$$

$$\text{Payment 2: } PV = FV_2[PVF_{12,2}] = \$7(.7972) = \$5.58$$

$$\text{Payment 7: } PV = FV_7[PVF_{12,7}] = \$6(.4523) = \$2.71$$

$$\text{Payment 8: } PV = FV_8[PVF_{12,8}] = \$7(.4039) = \$2.83$$

Next find PVA for the annuity at the beginning of period 3 (end of period 2), and bring it back two years as an amount.

$$PVA = PMT[PVFA_{12,4}] = \$3(3.0373) = \$9.11$$

and

$$PV = FV_2[PVF_{12,2}] = PVA(.7972) = \$9.11(.7972) = \$7.26$$

Now add up all the PVs to get the final answer of \$22.84.

### Calculator Solutions for Uneven Streams

Financial calculators have the ability to handle uneven streams with a limited number of payments. They're generally programmed to find the present value of the stream given an interest rate or the interest rate that will yield a particular present value. Your calculator's operating manual includes instructions on how to input uneven cash flows and produce these results.

Spreadsheets have functions that do the same thing and are generally easier to use than calculators.

Evaluating uneven streams is a key element of an important financial technique known as capital budgeting which we'll study in great detail in Chapters 10, 11, and 12. We'll look into calculator and spreadsheet solutions at that time.

### QUESTIONS

1. Why are time value concepts important in ordinary business dealings, especially those involving contracts?
2. Why are time value concepts crucial in determining what a bond or a share of stock should be worth?
3. In a retail store a discount is a price reduction. What's a discount in finance? Are the two ideas related?
4. Calculate the present value of one dollar 30 years in the future at 10% interest. What does the result tell you about very long-term contracts?
5. Write a brief verbal description of the logic behind the development of the time value formulas for annuities.
6. Deferred payment terms are equivalent to a cash discount. Discuss and explain this idea.
7. What's an opportunity cost interest rate?
8. Discuss the idea of a sinking fund. How is it related to time value?
9. The amount formulas share a closer relationship than the annuity formulas. Explain and interpret this statement.
10. Describe the underlying meaning of compounding and compounding periods. How does it relate to time value? Include the idea of an effective annual rate (EAR). What is the annual percentage rate (APR)? Is the APR related to the EAR?

11. What information are we likely to be interested in that's contained in a loan amortization schedule?
12. Discuss mortgage loans in terms of the time value of money and loan amortization. What important points should every homeowner know about how mortgages work? (*Hint*: Think about taxes and getting the mortgage paid off.)
13. Discuss the idea of capitalizing a stream of earnings in perpetuity. Where is this idea useful? Is there a financial asset that makes use of this idea?
14. When an annuity begins several time periods into the future, how do we calculate its present value today? Describe the procedure in a few words.

### BUSINESS ANALYSIS

1. A business can be valued by capitalizing its earnings stream (see Example 6.15, page 254). How might you use the same idea to value securities, especially the stock of large publicly held companies? Is there a way to calculate a value which could be compared to the stock's market price that would tell an investor whether it's a good buy? (If the market price is lower than the calculated value, the stock is a bargain.) What financial figures associated with shares of stock might be used in the calculation? Consider the per-share figures and ratios discussed in Chapter 3, including EPS, dividends, book value per share, etc. Does one measure make more sense than the others? What factors would make a stock worth more or less than your calculated value?

### PROBLEMS

#### Amount Problems

1. The Lexington Property Development Company has a \$10,000 note receivable from a customer due in three years. How much is the note worth today if the interest rate is
  - a. 9%?
  - b. 12% compounded monthly?
  - c. 8% compounded quarterly?
  - d. 18% compounded monthly?
  - e. 7% compounded continuously?
2. What will a deposit of \$4,500 left in the bank be worth under the following conditions?
  - a. Left for nine years at 7% interest
  - b. Left for six years at 10% compounded semiannually
  - c. Left for five years at 8% compounded quarterly
  - d. Left for 10 years at 12% compounded monthly
3. What interest rates are implied by the following lending arrangements?
  - a. You borrow \$500 and repay \$555 in one year.
  - b. You lend \$1,850 and are repaid \$2,078.66 in two years.

