

# Table of Contents

<b>Preface</b> . . . . .	xvii
<b>Part I. Deterministic Models</b> . . . . .	1
<b>Chapter 1. Introductory Elements to Financial Mathematics</b> . . . . .	3
1.1. The object of traditional financial mathematics . . . . .	3
1.2. Financial supplies. Preference and indifference relations . . . . .	4
1.2.1. The subjective aspect of preferences . . . . .	4
1.2.2. Objective aspects of financial laws. The equivalence principle . . . . .	10
1.3. The dimensional viewpoint of financial quantities . . . . .	11
<b>Chapter 2. Theory of Financial Laws</b> . . . . .	13
2.1. Indifference relations and exchange laws for simple financial operations . . . . .	13
2.2. Two variable laws and exchange factors . . . . .	17
2.3. Derived quantities in the accumulation and discount laws . . . . .	19
2.3.1. Accumulation . . . . .	19
2.3.2. Discounting . . . . .	22
2.4. Decomposable financial laws . . . . .	24
2.4.1. Weak and strong decomposability properties: equivalence relations . . . . .	24
2.4.2. Equivalence classes: characteristic properties of decomposable laws . . . . .	27
2.5. Uniform financial laws: mean evaluations . . . . .	32
2.5.1. Theory of uniform exchange laws . . . . .	32
2.5.2. An outline of associative averages . . . . .	35
2.5.3. Average duration and average maturity . . . . .	37
2.5.4. Average index of return: average rate . . . . .	38
2.6. Uniform decomposable financial laws: exponential regime . . . . .	39

<b>Chapter 3. Uniform Regimes in Financial Practice</b> . . . . .	41
3.1. Preliminary comments . . . . .	41
3.1.1. Equivalent rates and intensities . . . . .	42
3.2. The regime of simple delayed interest (SDI) . . . . .	42
3.3. The regime of rational discount (RD) . . . . .	45
3.4. The regime of simple discount (SD) . . . . .	48
3.5. The regime of simple advance interest (SAI) . . . . .	50
3.6. Comments on the SDI, RD, SD and SAI uniform regimes . . . . .	52
3.6.1. Exchange factors (EF). . . . .	52
3.6.2. Corrective operations . . . . .	53
3.6.3. Initial averaged intensities and instantaneous intensity . . . . .	53
3.6.4. Average length in the linear law and their conjugates . . . . .	54
3.6.5. Average rates in linear law and their conjugated laws . . . . .	54
3.7. The compound interest regime . . . . .	55
3.7.1. Conversion of interests . . . . .	55
3.7.2. The regime of discretely compound interest (DCI). . . . .	56
3.7.3. The regime of continuously compound interest (CCI). . . . .	63
3.8. The regime of continuously comound discount (CCD) . . . . .	69
3.9. Complements and exercises on compound regimes . . . . .	74
3.10. Comparison of laws of different regimes. . . . .	83
<b>Chapter 4. Financial Operations and their Evaluation:</b>	
<b>Decisional Criteria.</b> . . . . .	91
4.1. Calculation of capital values: fairness . . . . .	91
4.2. Retrospective and prospective reserve . . . . .	97
4.3. Usufruct and bare ownership in “discrete” and “continuous” cases. . . . .	104
4.4. Methods and models for financial decisions and choices . . . . .	107
4.4.1. Internal rate as return index . . . . .	107
4.4.2. Outline on GDCF and “internal financial law” . . . . .	109
4.4.3. Classifications and propert of financial projects . . . . .	111
4.4.4. Decisional criteria for financial projects . . . . .	114
4.4.5. Choice criteria for mutually exclusive financial projects . . . . .	124
4.4.6. Mixed projects: the TRM method . . . . .	127
4.4.7. Dicisional criteria on mixed projects . . . . .	133
4.5. Appendix: outline on numerical methods for the solution of equations. . . . .	138
4.5.1. General aspects. . . . .	138
4.5.2. The linear interpolation method . . . . .	139
4.5.3. Dichotomic method (or for successive divisions) . . . . .	141
4.5.4. Secants and tangents method. . . . .	142
4.5.5. Classical interation method. . . . .	143

