Chapter 7

Exchanges and Prices on the Financial Market

7.1. A reinterpretation of the financial quantities in a market and price logic: the perfect market

7.1.1. *The perfect market*

The relations examined in Chapter 2, which follow from indifference financial laws, give rise to models summarizing preferences in simple or complex exchange operations. Such models enable the measurement of the value given to the temporary availability of financial capital, by means of calculation of the interest on the landed principal or, more generally, of the return of a financial investment.

Many of these concepts can be reformulated more concretely, putting them in a *market logic*, in particular, of the exchange market, establishing the relations that link *prices* of assets, obtained by the meeting of global demand and supply in this market. Therefore, we consider now a different, but analogous, formulation of the theory of financial equivalence, which is helpful in understanding the exchanges taking place in the financial market. Exchange factors in the presence of effective transactions and the indifference relation (in particular, of equivalence if the right conditions, which realize the strong decomposability, exist) that links *financial values* of market referred to different times are considered in this formulation.

The point of view introduced here is therefore an inversion with respect to the settings of Chapter 2 and to the particular cases of Chapter 3, which gave rise to the results of Chapter 4, 5 and 6. In fact, in the previous chapters, on the basis of a *theoretical* approach, first a financial law is introduced; from there the *value* at a given time of the activities linked to an operation *O* net of the losses is obtained; finally the *price* is adapted to the found value. In this chapter we use an *empirical* approach in the sense that the initial input is the *price* of the activities in *O* net of the losses (price obtainable from "market surveys" in broad sense) and from it – on the basis of constraints and relations analogous to those that gave rise to the *value* – is found the coherent structure of return dynamics that links by equivalence the given price to *O*. From this last point of view, we can see, as a particular case of fixed rates, the calculation of the internal rate of return (IRR) of a *project O* (see section 4.4).

To fully understand the approach that we called *empiric,* it is convenient, for simplicity, to refer to the case of *bonds,* public or private, and to the specific market where they are traded. The management of loans shared in stocks and the connected financial valuation has already been discussed in Chapter 6, where, amongst other things, we considered drawing bonds. In addition, we analyze the properties of securities prices that come from a special hypothesis on the financial market, which enables us to speak about a *perfect capital market.*

We talk about a *perfect market* when it has the following features:

– *no friction,* i.e.:

- no transaction cost and taxes;

- the possibility of *short selling,* i.e. sale of securities not owned by the seller with delivery at sale date;

- no risk of default (thus certainty of results);

- homogenity of information;

– *continuity,* i.e.:

- securities are *infinitively divisible* and can be increased; there is no limitation in the trading quantities;

– *competitiveness,* i.e. each market operator:

- *maximizing his profit* – he prefers, all things being equal, to own higher quantities (see rule c of economic behavior in Chapter 1);

- is a *price taker*, i.e. he is a passive subject, not active, with respect to price, in the sense that his operation does not influence the stock price;

– *coherence,* i.e.:

– no-arbitrage opportunities[1].

We will call a market satisfying the coherence hypothesis *coherent*.

It is clear that the perfect market comes from ideal and theoretical conditions; such a market is a model for study. It will be interesting to analyze the properties that are valid for transactions in such markets, properties analogous to those considered for financial laws in a different content. Note that in a real market some of the hypotheses may not be true, as well as some of the properties.

7.1.2. *Bonds*

We will not consider in this chapter random operations, but only bonds with certain *maturity*, concentrating on the following basics types.

a) *Zero-coupon bond*

In such a security the investor returns are completely incorporated; from the financial viewpoint, the debtor, who is the issuer of N bonds with maturity t, issue value P and nominal (and redemption) value C, makes the pure exchange operation

$$(0, NP) \cup (t, -NC) \tag{7.1}$$

whereas each creditor, subscriber or purchaser of a bond, makes the operation

$$(0, -P) \cup (t, C) \tag{7.2}$$

Usually the zero-coupon bonds have maturities that are not too long. Referring to operation (7.2), the return rate for the length t is given by $i_t = (C-P)/P$. With reference to a regime of simple accumulation (being $t \leq 1$) and then to the intensity $j = i_t/t$, given on the basis of market considerations, we obtain

$$P = C/(1+jt) \tag{7.3}$$

1 To clarify this context, an operation O, defined in (4.1) or (4.1'), is called *arbitrage (non-risk)* if the amounts S_h, not all zero, have the same sign. Therefore, O is not fair with any financial law. There are two types of arbitrage:

a) purchase of non-negative amounts, with at least a positive one, at a non-positive price (for free or with an encashment);

b) purchase of non-negative amounts at a negative price (with an encashment). The market coherence is equivalent to the principle of “no arbitrage”.

i.e. the issue value is the discounted value of C in regime of rational discount (conjugated to the simple accumulation).

Example 7.1

In the issue of a semiannual zero-coupon bond for 181 days, let the purchase price for 100 nominal be 95.18. Not considering taxes, it follows that the per period return rate is (100-95.18)/95.18 = 5.0641% and has an intensity equal to 10.0722 years^{-1}.

Considering a taxation of 12.5%, the purchaser pays effectively 95.18+0.125 (100-95.18) = 95.7825, to which corresponds a net per period rate of 4.4032% and an intensity equal to 8.7578 years^{-1}.

b) *Coupon bond*

In such a security, described in Chapter 6 as a shared loan with certain maturity, the investor return has a component of *return* paid periodically, i.e. interest payments (*coupon* payment), to which can be added a component of *incorporated return,* positive or negative (the *capital gain* or *capital loss,* with issues or purchases respectively at discount or at premium). The investor lends to the issuer the amount *P (issue price),* or buys from the previous investor on the exchange market paying the price *P (purchase price)*. In both cases he periodically receives, for the residual life, the payment $I = C'j$ of the coupons, with j being the coupon rate and C' the *nominal value* redeemed at maturity. If the redemption value C is different from the nominal value, one considers C in the financial valuation. Here, we consider fixed coupon bonds, deferring to the following section 8.5 for a brief introduction to bonds with varying coupons.

Therefore, indicating with n the *maturity* of the loan, the financial operation for the bondholder is given by

$$\mathbf{T\&S} = (t, t+1, \ldots , n-1, n)\&(-P, I, \ldots, I, C+I) \tag{7.4}$$

where we assume $t = 0$ in the case of subscription at the issue date, $t \in \mathcal{N}$ in case of later purchase. In all cases $n-t$ is the *length* of the investment, equal to the bond maturity, if $t = 0$.

Fixed coupon bonds are widely used for long-term investments.

Example 7.2

Let us consider a 5-year coupon bond with semiannual coupon and nominal annual rate 2-convertible of 6%, issue price 96.2 for 100 nominal. Not considering taxes, this results in **T&S** = (0; 0.5;1; 1.5; 2; 2.5; 3; 3.5; 4; 4.5; 5) & (-96.2; 3; 3; 3; 3; 3; 3; 3; 3; 3; 103).

The rates are: coupon rate = 3 %, current semiannual rate = 3/96.2 = 3.1185% i.e. annual rate of 6.3343%, effective semiannual rate (with the capital gain), i.e. the semiannual IRR of the operation, which is solution x of the following equation:

$$-96.2 + 3\ a_{\overline{10}|x} + 100\ (1+x)^{-10} = 0$$

This results in x = IRR semiannual = 3.4559%, annual IRR 7.0312%.

Let us also consider a 5-year coupon bond at the annual nominal rate of 6%. It is issued at September 1, 2001, then with a maturity date of September 1, 2006, at the price of 95.35. The semiannual coupons are paid on March 1 and September 1 of each year until maturity. In t = January 14, 2003 the *ex-interest price* (EIP), which assigns to the buyer a share of the current coupon after purchase is 95.75.

Calculation of residual life in t: 3 years+230 days.

Calculation of net coupon: with taxes at 12.5%, we have: 3 (1-0.125) = 2.625.

Calculation of net redemption value: with taxes at 12.5%, we have: 100-0.125·(100-95.35) = 99.419.

Calculation at time t of the price, called the *flat price* (FP), which assigns to the buyer the whole current coupon, so it is given by the *ex-interest price* (EIP) plus the “before day-by-day interests” (b.dbdi) from the last coupon payment (September 1, 2002) until t, then for 135 days; we have: FP = EIP + b.dbdi = 95.75 + 2.625·135/181 = 97.708.

Calculation of “ex-coupon price” at time t: paying the *ex-coupon price* ECP the buyer obtains the bond without current coupon; so ECP is given by EIP minus “after- day by day interests” (a.dbdi) from t until the next coupon payment (March 1, 2003), then for 46 days; we have: ECP = EIP - a.dbdi = 95.75 - 2.625· 46/181 = 95.08. Obviously it then also results: ECP = FP - (net) coupon = FP - 2.625.

Other types of bonds can depend on the variability or randomness of the coupon. In fact, we can have:

– a *coupon with varying rate* according to a previous agreed rule;

– a *coupon with indexed rate*, linked to the future evolution of market or macroeconomic indices.

7.2. Spot contracts, price and rates. Yield rate

Using the theory of financial contract, we will develop a parallel discussion to that in Chapter 2 that will consider the price formation, in a perfect market or at least under the coherence hypothesis, in conditions of certainty. To better clarify the analogy, we will use the same symbols, but with a different meaning.