- c. You lend \$750 and are repaid \$1,114.46 in five years with quarterly compounding.  
 d. You borrow \$12,500 and repay \$21,364.24 in three years under monthly compounding.  
 (Note: In parts c and d, be sure to give your answer as the annual nominal rate.)
4. How long does it take for the following to happen?
- \$856 grows into \$1,122 at 7%
  - \$450 grows into \$725.50 at 12% compounded monthly
  - \$5,000 grows into \$6,724.44 at 10% compounded quarterly
5. Sally Guthrie is looking for an investment vehicle that will double her money in five years.
- What interest rate, to the nearest whole percentage, does she have to receive?
  - At that rate, how long will it take the money to triple?
  - If she can't find anything that pays more than 11%, approximately how long will it take to double her investment?
  - What kind of financial instruments do you think Sally is looking at? Are they risky? What could happen to Sally's investment?
6. Branson Inc. has sold product to the Brandywine Company, a major customer, for \$20,000. As a courtesy to Brandywine, Branson has agreed to take a note due in two years for half of the amount due.
- What is the effective price of the transaction to Branson if the interest rate is: (1) 6%, (2) 8%, (3) 10%, or (4) 12%?
  - Under what conditions might the effective price be even less as viewed by Brandywine?
7. John Cleaver's grandfather died recently and left him a trunk that had been locked in his attic for years. At the bottom of the trunk John found a packet of 50 World War I "liberty bonds" that had never been cashed in. The bonds were purchased for \$11.50 each in 1918 and pay 3% interest as long as they're held. (Government savings bonds like these accumulate and compound their interest, unlike corporate bonds which regularly pay out interest to bond holders.)
- How much were the bonds worth in 2007?
  - How much would they have been worth if they paid interest at a rate more like that paid during the 1970s and 80s, say 7%?
  - Comment on the difference between the answers to parts (a) and (b).
8. Paladin Enterprises manufactures printing presses for small-town newspapers that are often short of cash. To accommodate these customers, Paladin offers the following payment terms.
- $\frac{1}{3}$  on delivery  
 $\frac{1}{3}$  after six months  
 $\frac{1}{3}$  after 18 months

The Littleton *Sentinel* is a typically cash-poor newspaper considering one of Paladin's presses.

- What discount is implied by the terms from Paladin's point of view if it can invest excess funds at 8% compounded quarterly?

- b. The *Sentinel* can borrow limited amounts of money at 12% compounded monthly. What discount do the payment terms imply to the *Sentinel*?
  - c. Reconcile these different views of the same thing in terms of opportunity cost.
9. Charlie owes Joe \$8,000 on a note that is due in five years with accumulated interest at 6%. Joe has an investment opportunity now that he thinks will earn 18%. There's a chance, however, that it will earn as little as 4%. A bank has offered to discount the note at 14% and give Joe cash that he can invest today.
  - a. How much ahead will Joe be if he takes the bank's offer and the investment does turn out to yield 18%?
  - b. How much behind will he be if the investment turns out to yield only 4%?
10. Ralph Renner just borrowed \$30,000 to pay for a new sports car. He took out a 60-month loan and his car payments are \$761.80 per month. What is the effective annual interest rate (EAR) on Ralph's loan?