<b>Chapter 5. Annuities-Certain and their Value at Fixed Rate . . . . .</b>	<b>147</b>
5.1. General aspects. . . . .	147
5.2. Evaluation of constant installment annuities in the compound regime . .	150
5.2.1. Temporary annual annuity . . . . .	150
5.2.2. Annual perpetuity . . . . .	155
5.2.3. Fractional and pluriannual annuities . . . . .	156
5.2.4. Inequalities between annuity values with different frequency: correction factors . . . . .	166
5.3. Evaluation of constant installment annuities according to linear laws . .	172
5.3.1. The direct problem. . . . .	172
5.3.2. Use of correction factors . . . . .	174
5.3.3. Inverse problem . . . . .	175
5.4. Evaluation of varying installment annuities in the compound regime . .	176
5.4.1. General case . . . . .	176
5.4.2. Specific cases: annual annuities in arithmetic progression . . . . .	179
5.4.3. Specific cases: fractional and pluriannual annuities in arithmetic progression . . . . .	183
5.4.4. Specific cases: annual annuity in geometric progression . . . . .	190
5.4.5. Specific cases: fractional and pluriannual annuity in geometric progression . . . . .	196
5.5. Evaluation of varying installment annuities according to linear laws. . .	204
5.5.1. General case . . . . .	204
5.5.2. Specific cases: annuities in arithmetic progression. . . . .	205
5.5.3. Specific cases: annuities in geometric progression. . . . .	207
 <b>Chapter 6. Loan Amortization and Funding Methods . . . . .</b>	 <b>211</b>
6.1. General features of loan amortization . . . . .	211
6.2. General loan amortization at fixed rate . . . . .	213
6.2.1. Gradual amortization with varying installments . . . . .	213
6.2.2. Particular case: delayed constant installment amortization . . . . .	221
6.2.3. Particular case: amortization with constant principal repayments . .	225
6.2.4. Particular case: amortization with advance interests . . . . .	226
6.2.5. Particular case: “American” amortization . . . . .	228
6.2.6. Amortization in the continuous scheme . . . . .	232
6.3. Life amortization . . . . .	234
6.3.1. Periodic advance payments. . . . .	234
6.3.2. Periodic payments with delayed principal amounts . . . . .	241
6.3.3. Continuous payment flow . . . . .	242
6.4. Periodic funding at fixed rate . . . . .	244
6.4.1. Delayed payments . . . . .	244
6.4.2. Advance payments. . . . .	248
6.4.3. Continuous payments . . . . .	251