Referring initially to a unitary zero-coupon bond (UZCB) as a fundamental element (given that more complex transactions can be obtained as linear combinations of UZCB with increasing lengths), we indicate with small letters the times, i.e. the distances from the chosen origin 0. If

$$v(y,z)\ ,\quad y \le z \tag{7.5}$$

is the market price paid in y to purchase the unitary amount in z on the basis of a contract entered into at time y, then such a contract is called a *spot* contract and *$v(y,z)$ is the spot price (SP);* note that the supply $(y;v(y,z))$ can be exchanged with the supply (z;1). The interval (y,z) is called the *exchange horizon (e.h.).*

The analogy of $v(y,z)$ with the discount factor $a(z,y)$ defined in Chapter 2, going from *values*, following subjective valuations, to *prices*, following market laws, is obvious. The position of the variables, (y,z) instead of (z,y) for v, being $y<z$, is due to the prevalent use of operators that prefer a chronological order.

On the basis of the *money return principle* it follows that:

$$v(y,z) < 1\ ,\ \ \forall(y<z) \tag{7.6}$$

Although prices are formed in light of complex causes, the introduction of market hypothesis imposes conditions and constraints. Thus, from market coherence it follows that:

$$v(y,z) > 0 \quad \forall(y < z)\ ,\quad v(y,y) = 1 \tag{7.7}$$

In the same way, from coherence follows the *decreasing of prices with time to maturity* of the bond (that is the final time of the e.h), i.e.:

$$v(y,z') > v(y,z'')\ ,\quad \forall(y \le z' \le z'' \text{ [2]} \tag{7.7'}$$

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2 The proof follows *ab absurdo*, observing that if it were $v(y,z') \le v(y,z'')$, the composition of the three operations:

1) purchase in y of UZCB with maturity z';

The return inherent in the exchange between [y,v(y,z)] and (z,1) can be measured by the rate, which is defined by:

$$i(y,z) \quad = \quad [v(y,z)]^{-1/(z-y)} - 1 \qquad (7.8)$$

Equation (7.8) shows that in this context *the rate is not per period but is per unit of time,* i.e. *on unitary base,* i.e. *on a unitary basis,* in particular *on an annual basis* if the unit is a year.

By inversion of (7.8) the following is obtained:

$$v(y,z) = [1 + i(y,z)]^{-(z-y)} \qquad (7.8')$$

Observation

When the price v is a function of return variables, as in (7.8'), and such variables are expressed by the market, then v changes its nature, assuming that of value following a calculation.

We define intensity of return at maturity (intensity r.m.), referring to a spot contract, by the function:

$$\phi(y,z) \quad = \quad -\ln v(y,z)/(z-y) \qquad (7.9)$$

By inverting (7.9) f(y,z) satisfies:

$$v(y,z) = e^{-\phi(y,z)(z-y)} \qquad (7.10)$$

that is – recalling the definition of instantaneous intensity given in Chapter 2 for a discount law with two time-variables, to which those for the price formation are analogous – the intensity $\phi(y,z)$ coincides with the constant instantaneous intensity of the exponential law equivalent, in return terms, to the one obtainable from *v(y,z)* on the e.h. *(y,z)*. In addition, due to (7.9), being: $\ln v(y,z) = -\int_y^z \delta(y,u)du$, the formula $\phi(y,z) = (\int_y^z \delta(y,u)du)/(z-y)$ follows. Then the *intensity r.m.* $\phi(y,z)$ *is the mean of the instantaneous intensities* $\delta(y,u)$ *agreed in y and varying with u in the interval (y,z).*

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2) short sell in *y* of UZCB with maturity *z"*;
3) purchase in *z'* of UZCB with maturity *z"*;
is equivalent to the union of the supplies [*y*, *v*(*y*,*z"*) - *v*(*y*,*z'*)], [*z'*,1-*v*(*z'*,*z"*)]. Taking into account (7.6), in the hypothesis to verify the amount of the former supply is non-negative and that of the second one is positive, in contrast with the *no arbitrage principle*.

It follows that in the continuous time approach the return structure of the spot market can be fully found from the assumption of the financial law d*(y,u)* because from it we find for all intervals *(y,z)* the intensity r.m. and then the spot rates and prices according to the other constraints. In fact, taking into account (7.9) and (7.10) and the comparison between (7.8) and (7.9), we find the relation between rate and intensity r.m., expressed by:

$$\phi(y,z) = \ln[1+i(y,z)] \tag{7.11}$$

or, by inversion,

$$i(y,z) = e^{\phi(y,z)} - 1 \tag{7.11'}$$

From what has been said above, the analogy of (7.11) with that regarding the constant intensity of the exponential law as a function of the rate: d = ln(1+*i*) (i.e. *flat structure*) is obvious. Furthermore, due to (7.11), recalling the logarithmic series, it follows that:

$$\phi(y,z) = \sum_{k=1}^{+\infty} (-1)^{k-1} \frac{[i(y,z)]^k}{k}$$

where the quadratic approximation of $\phi(y,z)$ is: $i(y,z) - [i(y,z)]^2 < i(y,z)$. It follows that the exact value of $\phi(y,z)$ is between its quadratic approximation and $i(y,z)$.

Example 7.3

Assuming a (bank) year as the time unit, if 0.95 is the price paid on February 15, 2003 to purchase a UZCB with maturity June 30, 2003, the annual return rate of the operation is 14.6578% , while the intensity at maturity is 0.136782 per year.

In fact, we have:

$$v(y,z) = 0.95$$

$$\left.\begin{array}{r} y = (\text{February } 15,\ 2003) \\ z = (\text{June } 30,\ 2003) \end{array}\right\} \rightarrow z - y = \frac{135}{360} \text{ years}$$

from which, due to (7.8), it follows that:

$$i(y,z) = v(y,z)^{-\frac{1}{z-y}} - 1 = (0.95)^{-\frac{360}{135}} - 1 = 0.1465783$$

$$h(y,z) = -\ln\ v(y,z)/(z-y) = (-\ln\ 0.95)\frac{360}{135} = 0.1367821$$

Example 7.4

The SP of a quarterly UZCB with annual intensity of intensity r.m. at level of 0.03 is equal to 0.99252805. The annual return rate is 3.0454%.

In fact, we have:

$$\phi(y,z) = 0.03$$
$$z - y = \frac{1}{4} = 0.25 \text{ years}$$

from which, for (7.10), it follows that:

$$v(y,z) = e^{-\phi(y,z)(z-y)} = e^{-0.03 \cdot 0.25} = 0.99252805$$
$$i(y,z) = e^{\phi(y,z)} - 1 = (0.99252805)^{-4} - 1 = 0.03045453$$

If the zero-coupon bond is not of unitary type, having a redemption value S at maturity z, for the perfect market property (continuity and *price-taker*) it follows that the price of the security in y, indicated with $V(y,z;S)$, is equivalent to that of S UZCB, the price of which is $v(y,z)$, which then must be:

$$V(y,z;S) \quad = \quad S \ v(y,z) \qquad (7.12)$$ [3]

Example 7.5

Using data of Example 7.3, on February 15, 2003 the SP of a zero-coupon bond with redemption value 100 and maturing on June 30, 2003 is 95.

In fact, we have:

$$V(y,z;100) = 100\ v(y,z) = 100 \cdot 0.95 = 95$$

Example 7.6

If the SP of a two-yearly zero-coupon bond with redemption value 200 is 150, the SP of a corresponding two-yearly UZCB is 0.75.

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3 We can prove (7.12) *ab absurdo*, on the basis of the no arbitrage principle, showing that the hypothesis of inequality of prices between the non-unitary zero-coupon bond, whose value is $V(y,z;S)$, and the S UZCB allows an arbitrage, obtained with the short selling of stocks with higher price. So that if, in (7.12), the results are $V(y,z;S) < S\ v(y,z)$, the arbitrage is obtained by buying in y the bond that gives (z,S) and short selling in y the S UZCB with maturity z. We obtain an analogous conclusion if $V(y,z;S) > S\ v(y,z)$.

In fact, we have:

$$v(y,z) = \frac{V(y,z;200)}{200} = \frac{150}{200} = 0.75$$

We have highlighted the analogy of *v(y,z)* with the discount factor *a(z,y)* of a financial law with two time variables. However, we can also work in a market logic with the analogy of the accumulation factor *m(y,z)* resulting from the conjugated law *a(z,y)*, then given by its reciprocal. We can precisely define: *m(y,z)* = 1/*v(y,z)* as the ratio between the encashment *K* due to owning the bond at maturity *z* and the price *Kv(y,z)* paid for its purchase at a time *y*<*z*. It follows that *m(y,z)*-1 is the incorporated per period return rate, which refers to e.h. and is obtained by such investment.

Considering *complex securities* that regard inflow vectors $\{S_k\}$ of subsequent amounts according to the maturities $\{z_k\}$, i.e. operations that regard m the supplies:

$$\{(z_1, S_1), (z_2, S_2), \ldots.. , (z_n, S_n)\} \tag{7.13}$$

it is obvious that, given the infinite divisibility of securities in a perfect market, such amounts can also be obtained forming a *portfolio* ***S*** (= set of distinct securities) of $\sum_{k=1}^{n} S_k$ UZCB, divided amongst *n* maturities in order to have S_k UZCB with maturity z_k (*k*=1,2,...,*n*). If the operation is carried out at time *y*, the price of one UZCB that matures in z_k is given by $v(y,z_k)$, thus the price in *y* of the whole portfolio is:

$$\sum_{k=1}^{n} S_k \, v(y,z_k) \tag{7.14}$$

From the market coherence follows the *property of price linearity*: the price *V(y,**S**)* of the portfolio *S*, i.e. of the complex security, must coincide with the value (7.14). In formula:

$$V(y,\mathbf{S}) = \sum_{k=1}^{n} V(y,z_k;S_k) = \sum_{k=1}^{n} S_k \, v(y,z_k) \tag{7.15}$$ [4]

4 For proof, it is enough to repeat for each maturity the argument in footnote 3: if (7.15) is not satisfied, there is arbitrage with the buying (selling) in *t* of the complex security and the selling (buying) in *t* of S_k UZCB with maturity z_k .