### Annuity Problems

11. How much will \$650 per year be worth in eight years at interest rates of
  - a. 12%?
  - b. 8%?
  - c. 6%?
12. The Wintergreens are planning ahead for their son's education. He's eight now and will start college in 10 years. How much will they have to set aside each year to have \$65,000 when he starts if the interest rate is 7%?
13. What interest rate would you need to get to have an annuity of \$7,500 per year accumulate to \$279,600 in 15 years?
14. How many years will it take for \$850 per year to amount to \$20,000 if the interest rate is 8%?
15. What would you pay for an annuity of \$2,000 paid every six months for 12 years if you could invest your money elsewhere at 10% compounded semiannually?
16. Construct an amortization schedule for a four-year, \$10,000 loan at 6% interest compounded annually.
17. A \$10,000 car loan has payments of \$361.52 per month for three years. What is the interest rate? Assume monthly compounding and give the answer in terms of an annual rate.
18. Joe Ferro's uncle is going to give him \$250 a month for the next two years starting today. If Joe banks every payment in an account paying 6% compounded monthly, how much will he have at the end of *three* years?
19. How long will it take a payment of \$500 per quarter to amortize a loan of \$8,000 at 16% compounded quarterly? Approximate your answer in terms of years and months. How much less time will it take if loan payments are made at the beginning of each quarter rather than at the end?
20. Ryan and Laurie Middleton just purchased their first home with a traditional (monthly compounding and payments) 6% 30-year mortgage loan of \$178,000.
  - a. How much is their monthly payment?
  - b. How much interest will they pay the first month?

- c. If they make all their payments on time over the 30-year period, how much interest will they have paid?
  - d. If Ryan and Laurie decide to move after 7 years what will the balance of their loan be at that time?
  - e. If they finance their home over 15 rather than 30 years at the same interest rate, how much less interest will they pay over the life of the loan?
21. What are the monthly mortgage payments on a 30-year loan for \$150,000 at 12%? Construct an amortization table for the first six months of the loan.
  22. Construct an amortization schedule for the last six months of the loan in Problem 21. (*Hint*: What is the unpaid balance at the end of  $29\frac{1}{2}$  years?)
  23. How soon would the loan in Problem 21 pay off if the borrower made a single additional payment of \$17,936.29 to reduce principal at the end of the fifth year?
  24. What are the payments to interest and principal during the 25th year of the loan in Problem 21?
  25. Adam Wilson just purchased a home and took out a \$250,000 mortgage for 30 years at 8%, compounded monthly.
    - a. How much is Adam's monthly mortgage payment?
    - b. How much sooner would Adam pay off his mortgage if he made an additional \$100 payment each month?  
*The financial tables in Appendix A are not sufficiently detailed to do parts c and d. Solve them using a financial calculator.*
    - c. Assume Adam makes his normal mortgage payments and at the end of five years, he refinances the balance of his loan at 6%. If he continues to make the same mortgage payments, how soon after the first five years will he pay off his mortgage?
    - d. How much interest will Adam pay in the 10th year of the loan
      - i. If he does not refinance
      - ii. If he does refinance
  26. Amy's uncle died recently and left her some money in a trust that will pay her \$500 per month for five years starting on her 25th birthday. Amy is getting married soon, and she would like to use this money as a down payment on a house now. If the trust allows her to assign its future payments to a bank, and her bank is willing to discount them at 9% compounded monthly, how much will she have toward her down payment on home ownership? Amy just turned 23.
  27. Lee Childs is negotiating a contract to do some work for Haas Corp. over the next five years. Haas proposes to pay Lee \$10,000 at the end of each of the third, fourth, and fifth years. No payments will be received prior to that time. If Lee discounts these payments at 8%, what is the contract worth to him today?
  28. Referring to the previous problem, Lee wants to receive the payments for his work sooner than Haas proposes to make them. He has counterproposed that the payments be made at the beginning of the third, fourth, and fifth years rather than at the end. What will the contract be worth to Lee if Haas accepts his counterproposal?
  29. The Orion Corp. is evaluating a proposal for a new project. It will cost \$50,000 to get the undertaking started. The project will then generate cash inflows of \$20,000 in its first year and \$16,000 per year in the next five years, after which

it will end. Orion uses an interest rate of 15% compounded annually for such evaluations.

- a. Calculate the “net present value” (NPV) of the project by treating the initial cost as a cash outflow (a negative) in the present, and adding the present value of the subsequent cash inflows as positives.
- b. What is the implication of a positive NPV? (Words only.)
- c. Suppose the inflows were somewhat lower, and the NPV turned out to be negative. What would be the implication of that result? (Words only.)