6.5. Amortizations with adjustment of rates and values . . . . .	253
6.5.1. Amortizations with adjustable rate . . . . .	253
6.5.2. Amortizations with adjustment of the outstanding loan balance . . . . .	256
6.6. Valuation of reserves in unshared loans . . . . .	258
6.6.1. General aspects . . . . .	258
6.6.2. Makeham's formula . . . . .	259
6.6.3. Usufructs and bare ownership valuation for some amortization forms . . . . .	262
6.7. Leasing operation . . . . .	265
6.7.1. Ordinary leasing . . . . .	265
6.7.2. The monetary adjustment in leasing . . . . .	268
6.8. Amortizations of loans shared in securities . . . . .	268
6.8.1. An introduction on the securities . . . . .	268
6.8.2. Amortization from the viewpoint of the debtor . . . . .	270
6.8.3. Amortization from the point of view of the bondholder . . . . .	271
6.8.4. Drawing probability and mean life . . . . .	272
6.8.5. Adjustable rate bonds, indexed bonds and convertible bonds . . . . .	274
6.8.6. Rule variations in bond loans . . . . .	275
6.9. Valuation in shared loans . . . . .	276
6.9.1. Introduction . . . . .	276
6.9.2. Valuation of bonds with given maturity . . . . .	277
6.9.3. Valuation of drawing bonds . . . . .	280
6.9.4. Bond loan with varying rate or values adjusted in time . . . . .	286
<b>Chapter 7. Exchanges and Prices on the Financial Market . . . . .</b>	<b>289</b>
7.1. A reinterpretation of the financial quantities in a market and price logic: the perfect market . . . . .	289
7.1.1. The perfect market . . . . .	289
7.1.2. Bonds . . . . .	291
7.2. Spot contracts, price and rates. Yield rate . . . . .	294
7.3. Forward contracts, prices and rates . . . . .	302
7.4. The implicit structure of prices, rates and intensities . . . . .	304
7.5. Term structures . . . . .	310
7.5.1. Structures with "discrete" payments . . . . .	310
7.5.2. Structures with fractional periods . . . . .	324
7.5.3. Structures with flows "in continuum" . . . . .	327
<b>Chapter 8. Annuities, Amortizations and Funding in the Case of Term Structures . . . . .</b>	<b>331</b>
8.1. Capital value of annuities in the case of term structures . . . . .	331
8.2. Amortizations in the case of term structures . . . . .	336
8.2.1. Amortization with varying installments . . . . .	337

8.2.2. Amortization with constant installments . . . . .	343
8.2.3. Amortization with constant principal repayments . . . . .	348
8.2.4. Life amortization. . . . .	349
8.3. Updating of valuations during amortization. . . . .	352
8.4. Funding in term structure environments . . . . .	355
8.5. Valuations referred to shared loans in term structure environments. . . . .	358
8.5.1. Financial flows by the issuer's and investors' point of view . . . . .	359
8.5.2. Valuations of price and yield. . . . .	360
<b>Chapter 9. Time and Variability Indicators, Classical Immunization. . . . .</b>	<b>363</b>
9.1. Main time indicators . . . . .	363
9.1.1. Maturity and time to maturity . . . . .	364
9.1.2. Arithmetic mean maturity . . . . .	364
9.1.3. Average maturity. . . . .	364
9.1.4. Mean financial time length or "duration" . . . . .	366
9.2. Variability and dispersion indicators . . . . .	374
9.2.1. 2 <sup>nd</sup> order duration . . . . .	374
9.2.2. Relative variation . . . . .	376
9.2.3. Elasticity . . . . .	377
9.2.4. Convexity and volatility convexity . . . . .	377
9.2.5. Approximated estimations of price fluctuation . . . . .	380
9.3. Rate risk and classical immunization. . . . .	386
9.3.1. An introduction to financial risk . . . . .	386
9.3.2. Preliminaries to classic immunization . . . . .	392
9.3.3. The optimal time of realization . . . . .	393
9.3.4. The meaning of classical immunization . . . . .	395
9.3.5. Single liability cover . . . . .	396
9.3.6. Multiple liability cover . . . . .	400
<b>Part II. Stochastic Models . . . . .</b>	<b>409</b>
<b>Chapter 10. Basic Probabilistic Tools for Finance . . . . .</b>	<b>411</b>
10.1. The sample space. . . . .	411
10.2. Probability space . . . . .	412
10.3. Random variables . . . . .	417
10.4. Expectation and independence. . . . .	421
10.5. Main distribution probabilities. . . . .	425
10.5.1. The binominal distribution . . . . .	425
10.5.2. The Poisson distribution. . . . .	426
10.5.3. The normal (or Laplace Gauss) distribution . . . . .	427
10.5.4. The log-normal distribution. . . . .	430
10.5.5. The negative exponential distribution. . . . .	432