The supplies of the bonds with fixed coupon (*coupon bonds* or *bullet bonds*) or also bonds with varying coupon (for instance, if the coupon rate varies due to indexing or other reasons) are included in (7.13). Indicating with C the redemption value of the security at time z_n and with I_h the varying coupon at time z_h (I if constant), for such a security (7.13) becomes

$$\{(z_1, I_1), (z_2, I_2), ... , (z_{n-1}, I_{n-1}), (z_n, C_n + I_n)\} \tag{7.13'}$$

and the supplies can also be referred to the residual time after the buying on the market, not necessarily at the issue date. Therefore the value at time y, on the basis of (7.15), is given by

$$V(y,\mathbf{S}) = \sum_{k=1}^{n} I_k \, v(y,z_k) + C \, v(y,z_n) \tag{7.15'}$$

Until now we have considered prices and rates referred to a given maturity, to apply to securities already in the market. Let us now consider a change of the intensity r.m. ϕ regarding one security, or a set of homogenous securities, during its, or their, economic life. This is the *return rate* (or *yield rate*), defined as that rate, which, used to discount the cash flow produced by the security after its purchase and to its maturity, makes the result equal to its purchase *tel quel* price. Using, referring to the security purchased in 0:

- P = purchase *tel quel* price;
- n = residual length;
- Y = yield rate;
- S_k = net encashment at time $z_k > 0$ ($k = 1,...,n$).

The rate Y is the solution of the equation:

$$P = \sum_{k=1}^{n} \frac{S_k}{(1+Y)^{z_k}} \tag{7.15''}$$

It is immediately verified that the *yield rate* is the IRR on a time interval to maturity and is reduced to the *spot rate* $i(0,n)$ if the security is a zero-coupon bond with life n.

Observing a given number of almost homogenous bonds and calculating for each length the yield rate corresponding to the market price according to (7.15''), on a Cartesian diagram we obtain a set of points with the same number of points as the observed bonds. By means of an appropriate interpolation we find the *yield curve*, putting on the abscissa the residual length and on the ordinate the interpolated yield. Such a curve is a model that gives information on the behavior of the observed bond market if a representative sample is used. Obviously the obtained yields for each length can be different from those effectively obtainable from each security on the market.

We can usually say that if for the security its measured point is above (below) the *yield curve*, it is overestimated (underestimated) by the market, with the following input to sell (buy) if there is assumed a tendency for equilibrium.

In conclusion, the indication obtainable from the *yield curve* dot does not have the same coherence as the spot rates. However, theoretically we can say that the yield rate Y of a single bond is a functional mean (according to Chisini 1929) of the spot rate applied for the valuation of such a bond. In a formula, indicating with S_k the expected inflow due to the bond at time z_k , by definition this constraint:

$$\sum_{k=1}^{n} \frac{S_k}{[1+i(0,z_k)]^{z_k}} = \sum_{k=1}^{n} \frac{S_k}{(1+Y)^{z_k}}$$

follows.

Example 7.7

A bond issued on January 1, 2003 gives the right to the encashment sequence: 4; 2; 101, and according to the time schedule July 1, 2003, October 1, 2003, and November 15, 2003. If, at the issue date and according to the same time schedule, the spot prices structure of the UZCB is (0.96; 0.94; 0.93), the price of the bond is 99.65.

In fact, indicating with:

$$S = (S_1, S_2, S_3) \qquad S_1 = 4 \text{ , } S_2 = 2 \text{ , } S_3 = 101$$

$$y = 1.1.2003 \text{ ; } z_1 = 1.7.2003 \text{ ; } z_2 = 1.10.2003 \text{ ; } z_3 = 15.11.2003$$

$$z_1 - y = \frac{6}{12} \, year \text{ ; } z_2 - z_1 = \frac{3}{12} \, year \text{ ; } z_3 - z_2 = \frac{1.5}{12} \, year$$

$$v(y,z_1) = 0.96 \quad ; \quad v(y,z_2) = 0.94 \quad ; \quad v(y,z_3) = 0.93$$

we have:

$$\begin{aligned} V(y;S) &= V(y,z_1;S_1) + V(y,z_2;S_2) + V(y,z_3;S_3) = \\ &= S_1 v(y,z_1) + S_2 v(y,z_2) + S_3 v(y,z_3) = \\ &= 4 \cdot 0.96 + 2 \cdot 0.94 + 101 \cdot 0.93 = 99.65 \end{aligned}$$

Example 7.8

If the price of the complex security in Example 7.7 is 100 on January 1, 2003, keeping all the other conditions the same, we would realize a secure profit of 0.35 using the following arbitrage strategy:

– short selling the complex bond, with a return of 100;
– buying 4 UZCB maturing on July 1, 2003, with a cost of 4.0.96 = 3.84;
– buying 2 UZCB maturing on October 1, 2003, with a cost of 2.0.94=1.88;
– buying 101 UZCB maturing 15.11.2003, with a cost of 101.0.93= 93.93.

As the result, we would have: 100 -3.84 -1.88 -93.93 = 0.35 > 0.

Example 7.9

Given the function $v(y,z)=\left[1+1.06^z-1.06^y\right]^{-1}$ that defines the SP of a UZCB, where time is measured in years, the intensity r.m. of the spot contract, expressed in years^{-1} is given, due to (7.9), by:

$$\phi(y,z)=\frac{\ln\left(1+1.06^z-1.06^y\right)}{z-y}$$

If z-y is small, $1.06^z-1.06^y<<1$ results, then a good approximation is MacLaurin's formula:

$$\phi(y,z) = \frac{1.06^z-1.06^y}{z-y} = \text{(incremental ratio of } 1.06^x\text{)}$$

Using y =3 , z = 5.5, $v(y,z)$ = 0.842622 results and we obtain:

$$\phi(y,z)=\frac{0.171237}{2.5}=0.068495$$

Instead, using y=3 ; z=3+1/12 = 3.083333, we obtain: $v(y,z)$ = 0.994236 , $\phi(y,z)$ = 0.069367 , approximated $\phi(y,z) = \frac{1.196813-1.191016}{0.083333} = 0.069568$.

7.3. Forward contracts, prices and rates

We can now consider operations that include an exchange between two dates, both of which are after the time of the contract. Comparing with spot contracts, there is no more coincidence between the time of contract (in which the conditions are fixed) and the time of payment. In such a case, we talk about *forward contracts,* which give rise to *delayed sales,* agreed with time x and taking place in $y > x$. If, as we suppose, the sold asset, delivered and paid in y, is a security that gives right to an encashment at maturity $z \geq y$ (or many encashments at times $z_k \geq y$), then for each trade we consider three times, x, y, z. In addition, we can underline that in x (= contracting time) there is no money or asset transfer and that the price of the asset (in particular, of the security), agreed with x, is a *forward price* (f.p.).

The elementary contract that we consider here is the forward purchase, with conditions agreed at time x but with delivery and payment at time $y > x$, of a UZCB redeemed at time $z \geq y$. Let us indicate with:

$$s(x;y,z)\ , \quad x < y \leq z \quad ^{5} \tag{7.16}$$

the f.p., fixed in x, of the UZCB delivered in y and with maturity in z.

Also here is obvious the analogy of $s(x;y,z)$ with the continuing discount factor with the meaning specified in Chapter 2. Furthermore, for continuity reasons implied in the perfect market hypothesis for $x \to y$, it results in:

$$s(y;y,z) = v(y,z) \tag{7.17}$$

then the spot contract can be seen as a limit case of the forward one.

Example 7.10

It is agreed today to buy, after two months, at the price of 0.80, a UZCB with a residual life of four months at the time of purchase.

In symbols, expressing time in months, the agreed forward price is: $s(0;2,6)$ = 0.80. The financial operation can be written as (0,2,6)&(0,-0.80,+1) or $(0,0) \cup (2,-0.80) \cup (6,+=1)$.

5 In general, we can put: $x \leq y \leq z$, meaning that if $x = y \leq z$, the contract is a spot contract.

Analogously to what was seen for the spot contracts, the return in the exchange between [y, $s(x;y,z)$] and (z,1) can be measured by the *rate* referred to a time unit, i.e. on unitary base, in particular annual, defined by:

$$i(x;y,z) \quad = \quad [s(x;y,z)]^{-1/(z-y)} - 1 \qquad (7.8'')$$

where, by inversion,

$$s(x;y,z) = [1 + i(x;y,z)^{-(z-y)} \qquad (7.8''')$$

In addition we define *intensity r.m.*, referring to a forward contract, by the function:

$$\phi(x;y,z) = -\ln [s(x;y,z)]/(z-y) \qquad (7.9')$$

By inversion:

$$s(x;y,z) = e^{-\phi(x;y,z)(z-y)} \qquad (7.10')$$

therefore $\phi(x;y,z)$ coincides with the constant instantaneous intensity of the exponential equivalent law in terms of a return to the one obtainable from $s(x;y,z)$ on the e.h. (y,z). Furthermore, due to (7.9'), owing to

$$\ln s(x;y,z) = -\int_y^z \delta(x,u)du \ ,$$

we have $\phi(x;y,z) = (\int_y^z \delta(x,u)du)/(z-y)$. Thus, *the intensity r.m.* $\phi(x;y,z)$ *in forward contracts is the mean of the instantaneous interest intensity* $\delta(x,u)$ *fixed in x and varying with u in the interval (y,z).* Also in the forward market with a continuous time approach, the return structure is given starting from an instantaneous intensity function $\delta(x,u)$. In fact, from $\{\delta(x,u)\}$ we find $\phi(x;y,z)$ on the basis of the aforementioned formula. From the comparison of (7.8'') and (7.9') we find the relation between $\phi(x;y,z)$ and rate, expressed by:

$$\phi(x;y,z) \quad = \ln [1 + i(x;y,z)] \qquad (7.11'')$$

or, by inversion:

$$i(x;y,z) \;=\; e^{\,\phi(x;y,z)} - 1 \qquad (7.11''')$$

On the analogy of the conclusions obtained about the spot contracts and the (7.11), for $\phi(x;y,z)$, due to (7.11"), the quadratic approximation of the logarithmic series holds; therefore $\phi(x;y,z)$ is included from its quadratic approximation $i(x;y,z)-[i(x;y,z)]^2$ to $i(x;y,z)$.