(This problem is a preview of a technique called capital budgeting, which we’ll study in detail in Chapters 10, 11, and 12.)

### Multipart Problems

30. The Tower family wants to make a home improvement that is expected to cost \$60,000. They want to fund as much of the cost as possible with a home equity loan but can afford payments of only \$600 per month. Their bank offers equity loans at 12% compounded monthly for a maximum term of 10 years.
  - a. How much cash do they need as a down payment?
  - b. Their bank account pays 8% compounded quarterly. If they delay starting the project for two years, how much would they have to save each quarter to make the required down payment if the loan rate and estimated cost remain the same?
31. The Stein family wants to buy a small vacation house in a year and a half. They expect it to cost \$75,000 at that time. They have the following sources of money.
  1. They have \$10,000 currently in a bank account that pays 6% compounded monthly.
  2. Uncle Murray has promised to give them \$1,000 a month for 18 months starting today.
  3. At the time of purchase, they’ll take out a mortgage. They anticipate being able to make payments of about \$300 a month on a 15-year, 12% loan.

In addition, they plan to make quarterly deposits to an investment account to cover any shortfall in the amount required. How much must those additions be if the investment account pays 8% compounded quarterly?

32. Clyde Atherton wants to buy a car when he graduates from college in two years. He has the following sources of money.
  1. He has \$5,000 now in the bank in an account paying 8% compounded quarterly.
  2. He will receive \$2,000 in one year from a trust.
  3. He’ll take out a car loan at the time of purchase on which he’ll make \$500 monthly payments at 18% compounded monthly over four years.
  4. Clyde’s uncle is going to give him \$1,500 a quarter starting today *for one year*.

In addition, Clyde will save up money in a credit union through monthly payroll deductions at his part-time job. The credit union pays 12% compounded monthly. If the car is expected to cost \$40,000 (Clyde has expensive taste!), how much must he save each month?

33. Joe Trenton expects to retire in 15 years and has suddenly realized that he hasn’t saved anything toward that goal. After giving the matter some thought, he has decided that he would like to retire with enough money in savings to withdraw \$85,000 per year for 25 years after he retires. Assume a 6% return on investment before and after retirement and that all payments into and withdrawals from savings are at year end.

- a. How much does Joe have to save in each year for the next 15 years to reach this goal?
  - b. How much would Joe have needed to save each year if he had started when retirement was 25 years away?
  - c. Comment on the difference between the results of parts a and b.
34. Janet Elliott just turned 20 and received a gift of \$20,000 from her rich uncle. Janet plans ahead and would like to retire on her 55th birthday. She thinks she'll need to have about \$2 million saved by that time in order to maintain her lavish lifestyle. She wants to make a payment at the end of each year until she's 50 into an account she'll open with her uncle's gift. After that she'd like to stop making payments and let the money grow with interest until it reaches \$2 million when she turns 55. Assume she can invest at 7% compounded annually. Ignore the effect of taxes.
- a. How much will she have to invest each year in order to achieve her objective?
  - b. What percent of the \$2 million will have been contributed by Janet (including the \$20,000 she got from her uncle)?
35. Merritt Manufacturing needs to accumulate \$20 million to retire a bond issue that matures in 13 years. The firm's manufacturing division can contribute \$100,000 per quarter to an account that will pay 8%, compounded quarterly. How much will the remaining divisions have to contribute every month to a second account that pays 6% compounded monthly in order to reach the \$20 million goal?
36. Carol Pasca just had her fifth birthday. As a birthday present, her uncle promised to contribute \$300 per month to her education fund until she turns 18 and starts college. Carol's parents estimate college will cost \$2,500 per month for four years, but don't think they'll be able to save anything toward it for eight years. How much will Carol's parents need to contribute to the fund each month starting on her 13th birthday to pay for her college education? Assume the fund earns 6% compounded monthly.
37. Joan Colby is approaching retirement and plans to purchase a condominium in Florida in three years. She now has \$40,000 saved toward the purchase in a bank account that pays 8% compounded quarterly. She also has five \$1,000 face value corporate bonds that mature in two years. She plans to deposit the bonds' principal repayments in the same account when they're paid. Joan also receives \$1,200 per month alimony from her ex-husband which will continue for two more years until he retires (24 checks including one that arrived today). She's decided to put her remaining alimony money toward her condo, depositing it as received in a credit union account that pays 8% compounded monthly. She'll make the first deposit today with the check she already has. Joan anticipates buying a \$200,000 property. What will her monthly payment be on a 15-year mortgage at 6%? What would the payment be on a 30-year loan at the same interest rate?