10.5.6. The multidimensional normal distribution . . . . .	433
10.6. Conditioning . . . . .	435
10.7. Stochastic processes . . . . .	446
10.8. Martingales . . . . .	450
10.9. Brownian motion. . . . .	453
<b>Chapter 11. Markov Chains . . . . .</b>	<b>457</b>
11.1. Definitions . . . . .	457
11.2. State classification . . . . .	462
11.3. Occupation times. . . . .	467
11.4. Absorption probabilities . . . . .	468
11.5 Asymptotic behavior . . . . .	469
11.6 Examples. . . . .	474
11.6.1. A management problem in an insurance company . . . . .	474
11.6.2. A case study in social insurance . . . . .	476
<b>Chapter 12. Semi-Markov Processes . . . . .</b>	<b>481</b>
12.1. Positive (J-X) processes. . . . .	481
12.2. Semi-Markov and extended semi-Markov chains . . . . .	482
12.3. Primary properties . . . . .	484
12.4. Examples . . . . .	488
12.5. Markov renewal processes, semi-Markov and associated counting processes . . . . .	491
12.6. Particular cases of MRP. . . . .	493
12.6.1. Renewal processes and Markov chains . . . . .	493
12.6.2. MRP of zero order . . . . .	494
12.6.3. Continuous Markov processes . . . . .	495
12.7. Markov renewal functions . . . . .	496
12.8. The Markov renewal equation . . . . .	500
12.9. Asymptotic behavior of an MRP . . . . .	502
12.10. Asymptotic behavior of SMP. . . . .	503
12.10.1. Irreducible case . . . . .	503
12.10.2. Non-irreducible case . . . . .	506
12.11. Non-homogenous Markov and semi-Markov processes . . . . .	508
12.11.1. General definitions . . . . .	508
<b>Chapter 13. Stochastic or Itô Calculus . . . . .</b>	<b>517</b>
13.1. Problem of stochastic integration . . . . .	517
13.2. Stochastic integration of simple predictable processes and semi-martingales . . . . .	519
13.3. General definition of the stochastic integral . . . . .	523

13.4. Itô's formula . . . . .	529
13.4.1. Quadratic variation of a semi-martingale. . . . .	529
13.4.2. Itô's formula . . . . .	531
13.5. Stochastic integral with standard Brownian motion as integrator process . . . . .	532
13.5.1. Case of predictable simple processes . . . . .	533
13.5.2. Extension to general integrand processes. . . . .	535
13.6. Stochastic differentiation . . . . .	536
13.6.1. Definition . . . . .	536
13.6.2. Examples . . . . .	536
13.7. Back to Itô's formula . . . . .	537
13.7.1. Stochastic differential of a product . . . . .	537
13.7.2. Itô's formula with time dependence. . . . .	538
13.7.3. Interpretation of Itô's formula . . . . .	540
13.7.4. Other extensions of Itô's formula . . . . .	540
13.8. Stochastic differential equations. . . . .	545
13.8.1. Existence and unicity general theorem. . . . .	545
13.8.2. Solution of stochastic differential equations. . . . .	549
13.9. Diffusion processes . . . . .	550
<b>Chapter 14. Option Theory . . . . .</b>	<b>553</b>
14.1. Introduction . . . . .	553
14.2. The Cox, Ross, Rubinstein (CRR) or binomial model. . . . .	557
14.2.1. One-period model . . . . .	557
14.2.2. Multi-period model . . . . .	561
14.3. The Black-Scholes formula as the limit of the binomial model . . . . .	564
14.3.1. The lognormality of the underlying asset. . . . .	564
14.3.2. The Black-Scholes formula. . . . .	567
14.4. The Black-Scholes continuous time model . . . . .	568
14.4.1. The model . . . . .	568
14.4.2. The Solution of the Black-Scholes-Samuelson model . . . . .	569
14.4.3. Pricing the call with the Black-Scholes-Samuelson model . . . . .	570
14.5. Exercises on option pricing. . . . .	576
14.6. The Greek parameters . . . . .	577
14.6.1. Introduction . . . . .	577
14.6.2. Values of the Greek parameters . . . . .	579
14.6.3. Excercises . . . . .	581
14.7. The impact of dividend repartition . . . . .	583
14.8. Estimation of the volatility . . . . .	584
14.8.1. Historic method. . . . .	584
14.8.2. Implicit volatility method. . . . .	586
14.9. Black-Scholes on the market. . . . .	587