Furthermore, for forward contracts, as well as for the spot contracts, we can work in a market logic in terms of accumulation and accumulation factors on the basis of conjugated law. Therefore, we can define, analogously to the continuing accumulation factor defined in Chapter 2, the factor $r(x;y,z) = 1/s(x;y,z)$ defined as the ratio between the encashment K, due to the ownership of the bond at maturity z, and the price $Ks(x;y,z)$ paid for its purchase at time $y<z$ in a forward contract with conditions agreed in x. It follows that $r(x;y,z)$-1 is the per period incorporated return rate, referred to the e.h., obtained from such an investment.

Example 7.11

Considering with the same spot price function defined in Example 7.10 the forward contract with $x = 1$; $y = 3$; $z = 5.5$, the intensity r.m. is given by

$$\phi(x;y,z) = -\ln \frac{v(x,z)}{v(x,y)} \Big/ (z-y) = \frac{-\ln\ 0.852869}{2.5} = 0.063660$$

The corresponding annual interest rate $i(x;y,z)$, given in the forward contract, satisfies relation (7.11"), i.e.: $0.063660 = \ln\ [1+i(x;y,z)]$, from which: $i(x;y,z) =$ 0.065730 = 6.5730%

7.4. The implicit structure of prices, rates and intensities

It is fundamental that the following property of the *implicit structure*, if the market coherence holds true, links the parameters of forward contracts to those of spot contracts, propriety that can be summarized regarding prices with the formula:

$$s(x;y,z) \;=\; v(x,z)/v(x,y) \quad , \quad \forall\ (x \le y \le z) \qquad (7.18)$$

Equation (7.18) expresses briefly the fact that forward prices are obtained *implicitly* from the spot ones on the basis of the constraint

$$v(x,z) = v(x,y)\, s(x;y,z) \quad , \quad \forall\, (x \le y \le z) \qquad (7.18')$$

equivalent to (7.18) and analogous to that valid for continuing factors.

Thus, we speak about the *theorem of implicit prices,* observing that (7.18) follows from the coherence hypothesis[6]. This hypothesis leads us to assert that the *applied forward rates are those implicit in the spot structure.*

Following from (7.18) and (7.7'), the relations that summarize the main properties of f.p. implicit in SP are as follows:

$\forall (x \le y \le z)$: $s(x;y,z) > 0$ (positive f.p.)

$\forall\ (x \le y)$: $\begin{cases} s(x;y,z) < 1, & if \quad y < z \\ s(x;y,y) = 1 \end{cases}$ (f.p. not greater than the profit at maturity)

$\forall (x \le y' < y'' \le z)$: $s(x;y',z) < s(x;y'',z)$ (increasing of f.p. with initial time of e.h.)

$\forall (x \le y \le z' < z'')$: $s(x;y,z') > s(x;y,z'')$ (decreasing of f.p. with final time of e.h.)

Furthermore, the perfect market *hypothesis in conditions of certainty* implies the property, analogous to decomposability (for which, as specified in Chapter 2, the initial discount factor is equal to the continuing one), thus called: *independency from contractual time,* on the basis of which

$$s(x;y,z) = s(y;y,z) = v(y,z)\ , \ \forall (x \le y \le z) \qquad (7.19)$$

follows. Due to this equation, the f.p. $s(x;y,z)$ in x to pay in y the UZCB redeemed in z, must coincide with the SP $v(y,z)$ of such UZCB; this is according to the principle of *price uniqueness* of exchange on the horizon (y,z), i.e. of its invariance with respect to x.

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6 Also in this case the proof holds *ab absurdo*, observing that the lack of (7.18) effectiveness leads to certain profit. Indeed, if $v(x,z) > v(x,y)\, s(x;y,z)$, we would obtain a certain profit from the composition of the following three operations at time x:

1) short selling of UZCB redeemed in z;
2) spot purchase of $s(x;y,z)$ unit of UZCB redeemed in y;
3) forward purchase, with delivery in y, of the UZCB redeemed in z.

The result of this composition is a certain profit of the amount $v(x,z) - v(x,y) \cdot s(x;y,z)$ in x, owing to the set-off among other supplies. We obtain a certain loss in the hypothesis $v(x,z) < v(x,y)\, s(x;y,z)$, because there is a certain profit inverting the sign of each price. (7.18) is also justified by the fact that it must be equivalent to pay in x the spot price $v(x,z)$ to purchase a unitary amount in z or investing it in x to purchase in $y \le z$ at forward price $s(x;y,z)$, but to obtain the required amount $s(x;y,z)$ in y we have to pay in x the spot price $v(x,y) \cdot s(x;y,z)$.

In other words, in light of the hypothesis of independence from contractual time, (7.18) becomes

$$v(x,z) = v(x,y)\, v(y,z) \tag{7.20}$$

Then the financial law induced by the spot structure is decomposable.

Note that in practice the following can happen:

– at time $x<y$ the "future" price $v(y,z)$ has to be considered random, then (7.19) does not hold;

– we can find, *a posteriori*, SP and f.p. not satisfying (7.18);

– the f.p. are not implicit by SP, then there are arbitrage possibilities.

In this case, the ideal situation of a perfect market does not hold.

Example 7.12

Referring to data from Example 7.7, the structure of forward prices, implicit in that of the given spot prices, is:

$$s(y;y,z_1) = \frac{v(y,z_1)}{v(y,y)} = v(y,z_1) = 0.96 \quad \text{(being } v(y,y) = 1\text{)}$$

$$s(y;z_1,z_2) = \frac{v(y,z_2)}{v(y,z_1)} = \frac{0.94}{0.96} = 0.97916666$$

$$s(y;z_2,z_3) = \frac{v(y,z_3)}{v(y,z_2)} = \frac{0.93}{0.94} = 0.98936170$$

Obviously the price on January 1, 2003 of the complex examined security is always 99.65. In fact, we have:

$$\begin{aligned} V(y;S) &= S_1\ s(y;y,z_1) + S_2\ s(y;z_1,z_2)s(y;y,z_1) + S_3\ s(y;z_2,z_3)s(y;z_1,z_2)s(y;y,z_1) \\ &= 3.84000000 + 1.88000000 + 93.92999992 = 99.64999992 \cong 99.65 \end{aligned}$$

In light of forward prices $s(x;y,z)$ given by the market, we can define the *implicit forward rates,* considered as (mean) rates on unitary basis (in particular, annual), that express the return given by the market, and that are obviously linked to $s(x.y,z)$ by (7.8") and (7.8"').

If market returns are defined by means of spot rates (7.8), the implicit forward structure can be expressed in terms of rates using the following fundamental relation that follows from (7.18), applying (7.8) and (7.8"):

$$[1+i(x;y,z)]^{z-y} = \frac{[1 + i(x,z)]^{z-x}}{[1 + i(x,y)]^{y-x}} \tag{7.21}$$

As already mentioned, we can also define the rate structure as a function of the intensities r.m. defined in forward contracts, adopting suitable symbols and changing the definitions (7.9) and (7.9'). Thus, due to (7.11) and using natural logarithms in (7.21) we find

$$\phi(x; y, z)\,(z - y) = \phi(x, z)(z - x) - \phi(x, y)\,(y - x) \tag{7.22}$$

from which

$$\phi(x,z) = \phi(x,y)\,\frac{(y - x)}{(z - x)} + \phi(x;y,z)\,\frac{(z - y)}{(z - x)} \tag{7.23}$$

where the spot intensity r.m. in the total interval *(x,z)* is the weighted mean of the intensities r.m. (where the former is a spot intensity and the latter is a forward intensity) for the partial intervals *(x,y)* and *(y,z)* by which the total interval can be decomposed.