### INTERNET PROBLEM

38. Assume you will retire at age 65. Use the "investment" calculator at <http://www.tcalc.com> to determine how much you would need to save each *month* if your goal is to accumulate a \$1 million retirement nest egg. Plan a 6% annual return and a 0% tax rate assuming you'll invest in tax exempt municipal bonds. Do the calculation twice. First assume you won't retire for 45 years and then in just 15 years. What does this tell you about starting to plan for retirement early?

### COMPUTER PROBLEMS

39. At any particular time, home mortgage rates are determined by market forces, and individual borrowers can't do much about them. The length of time required to pay off a mortgage loan, however, varies a great deal with the size of the monthly payment made, which is under the borrower's control.

You're a junior loan officer for a large metropolitan bank. The head of the mortgage department is concerned that customers don't fully appreciate that a relatively small increase in the size of mortgage payments can make a big difference in how long the payments have to be made. She feels homeowners may be passing up an opportunity to make their lives better in the long run by not choosing shorter-term mortgages that they can readily afford.

To explain the phenomenon to customers, she's asked you to put together a chart that displays the variation in payment size with term at typical interest rates. The starting point for the chart should be the term for a typical 30-year (360-month) loan. Use the TIMEVAL program to construct the following chart.

#### Mortgage Payments per \$100,000 Borrowed as Term Decreases

		Mortgage Term in Years			
		30	25	20	15
Rates	6%				
	8%				
	10%				
	12%				

Write a paragraph using the chart to explain the point. What happens to the effect as interest rates rise? Why?

40. Amitron Inc. is considering an engineering project that requires an investment of \$250,000 and is expected to generate the following stream of payments (income) in the future. Use the TIMEVAL program to determine if the project is a good idea in a present value sense. That is, does the present value of expected cash inflows exceed the value of the investment that has to be made today?

Year	Payment
1	\$63,000
2	69,500
3	32,700
4	79,750
5	62,400
6	38,250

- a. Answer the question if the relevant interest rate for taking present values is 9%, 10%, 11%, and 12%. In the program, notice that period zero represents a cash flow made at the present time, which isn't discounted. The program will do the entire calculation for you if you input the initial investment as a negative number in this cell.

- b. Use trial and error in the program to find the interest rate (to the nearest hundredth of a percent) at which Amitron would be just indifferent to the project.

This problem is a preview of an important method of evaluating projects known as *capital budgeting*. We'll study the topic in detail in Chapters 10, 11, and 12. In part a of this problem, we find the *net present value (NPV)* of the project's cash flows at various interest rates and reason intuitively that the project is a good idea if that figure is positive. In part b, we find the return inherent in the project itself, which is called the *internal rate of return (IRR)*. We'll learn how to use that in Chapter 10.

41. The Centurion Corp. is putting together a financial plan for the company covering the next three years, and it needs to forecast its interest expense and the related tax savings. The firm's most significant liability is a fully amortized mortgage loan on its real estate. The loan was made exactly ten and one-half years ago for \$3.2M at 11% compounded monthly for a term of 30 years. Use the AMORTIZ program to predict the interest expense associated with the real estate mortgage over the next three years. (*Hint*: Run AMORTIZ from the loan's beginning and add up the months in each of the next three years.)

#### DEVELOPING SOFTWARE

42. Write your own program to amortize a 10-year, \$20,000 loan at 10% compounded annually. Input the loan amount, the payment, and the interest rate. Set up your spreadsheet just like Table 6.4, and write your program to duplicate the calculation procedure described.