14.9.1. Empirical studies . . . . .	587
14.9.2. Smile effect . . . . .	587
14.10. Exotic options . . . . .	588
14.10.1. Introduction . . . . .	588
14.10.2. Garman-Kohlhagen formula . . . . .	589
14.10.3. Greek parameters . . . . .	590
14.10.4. Theoretical models . . . . .	590
14.10.5. Binary or digital options . . . . .	592
14.10.6. “Asset or nothing” options . . . . .	595
14.10.7. The barrier options . . . . .	599
14.10.8. Lockback options . . . . .	601
14.10.9. Asiatic (or average) options . . . . .	601
14.10.10. Rainbow options . . . . .	602
14.11. The formula of Barone-Adesi and Whaley (1987): formula for American options . . . . .	603
<b>Chapter 15. Markov and Semi-Markov Option Models . . . . .</b>	<b>607</b>
15.1. The Janssen-Manca model . . . . .	607
15.1.1. The Markov extension of the one-period CRR model . . . . .	608
15.1.2. The multi-period discrete Markov chain model . . . . .	616
15.1.3. The multi-period discrete Markov chain limit model . . . . .	619
15.1.4. The extension of the Black-Scholes pricing formula with Markov environment: the Janssen-Manca formula . . . . .	621
15.2. The extension of the Black-Scholes pricing formula with a semi-Markov environment: the Janssen-Manca-Volpe formula . . . . .	624
15.2.1. Introduction . . . . .	624
15.2.2. The Janssen-Manca-Çınlar model . . . . .	625
15.2.3. Call option pricing . . . . .	628
15.2.4. Stationary option pricing formula . . . . .	630
15.3. Markov and semi-Markov option pricing models with arbitrage possibility . . . . .	631
15.3.1. Introduction . . . . .	631
15.3.2. The homogenous Markov model for the underlying asset . . . . .	633
15.3.3. Particular cases . . . . .	634
15.3.4. Numerical example for the Markov model . . . . .	635
15.3.5. The continuous time homogenous semi-Markov model for the underlying asset . . . . .	637
15.3.6. Numerical example for the semi-Markov model . . . . .	639
15.3.7. Conclusion . . . . .	640



<b>Chapter 16. Interest Rate Stochastic Models – Application to the Bond Pricing Problem</b> . . . . .	641
16.1. The bond investments . . . . .	641
16.1.1. Introduction . . . . .	641
16.1.2. Yield curve . . . . .	642
16.1.3. Yield to maturity for a financial investment and for a bond. . . . .	643
16.2. Dynamic deterministic continuous time model for instantaneous interest rate. . . . .	644
16.2.1. Instantaneous interest rate. . . . .	644
16.2.2. Particular cases . . . . .	645
16.2.3. Yield curve associated with instantaneous interest rate . . . . .	645
16.2.4. Example of theoretical models . . . . .	646
16.3. Stochastic continuous time dynamic model for instantaneous interest rate. . . . .	648
16.3.1. The OUV stochastic model . . . . .	649
16.3.2. The CIR model (1985). . . . .	655
16.3.3. The HJM model (1992) . . . . .	658
16.4. Zero-coupon pricing under the assumption of no arbitrage . . . . .	666
16.4.1. Stochastic dynamics of zero-coupons. . . . .	667
16.4.2. Application of the no arbitrage principle and risk premium. . . . .	668
16.4.3. Partial differential equatin for the structure of zero coupons . . . . .	670
16.4.4. Values of zero coupons without arbitrage opportunity for particular cases . . . . .	672
16.4.5. Values of a call on zero-coupon . . . . .	681
16.4.6. Option on bond with coupons . . . . .	682
16.4.7. A numerical example . . . . .	683
16.5. Appendix (solution of the OUV equation) . . . . .	684
<b>Chapter 17. Portfolio Theory</b> . . . . .	687
17.1. Quantitative portfolio management . . . . .	687
17.2. Notion of efficiency . . . . .	688
17.3. Exercises. . . . .	693
17.4. Markowitz theory for two assets. . . . .	694
17.5. Case of one risky asset and one non-risky asset. . . . .	698
<b>Chapter 18. Value at Risk (VaR) Methods and Simulation</b> . . . . .	703
18.1. VaR of one asset . . . . .	703
18.1.1. Introduction . . . . .	703
18.1.2. Definition of VaR for one asset . . . . .	704
18.1.3. Case of the normal distribution . . . . .	705