Example 7.13

Referring to the data from Example 7.7, the spot rate structure is:

$$i(y,z_1) = v(y,z_1)^{-\frac{1}{z_1-y}} - 1 = (0.96)^{-\frac{12}{6}} - 1 = 0.085069444$$

$$i(y,z_2) = v(y,z_2)^{-\frac{1}{z_2-y}} - 1 = (0.94)^{-\frac{12}{9}} - 1 = 0.085999258$$

$$i(y,z_3) = v(y,z_3)^{-\frac{1}{z_3-y}} - 1 = (0.93)^{-\frac{12}{10{,}5}} - 1 = 0.086474374$$

Obviously, rates increase with decreasing prices. The corresponding structure of implicit forward rates is:

$$i(y;y,z_1)=\frac{[1+i(y,z_1)]^{\frac{z_1-y}{z_1-y}}}{[1+i(y,y)]^{\frac{y-y}{z_1-y}}}-1=i(y,z_1)=0.08506944$$

$$i(y;z_1,z_2)=\frac{[1+i(y,z_2)]^{\frac{z_2-y}{z_2-z_1}}}{[1+i(y,z_1)]^{\frac{z_1-y}{z_2-z_1}}}-1=0.08786128$$

$$i(y;z_2,z_3)=\frac{[1+i(y,z_3)]^{\frac{z_3-y}{z_3-z_2}}}{[1+i(y,z_2)]^{\frac{z_2-y}{z_3-z_2}}}-1=0.089329438$$

or, equivalently, using the results from Example 7.12:

$$i(y;y,z_1)=s(y,y,z_1)^{-\frac{1}{z_1-y}}-1=i(y,z_1)=0.085069444$$

$$i(y;z_1,z_2)=s(y,z_1,z_2)^{-\frac{1}{z_2-z_1}}-1=0.087861277$$

$$i(y;z_2,z_3)=s(y,z_2,z_3)^{-\frac{1}{z_3-z_2}}-1=0.089329438$$

Recalling that in the hypothesis of deterministic perfect market we have:

$$s(y,y,z_1)=v(y,z_1)$$
$$s(y,z_1,z_2)=v(z_1,z_2)$$
$$s(y,z_2,z_3)=v(z_2,z_3)$$

it follows that:

$$i(y,z_1)=i(y,y,z_1)=0.08506944$$
$$i(z_1,z_2)=i(y,z_1,z_2)=0.08786128$$
$$i(z_2,z_3)=i(y,z_2,z_3)=0.08932944$$

This means that borrowing, at market conditions, the amount 99.65, which is needed to purchase the security, and paying back at due dates the amounts to which it is entitled, at the security's maturity, the debt will all be paid back in full without adding any cost or profit.

In fact, we have:

Time	*Outstanding balance*
January 1, 2003	99.65
July 1, 2003	$99.65\,[1+i(y,z_1)]^{z_1-y}-4 = 99.80208327$
October 1, 2003	$99.80208327\,[1+i(z_1,z_2)]^{z_2-z_1}-2 = 99.92553188$
November 15, 2003	$99.92553188\,[1+i(z_2,z_3)]^{z_3-z_2}-101 = 0$

In practice, the spot rates, that are realized on the market on the subsequent due dates, can be different from the foreseen ones on the basis of the above-mentioned hypothesis.

If, for example, the observed spot prices are higher than the foreseen rates and are equal to:

$$i_{eff}(y,z_1) = i(y,z_1) = 0.085069444$$
$$i_{eff}(z_1,z_2) = 0.088865467$$
$$i_{eff}(z_2,z_3) = 0.089432222$$

then the described operation would imply, for the debtor, a loss of 0.02495764 which is equal to the outstanding balance at maturity. In fact we have:

Time	*Outstanding balance*
January 1, 2003	99.65
July 1, 2003	$99.65\,[1+i(y,z_1)]^{z_1-y}-4 = 99.80208327$
October 1, 2003	$99.80208327\,[1+i(z_1,z_2)]^{z_2-z_1}-2 = 99.92553188$
November 15, 2003	$99.94904525\,\left[1+i_{eff}(z_2,z_3)\right]^{z_3-z_2}-101 = 0.02495764$

If the observed spot prices were lower than the foreseen ones, then the operation described above would imply a profit for the debtor.

7.5. Term structures

7.5.1. *Structures with "discrete" payments*

The previous formulae gave prices, rates and intensities for spot or forward contracts, related to payment dates in $\mathcal{R}$.

According to market practice, without loss of generality, we now suppose for the payment dates a discrete "lattice" type distribution, i.e. such that the payments are made at the beginning (or the end) of periods of the same length, that we assume to be unitary[7]. Then, referring to a contract time $t \in \mathcal{R}$, let us consider a complex security with n equi-staggered maturities, that we assume as positive integers, starting from t; then we use a payment schedule (t, t+1,..., t+n). It follows that for financial valuations made in the previous section, we are interested in *spot prices* $v(t,t+k)$ and *forward prices* $s(t;t+h,t+k)$, *spot rates* $i(t,t+k)$ and *forward rates* $i(t;t+h,t+k)$ (referred to the year or, as a general case, to any unit of time), *spot intensities r.m.* $\phi(t,t+k)$ and *forward intensities r.m.* $\phi(t;t+h,t+k)$, where:

$$h \in \mathcal{N}\ ,\quad k \in \mathcal{N}\ ;\ \ 1 \le h \le k \le n\ . \tag{7.24}$$

In this case the definitions and coherence relations seen for the general case are valid. In particular, if h=k the forward prices have value 1 and the forward rates 0.

If the referring time t (i.e. of contract or valuation) is only one, in the sense that no other date is simultaneously considered, it is convenient to put t=0. Such a choice

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7 No necessarily annual. As an example, with semiannual, quarterly, etc., due dates in the market, it is enough to assume semester, quarter, etc., as the unit of measure adjusting times and equivalent rates and assuming a semiannual, quarterly etc., structure of prices and rates. We will develop this in section 7.5.2 in more details. The only restriction to such measure variation is that the due dates are rational numbers. In such a case, written in the form m_i/d_i , (i =1....n) and indicating by *lcm* the least common multiple of the denominators $d_1,...,d_n$ (obtained, as known, decomposing them into factors and taking the product of common and non-common factors, each with the highest exponent), it is enough to reduce the unit of measure according to the ratio *lcm*, where – using k_i = *lcm* $/d_i$ – the new maturities are the integers m_ik_i . By filling the tickler with all other integers in the interval where we put payments equal to zero, we obtain the wanted tickler with a unitary period. For example, assuming the maturities December 7, August 13, May 22, the *lcm* of denominators is $3.5.2^3$ = 120. Therefore, reducing by 120 the unit of measure, we have: k_1 = 10, k_2 = 15 , k_3 = 24 and the new maturities are 70, 195, 528. By completing with natural numbers the interval (70, 528) (in which, except for the three given times, we put no payments), we obtain the wanted distribution. In addition, we have to observe that more often the market gives returns by means of annual rates (or intensities) where in such cases we have to find the equivalent rates between year and the period here used as unitary, if it is subdivision of a year.

is not restrictive, if we consider the arbitrary time origin,[8] and it allows the reduction of the symbols, where the first variable is implicit and the other time variables are written by bottom indexes. It is then enough, with the aforesaid meaning of the symbols, to use[9]:

$$\begin{aligned} &v(t,t+k) = v_k\ ;\ \ s(t;t+h,t+k) = s_{h,k} \\ &i(t,t+k) = i_k\ ;\ \ i(t;t+h,t+k) = i_{h,k} \\ &\phi(t,t+k) = \phi_k\ ;\ \ \phi(t;t+h,t+k) = \phi_{h,k} \end{aligned} \tag{7.25}$$

Then we assume, unless stated otherwise, the symbols in (7.25) and the hypothesis that encashment on securities can occur only on the dates

$$T = (1,...,k,...,n) \tag{7.26}$$

It follows that we measure e.h. with natural numbers. Let us also assume a market *complete* and *perfect* (or, at least, *coherent*) in the sense that there is the possibility of having a zero-coupon bond at each time in (7.26) and the known properties hold true, amongst which is the property of *independence from the amount and coherence*.

We can then outline the term structures for prices and rate *in case of discrete dates*, deducing the formulas that, referring to prices, rates and intensities express each of them as function of the others. They are obtained from those seen in section 7.2 and 7.3 considered for the discrete case, i.e. using x=0 and $y,z \in \mathcal{N}$. For simplicity, from now on we will assume in the application the annual unit of measure, but it is easy to also consider multiples or submultiples periods, as we will see in section 7.5.2.

Spot structures

The symbols in (7.12) and (7.15) give the *spot prices* (SP) $V(0,k;S_k)$ in t=0 of the zero-coupon bonds that pay the amount S_k in k. It follows that

$$v_k = V(0,k;S_k) / S_k \tag{7.27}$$

8 The position t=0 does not imply uniformity in time of the financial law underlying the rates term structure. However, if we assume uniformity of time, the financial results do not depend on the choice of the time origin. In more general cases, for instance, if we have to compare or take into account in the same context different structures with different transaction times t_1, t_2, ..., we have to refer to the general case defined above.

9 The single time subscript of the spot rate are not to be confused with those used in Chapter 3 and 5, which have a different meaning. In the same way the double subscript in the forward parameters with integer time are not to be confused with the pairs of variables of the spot parameters with real times seen in section 7.2.

is the SP of the corresponding UZCB. For the linearity property of price it is not restrictive to refer only to UZCB with SP v_k. Thus, it is clear that for a portfolio of n zero-coupon bonds with amounts S_k payable in k, ($k = 1,...,n$) – where the security that is entitled to the supply (k,S_k) is equivalent to a given number of zero-coupon bonds with amount payable in k the sum of which is S_k – the SP is:

$$V(0,\mathbf{S}) = \sum_{k=1}^{n} S_k \, v_k \qquad (7.27')$$

From the sequence $\{v_k\}$ we find the *rate structure (on annual bases) of spot interest* $\{i_k\}$ in t=0 by means of the following formula that describes equivalently the scenario of the SP:

$$i_k = v_k^{-1/k} - 1 \;, \; (k = 1, \ldots ,n) \qquad (7.28)$$

If the spot rates are i_k in t=0, we find the unitary price, inverting (7.28):

$$v_k = [1 + i_k]^{-k} , \; (k = 1, \ldots , n) \qquad (7.28')$$

From the sequence $\{v_k\}$ or the sequence $\{i_k\}$ we find the *instantaneous intensities r.m. structure* ϕ_k in t=0 for spot contracts. It is enough to modify (7.28) or (7.28') and consider the natural logarithm, resulting in:

$$\phi_k = - \ln v_k/k = \ln [1 + i_k] \qquad (7.29)$$

Inverting (7.29) we find:

$$v_k = e^{-k\,\phi_k} \; ; \; i_k = e^{\phi_k} - 1 \qquad (7.29')$$

Example 7.14

In the market of zero-coupon bond with redemption value 100, are fixed today (t=0) the following SP dependent on annual payments dates, which, divided by 100, define v_k:

96.28 with maturity 1; 93.71 with maturity 2;
90.08 with maturity 3; 87.88 with maturity 4.

The corresponding spot rates structure in 0 is as follows:

$i_1 = v_1^{-1} - 1$ = 0.038637 = 3.8637%;
$i_2 = v_2^{-0.5} - 1$ = 0.033016 = 3.3016%
$i_3 = v_3^{-0.333} - 1$ = 0.035437 = 3.5437%
$i_4 = v_4^{-0.25} - 1$ = 0.032827 = 3.2827%

The intensities r.m. structure is as follows:

$$\phi_1 = -\ln v_1 \quad = \ln(1+i_1) = 0.037910$$
$$\phi_2 = -\ln v_2/2 \quad = \ln(1+i_2) = 0.032483$$
$$\phi_3 = -\ln v_3/3 \quad = \ln(1+i_3) = 0.033424$$
$$\phi_4 = -\ln v_4/4 \quad = \ln(1+i_4) = 0.032299 \,.$$

From the sequence $\{v_k\}$ we find *the spot discount (or advance interests) rate (on an annual basis) structure* $\{d_k\}$ in $t = 0$ on the interval $(0,k)$ by means of the following formula:

$$d_k = 1 - v_k^{1/k}, \qquad (k = 1, \ldots, n) \tag{7.28''}$$

which is obtained by inverting: $v_k = (1 - d_k)^k$.

By comparing (7.28') and (7.28'') we find

$$d_k = i_k / (1+i_k), \qquad (k = 1, \ldots, n) \tag{7.28'''}$$

that generalize a well known formula valid for flat structure, obtainable form (3.53).

Example 7.15

With the same value v_k as in Example 7.14, the annual spot discount rates are, according to (7.28''):

$$d_1 = 1 - v_1^{1.000} = 0.037200 \;;\; d_2 = 1 - v_2^{0.500} = 0.031961;$$

$$d_3 = 1 - v_3^{0.333} = 0.034225 \;;\; d_4 = 1 - v_4^{0.250} = 0.031783.$$

The results for the spot structure obtained in Examples 7.14 and 7.15 can be easily found using an *Excel* spreadsheet as shown below.

Maturity	Spot price %	Delayed spot rate	Spot intensity r.m.	Advance spot rate
1	96.28	0.0386373	0.0379096	0.0372000
2	93.71	0.0330160	0.0324826	0.0319607
3	90.08	0.0354375	0.0348240	0.0342246
4	87.88	0.0328268	0.0322995	0.0317834

Table 7.1. *Spot structure*

The *Excel* instructions are as follows. 2nd row: titles; from the 3rd row:

column A	A3: 1; A4:= A3+1; copy A4, then paste on A5 to A6;
column B	insert data (spot prices %) on B3-B6;
column C	C3:= (B3/100)^-(1/A3)-1; copy C3, then paste on C4 to C6;
column D	D3:= ln (1+C3); copy D3, then paste on D4 to D6;
column E	E3:= 1-(B3/100)^(1/A3); copy E3, then paste on E4 to E6.

Forward structures

The market fixes the *structure of prices, rates and intensities for forward contracts*. In a coherent market the *implicit* forward structure is assumed, i.e. derived from the spot structures on the basis of formulae that adapt (7.18) to (7.24).

Always using the contract time in t=0, we obtain the following basic relation that expresses *the forward price* (f.p.) *structure* $s_{k-1,k}$ of UZCB according to the spot structure v_k for annual e.h. (or *uniperiod*):

$$s_{k-1,k} = \frac{v_k}{v_{k-1}}, \qquad (k = 1,...,n) \tag{7.30}$$

which, for k=1, simply expresses the known relation for the SP: $s_{0,1} = v_1$.

The corresponding *structure of forward (implicit) interest rates* for annual e.h. is given by

$$i_{k-1,k} = s_{k-1,k}^{-1} - 1 = \frac{v_{k-1}}{v_k} - 1, \quad (k = 1,...,n) \tag{7.31}$$

By inversion we find

$$s_{k-1,k} = (1 + i_{k-1,k})^{-1} \tag{7.31'}$$

From (7.31), and recalling (7.28), the *implicit rates theorem*, which is expressed by the following equation, is obtained:

$$1 + i_{k-1,k} = \frac{\left[1+i_k\right]^k}{\left[1+i_{k-1}\right]^{k-1}}, \qquad (k = 1,...,n) \tag{7.32}$$

which gives the implicit forward structure according to the spot structure.

The forward market structure can be expressed according to the spot structure also using the *intensity r.m.* $\phi_{k-1,k}$, obtainable from $i_{k-1,k}$ and $s_{k-1,k}$ using

$$\phi_{k-1,k} = \ln(1+i_{k-1,k}) = -\ln s_{k-1,k}\ , \ (k = 1,..., n) \qquad (7.29'')$$

In fact it is possible to show the validity of the formula:

$$\phi_{k-1,k} = k\,\phi_k - (k-1)\,\phi_{k-1}\ , \quad (k = 1,...,n) \qquad (7.22')$$

which particularizes (7.22). Applying this formula recursively with varying k, by adapting (7.23) to discrete times, the following is obtained:

$$\phi_k = \sum_{r=1}^{k} \phi_{r-1,r} / k\ , \ (k = 1,..., n) \qquad (7.23')$$

i.e. the spot intensity for k periods is the arithmetic mean of the forward intensities in the unitary periods of such horizon (spot in the first of them).

By applying (7.32) recursively, we finally find that

$$(1 + i_k)^k = \prod_{r=1}^{k} (1+i_{r-1,r})\ , \ (k = 1,...n) \qquad (7.33)$$

i.e. the spot accumulation factor $1+i_k$, with reference to the horizon of k unitary consecutive periods, is the geometric mean of k forward accumulation factors relative to the single periods. In this sense the rate i_k on the e.h. $(0,k)$ in a coherent market is a functional mean, according to Chisini, of the forward rates $i_{r-1,r}$.

If, instead, the rates varying for unitary horizons are given as $i_{k--1,k}$, we implicitly find the spot prices, expressed by

$$v_k = (1 + i_k)^{-k} = \prod_{r=1}^{k} (1+i_{r-1,r})^{-1}\ ; \ (k = 1,...,n) \qquad (7.30')$$

Sometimes it is convenient to highlight the corresponding *forward discount (or advance interest) rate (implicit) structure* for annual e.h., expressed by

$$d_{k--1,k} = 1 - s_{k--1,k} = 1 - \frac{v_k}{v_{k-1}}\ , \quad (k = 1,...,n) \qquad (7.31'')$$

from which, recalling (7.28"), we find

$$1 - d_{k-1,k} = \frac{[1-d_k]^k}{[1-d_{k-1}]^{k-1}}\,, \qquad (k = 1,...,n) \qquad (7.32')$$

which links forward discount rate structure as a function of the spot ones. Applying recursively (7.32'), we finally find

$$[1 - d_k]^k = \prod_{r=1}^{k} [1-d_{r-1,r}]\,, \qquad (k = 1,...,n) \qquad (7.33')$$

analogous to (7.33), i.e. the spot discount factor $(1-d_k)$ referred to the horizon of k unitary consecutive periods is the geometric mean of the k forward discount factors of each period.

Example 7.16

In a coherent market the discount factors v_k , $(k = 1,...,4)$, obtained from the spot prices for annual horizons up to 4 years, given in Example 7.14, are fixed. The forward price structure $s_{k-1,,k}$ for unitary securities for annual horizons is as follows:

$$s_{0,1} = 0.9628/1.0000 = 0.962800$$
$$s_{1,2} = 0.9371/0.9628 = 0.973307$$
$$s_{2,3} = 0.9008/0.9371 = 0.961263$$
$$s_{3,4} = 0.8788/0.9008 = 0.975577$$

The corresponding implicit forward interest rate structure is

$$i_{k-1,k} = [s_{k-1,k}]^{-1} - 1 = \frac{[1+i_k]^k}{[1+i_{k-1}]^{k-1}} - 1$$

and recalling the results of Example 7.14, the structure assumes the values:

$$i_{0,1} = 0.962800^{-1} -1 = 0.038637 = \frac{1.038637}{1} - 1$$
$$i_{1,2} = 0.973307^{-1} -1 = 0.027425 = \frac{1.033016^{\,2}}{1.038637} - 1$$
$$i_{2,3} = 0.961263^{-1} -1 = 0.040298 = \frac{1.035437^{\,3}}{1.033016^{\,2}} - 1$$
$$i_{3,4} = 0.975537^{-1} -1 = 0.025434 = \frac{1.032827^{\,4}}{1.035437^{\,3}} - 1$$

Let us verify (7.33) for the values obtained here:

k=1: $1.038637 = 1.038637$
k=2: $1.033016^2 = 1.038637 \cdot 1.027425$
k=3: $1.035437^3 = 1.038637 \cdot 1.027425 \cdot 1.040298$
k=4: $1.032827^4 = 1.038637 \cdot 1.027425 \cdot 1.040298 \cdot 1.025034$

The corresponding implicit forward discount rate structure (which is seldom used)

$d_{k\text{-}1,k} = 1 - s_{k\text{-}1,k}$ assumes the following values:

$d_{0,1} = 0.037200$; $d_{1,2} = 0.026693$;
$d_{2,3} = 0.038737$; $d_{3,4} = 0.024423$.