18.1.4. Example II: an internal model in case of the lognormal distribution . . . . .	707
18.1.5. Trajectory simulation . . . . .	712
18.2. Coherence and VaR extensions . . . . .	712
18.2.1. Risk measures . . . . .	712
18.2.2. General form of the VaR . . . . .	713
18.2.3. VaR extensions: TVaR and conditional VaR . . . . .	716
18.3. VaR of an asset portfolio . . . . .	721
18.3.1. VaR methodology . . . . .	722
18.3.2. General methods for VaR calculation . . . . .	724
18.3.3. VaR implementation . . . . .	725
18.3.4. VaR for a bond portfolio . . . . .	732
18.4. VaR for one plain vanilla option . . . . .	734
18.5. VaR and Monte Carlo simulation methods . . . . .	737
18.5.1. Introduction . . . . .	737
18.5.2. Case of one risk factor . . . . .	737
18.5.3. Case of several risk factors . . . . .	738
18.5.4. Monte Carlo simulation scheme for the VaR calculation of an asset portfolio . . . . .	741
<b>Chapter 19. Credit Risk or Default Risk . . . . .</b>	<b>743</b>
19.1. Introduction . . . . .	743
19.2. The Merton model . . . . .	744
19.2.1. Evaluation model of a risky debt . . . . .	744
19.2.2. Interpretation of Merton's result . . . . .	746
19.2.3. Spreads . . . . .	747
19.3. The Longstaff and Schwartz model (1995) . . . . .	750
19.4. Construction of a rating with Merton's model for the firm . . . . .	752
19.4.1. Rating construction . . . . .	752
19.4.2. Time dynamic evolution of a rating . . . . .	756
19.5. Discrete time semi-Markov processes . . . . .	763
19.5.1. Purpose . . . . .	763
19.5.2. DTSMP definition . . . . .	765
19.6. Semi-Markov credit risk models . . . . .	768
19.7. NHSMP with backward conditioning time . . . . .	770
19.8. Examples . . . . .	772
19.8.1. Homogenous SMP application . . . . .	772
19.8.2. Non-homogenous downward example . . . . .	776
19.8.3. Non-homogenous downward backward example . . . . .	784

<b>Chapter 20. Markov and Semi-Markov Reward Processes and Stochastic Annuities</b> . . . . .	791
20.1. Reward processes . . . . .	791
20.2. Homogenous and non-homogenous DTMRWP. . . . .	795
20.3. Homogenous and non-homogenous DTSMRWP. . . . .	799
20.3.1. The immediate cases. . . . .	799
20.3.2. The due cases . . . . .	807
20.4. MRWP and stochastic annuities . . . . .	811
20.4.1. Stochastic annuities . . . . .	811
20.4.2. Motorcar insurance application . . . . .	814
20.5. DTSMRWP and generalized stochastic annuities (GSA) . . . . .	822
20.5.1. Generalized stochastic annuities (GSA) . . . . .	822
20.5.2. GSA examples . . . . .	824
<b>References</b> . . . . .	831
<b>Index</b> . . . . .	839