It is left as an exercise for the reader to verify (7.32') and (7.33'), recalling the results of Example 7.15.

The developments of the results obtained in Example 7.16, can be easily obtained using an *Excel* spreadsheet as follows, as can a comparison of the spot rates given by (7.28) and reported in Example 7.16.

Maturity	Spot price%	Fwd price	Spot rate	Fwd delayed rate	Fwd intensity	Fwd advance rate
1	96.28	0.962800	0.038637	0.038637	0.037910	0.037200
2	93.71	0.973307	0.033016	0.027425	0.027056	0.026693
3	90.08	0.961263	0.035437	0.040298	0.039507	0.038737
4	87.88	0.975577	0.032827	0.025034	0.024726	0.024423

Table 7.2. *Spot and uniperiod forward structure*

The *Excel* instructions are as follows: 2nd row: titles; from the 3rd row:

column A: A3: 1; A4:= A3+1; copy A4-paste on A5 to A6;
column B: insert date (spot price %) on B3 to B6;
column C: C3:= B3/100; C4:= B4/B3; copy C4, then paste on C5 to C6;
column D: D3:= (B3/100)^-(1/A3)-1; copy D3, then paste on D4 to D6;
column E: E3:= (1/C3)-1; copy E3, then paste on E4 to E6;
column F: F3:= ln (1+E3); copy F3, then paste on F4 to F6;
column G: G3:= 1-C3; copy G3, then paste on G4 to G6.

The description of a forward structure can be completed with the extension to prices and interest rates for e.h. not only unitary but *of integer positive length* (then *uni- and multi-period*). The market gives at contract time t=0 the forward prices $s_{h,k}$ of the UZCBs paid in h and entitle us to the unitary amount in k, with h,k specified in (7.24).

Such prices $s_{h,k}$ can be expressed as elements of an $n \times n$ upper triangular matrix (if h is the row index and k the column index), i.e.

$$\left\| \begin{array}{ccccc} s_{1,1} & s_{1,2} & s_{1,3} & \cdot\cdot & s_{1,n} \\ & s_{2,2} & s_{2,3} & \cdot\cdot & s_{2,n} \\ & & s_{3,3} & \cdot\cdot & s_{3,n} \\ & & & \cdot\cdot & \cdot\cdot \\ & & & & s_{n,n} \end{array} \right\| \qquad (7.34)$$

where:

$$s_{h,k} \begin{cases} = 1 \ , if \ \ h = k \\ < 1 \ , if \ \ h < k \end{cases} \quad , \qquad 1 \le h \le k \le n \qquad (7.34')$$

The number of elements in (7.34) is $n/(n+1)/2$, but the meaningful prices ($\neq 1$) are those for which $h<k$, the number of which is $n(n-1)/2$.

In the coherent market hypothesis, due to (7.18), for the dates (7.24) the *general formula* holds, that is the basis of a forward structure for a transaction made in t=0:

$$s_{h,k} = \frac{v_k}{v_h} \quad , \quad (1 \le h < k \le n) \qquad (7.35)$$

Owing to:

$$\frac{v_k}{v_h} = \frac{v_k}{v_{k\text{-}1}} \frac{v_{k\text{-}1}}{v_{k\text{-}2}} \dots \frac{v_{h+1}}{v_h}$$

we find, adapting the indices in (7.30), that:

$$s_{h,k} = \prod_{r=h+1}^{k} s_{r\text{-}1,r} = \prod_{r=h+1}^{k} (1 + i_{r\text{-}1,r})^{-1} \qquad (7.36)$$

If we want to express the constraint of the forward structure working on the rates, then indicating with $i_{h,k}$ the agreed interest rate in 0 on the e.h. (h,k), using (7.31) we obtain:

$$[1+i_{h,k}]^{k-h} = \frac{v_h}{v_k} = \left(\frac{v_k}{v_{k-1}} \frac{v_{k-1}}{v_{k-2}} \cdots \frac{v_{h+1}}{v_h} \right)^{-1} = \prod_{r=h+1}^{k} [1 + i_{r-1,r}] \quad (7.37)$$

Equation (7.37) expresses the annual forward accumulation factor, averaged on e.h., as a geometric mean of the forward accumulation factors for each period and links it to the f.p. of the UZCB. In the last term of (7.37), we can read the *varying per period rates* applicable in the market; the comparison with the left side shows that $i_{h,k}$ is equivalent to the mean forward rate on the horizon (h,k) in the flat structure that follows from the exponential regime.

From (7.37) we find the delayed forward interest rate (annual base) on the e.h. (h,k):

$$i_{h,k} = s_{h,k}^{-1/(k-h)} - 1 \quad (7.38)$$

or advance

$$d_{h,k} = 1 - s_{h,k}^{1/(k-h)} \quad (7.39)$$

(obtaining this last formula using a generalization of (7.28")).

For a comparison between (7.38) and (7.39), we find the relation between rates, analogous to (7.28"')

$$d_{h,k} = i_{h,k} /(1+i_{h,k}) \quad (7.39')$$

Example 7.17

Still using the structure of the SP given in Example 7.14, let us find the corresponding structure of the f.p. in a coherent market, leaving out the restriction of annual horizons. In this case, the upper triangular matrix $s_{h,k}$ ($1 \le h \le k \le 4$), with h = row index and k = column index, is of order 4 and assumes the values given below, found through (7.35).

Using v_1 = 0.9628; v_2 = 0.9371; v_3 = 0.9008; v_4 = 0.8788, let us find the prices matrix $s_{h,k}$ and forward rates $i_{h,k}$ by means of an *Excel* spreadsheet that has the following form.

	Prices structure $s_{h,k}$				
H	Price sp %	*k*=1	*k*=2	*k*=3	*k*=4
1	96.28	1.000000	0.973307	0.935604	0.912754
2	93.71		1.000000	0.961263	0.937787
3	90.08			1.000000	0.975577
4	87.88				1.000000

	Rates structure $i_{h,k}$				
H		*k*=1	*k*=2	*k*=3	*k*=4
1			0.027425	0.033841	0.030897
2				0.040298	0.032638
3					0.025034
4					

Table 7.3. *Uni- and multi-period forward structure*

The *Excel* instructions are as follows: 1st and 2nd row: empty; then:

price structure: from 3rd to 8th row. 3rd and 4th row: titles; rows 5-8:

column A	A5: 1; A6:= A5+1; copy A6, then paste on A7 to A8;
column B	insert data (spot prices %) on B5 to B8;
diagonal (k=h)	C5:1; D6:1; E7:1; F8:1;
1° supradiagonal (k=h+1):	D5:= $B6/$B5; copy D5, then paste on E6, F7;
2° supradiagonal (k=h+2)	E5:= $B7/$B5; copy E5, then paste on F6;
3° supradiagonal (h+2=k)	F5:= B8/B5;

rate structure: from 10th to 15th row. 10th and 11th row: titles; rows 12-15:

column A	A12: 1; A13:= A12+1; copy A13, then paste on A14 to A15;
1° supradiagonal (k=h+1)	D12:= D5^-(1/($A6-$A5))-1; copy D12, then paste on E13, F14;
2° supradiagonal (k=h+2)	E12:= E5^-(1/($A7-$A5))-1; copy E12, then paste on F13;
3° supradiagonal (h+2=k)	F12:= F5^-(1/(A8-A5))-1;

oher cells: empty.

In the price matrix the values on the first supradiagonal are obviously the prices $s_{k-1,k}$ obtained in Example 7.16. The corresponding implicit rates (excluding those that are reduced to spot rates) are found by means of (7.37).

Let us verify the properties of (7.37). With the numbers obtained by the previous matrix $\|i_{h,k}\|$ we find:

e.h. 1-3:	1.033841^{2}	=	$1.027425 \cdot 1.040298$
e.h. 2-4:	1.032638^{2}	=	$1.040298 \cdot 1.025034$
e.h. 1-4:	1.030897^{3}	=	$1.027425 \cdot 1.040298 \cdot 1.025034$

If we complete the matrix $\|i_{h,k}\|$ using $i_{h,k} = 0$ if $h \geq k$, and add 1 to each element of the square matrix $\|i_{h,k}\|$ thus obtained, then in each square submatrix, extracted from the new matrix and such that its main diagonal has elements $i_{h,k}$ satisfying $h<k$, the number written in the NE vertex is the geometric mean of those which appear on the main diagonal of the submatrix.

The previous considerations show how the gathering of market prices implicitly leads us to formalize on the given time horizon (h,k) a financial exchange law, defined only on integer time variables, that can be expressed by means of discount factors $s_{h,k}$ (<1 if $h<k$) defined in (7.5) or analogously by means of accumulation factors $r_{h,k} = 1/s_{h,k}$ or interest rates $i_{h,k}$ or discount rates $d_{h,k}$.

On the contrary, we can think – as was already observed at the beginning of this chapter – that the term structure valid in a market follows the definition of an empirical financial law that in a given time interval holds on the market for simple operations. Such a formulation can be extended to complex operations, in particular to *annuities* and *amortizations*. This will be considered in Chapter 8.

Observation

From the previous formulations, in particular from (7.33), it is obvious that the term structure maintains the principle of compound accumulation, even if in a more general way that leads to varying rates.

Building up the term structure of spot and forward rates.

Referring to the *bond market*, the use of spot rates implies that the financial flows generated by different securities are assumed to be discounted at the same rate. It is then essential to deduce from the available data the so-called *term structure of spot rates* applicable to all securities of the market as a function of the different evaluation time interval.

It is possible to find this structure, expressed by *spot rates* for the same interval terms, starting from an observation taken from the market on the issues prices of bonds with maturities increasing in arithmetic progression (according to natural numbers, on the basis of an appropriate choice of the unit time). We can then find a sequence of spot rates applicable to the examined market.

The calculation procedure can be described as follows: decide the time unit and change the rates accordingly; carry out a statistical observation of the issue prices (i.e. at time 0) of the securities in a bond market with different financial profiles; obtain a price for each of the maturities $h = 1,...,n$. If we refer to a *coupon bond with maturity h*, knowing the price $V_{(h)}$ and the coupon amounts $\{I_k\}$, $k = 1,...,h$, and the redemption C_h for the bond with maturity h [10] we can write the solving system, where the price $V_{(h)}$ of the bonds maturing after h periods is made equal to the present value according to the unknown rate structure. This system is given by:

$$V_{(h)} - H_{h-1} = \frac{I_h + C_h}{(1+i_h)^h} \quad , \quad h = 1,...,n \tag{7.40}$$

where:

$$H_0 = 0 \ \ , \ \ H_h = \sum_{k=1}^{h} \frac{I_k}{(1+i_k)^k} \ , \ h = 1,...,n\text{-}1 \tag{7.41}$$

In (7.40), the unknowns i_h appear in a triangular way, in the sense that in the 1st equation (h=1) we have only i_1 which is then found directly, in the 2nd equation (h=2) we have i_1 and i_2 which are again found directly knowing i_1 in H_1; then, in the h^{th} equation the first term is found using (7.41) and in the second term the only unknown is i_h which is found immediately.

This procedure assumes the existence of a sequence of securities with maturities distributed at regular intervals and quoted at equilibrium prices. Note that, different from the *yield rates*, there exists for each time interval a biunivocal correspondence between *spot rates* and prices.

Example 7.18

Let us apply the procedure to build up the spot rate structure, starting from the price sequence, referring to five bond types (which are not all zero-coupon bonds),

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10 For the ZCB it is enough to set all the values $\{I_k\}$ at zero.

with equally spaced maturities. Data are summarized in the first four columns of the following tables; each row is referred to one bond; the first three are zero-coupon bonds and the other two have fixed coupons. The last column gives the results, obtained as specified below, i.e. the *spot* rates referred to the length of the bond specified in the here 2nd column (but valid in the market of the bonds considered).

Nominal value	Residual length (in years)	Semiannual coupon	Market price	Spot rate (%) on given length
100	0.25	0%	98.90	4.524
100	0.50	0%	97.64	4.893
100	1.00	0%	95.11	5.141
100	1.50	3%	100.84	5.495
100	2.00	2.50%	98.80	5.741

Table 7.4. *Computation of spot rate structure*

For each of the zero-coupon bonds, the price is given by $100 \cdot v_k$ where k=0.25; 0.50; 1.00 and the *spot* rate is found applying (7.8), i.e.

$$0.9890^{-1/0.25} = 0.04524\ ;\ 0.9764^{-1/0.50} = 0.04893\ ;\ 0.9511^{-1} = 0.05141$$

The first three *spot rates* are then obtained in the last column. The 4th bond, with fixed coupon, is entitled supplies: (0.5,3), (1,3), (1.5,103), and the price is the sum of the present values of each amount using the *spot* rate referred to its time. The first two rates (in 2nd and 3rd row) are already known: their values are $i_{0.50}$ = 4.893% and $i_{1.00}$ = 5.141%. The third, i.e. $i_{1.50}$, is obviously the solution to the following equation:

$$\frac{3}{1.04893^{0.50}} + \frac{3}{1.05141} + \frac{103}{(1+i_{1.50})^{1.50}} = 100.84$$

from which $i_{1.50}$ = 0.05495. The 5th bond, with fixed coupon, is entitled to the following supplies: (0.5; 2.5), (1; 2.5), (1.5; 2.5), (2; 102.5), and here the price is the sum of the discounted values with four spot rates referred to the intervals which are multiples of a half-year. The first three, indicated with $i_{0.50}$, $i_{1.00}$ and $i_{1.50}$, are already known. The fourth, i.e. $i_{2.00}$, is obtained analogously as the solution of the following equation:

$$\frac{2.5}{1.04893^{0.5}} + \frac{2.5}{1.05141} + \frac{2.5}{1.05495^{1.5}} + \frac{102.5}{(1+i_{2.00})^{2}} = 98.80$$

from which $i_{2.00} = 0.05741$. In this way, substituting the results found in the subsequent equations for fixed coupon bonds, we find the whole *term structure* of *spot rates* corresponding to the price gathered on the market for the examined securities.

Also for a forward contract, we can build up a *term structure of forward rates*. It is enough to refer to the building up of a sequence of *spot rates* seen before and obtaining from them the implicit *forward* rates, on the basis of market coherence.

7.5.2. *Structures with fractional periods*

As already shown at the beginning of section 7.5.1, we clarified that the time structure in "discrete" scheme is referred to unitary periods, but the unit of times is not necessarily a year. In financial practice, there are market structures with a period which is not annual, but fractional, in which spot and forward prices, rates, intensities r.m have as a basis a fraction of a year (semester, quarter, month, etc.), while pluriennial periods are not used. In such a case, the e.h. and the bond maturities will be multiples of such fractional periods. Let us give a brief insight into this argument.

We must observe that the definition and transformation formulae given in section 7.5.1 are still valid, without any modification, with fractional periods, except for the time measure that is no longer a year, but a fraction of a year. Then the prices concern assets with fractional maturities and the rates refer to periods that are fractions of a year.

It is unnecessary to repeat here the formulae to adapt them to this case: it is enough to declare the different time unit. The argument will then be clarified developing, using *Excel*, Examples 7.19, 7.20 and 7.21, which closely follow Examples 7.15, 7.16 and 7.17, which refer to annual bases.

Example 7.19

On the UZCB market there is fixed today (t=0) the following SP as a function of the quarterly maturities, which define v_k , assuming the quarter as the unit to measure time:

– 0.9866 with maturity after one quarter; 0.9788 with maturity after two quarters;

– 0.9654 with maturity after three quarters; 0.9521 with maturity after four quarters.

In the following *Excel* table, with formulations analogous to those see in Example 7.15, the corresponding structures of spot delayed and advance rate and also of the intensity r.m. are set out.

Maturity	Spot price	Delayed spot rate	Spot intensity r.m.	Advance spot rate
1	0.9866	0.0135820	0.0134906	0.0134000
2	0.9788	0.0107716	0.0107140	0.0106568
3	0.9654	0.0118067	0.0117376	0.0116690
4	0.9521	0.0123469	0.0122713	0.0121963

Table 7.5. *Quarterly basis spot structure*

Comparing with the *Excel* instruction of Example 7.15, to go from the 2nd column to 3rd and 5th column we do not have to divide by 100, because they are prices of UZCB.

Example 7.20

Using the data on prices given in Example 7.19, in the following *Excel* table are fixed starting form the spot structure, with formulations analogous to what was seen in Example 7.16, the corresponding one period structure of forward prices and rates, delayed and advance, and also the intensity r.m.

Maturity	Spot price	Fwd price	Spot rate	Fwd delayed rate	Fwd intensity	Fwd advance rate
1	0.9866	0.986600	0.013582	0.013582	0.013491	0.013400
2	0.9788	0.992094	0.010772	0.007969	0.007937	0.007906
3	0.9654	0.986310	0.011807	0.013880	0.013785	0.013690
4	0.9521	0.986223	0.012347	0.013969	0.013872	0.013777

Table 7.6. *Quarterly basis spot and uni-period forward structure*

Comparing with the *Excel* instruction of Example 7.16 to go from the 2nd column to 3rd and 4th column we do not have to divide by 100, given that one considers prices of UZCB.

Example 7.21

Using the data on prices given in Example 7.19, in the following *Excel* table with formulations analogous to what seen in Example 7.17, the corresponding multiperiod structure of forward prices and rates is set out.

Prices structure $s_{0;h,k}$

h	Spot price	k=1	k=2	k=3	k=4
1	0.9866	1.000000	0.992094	0.978512	0.965031
2	0.9788		1.000000	0.986310	0.972722
3	0.9654			1.000000	0.986223
4	0.9521				1.000000

Rates structure $i_{0;h,k}$

H		k=1	k=2	k=3	k=4
1			0.007969	0.010920	0.011936
2				0.013880	0.013925
3					0.013969
4					

Table 7.7. *Quarterly basis uni- and multi-period forward structure*

The *Excel* instructions are those in Example 7.17.

Observations

In banks and Stock Exchange markets it is used to consider nominal annual return rates even in case of fractional structures. In the considered case, with quarterly structure and data from Example 7.20 (with structures of any frequency, it is enough to use *m* instead of 4), given the *uniperiod forward rates* in the 5th column of the following *Excel* table, it is enough to multiply by four to have (in the 6th column) the nominal annual return in the current quarters.

However, these values show on an annual basis the return of each quarter, but do not give the effective return rate obtained on an annual e.h. To obtain this, starting from an investment in 0 with a given structure, we proceed as follows. The return rate r_k on an e.h. of k periods is found from

$$1+r_k = (1+i_k)^k = \prod_{r=1}^{k}(1+i_{r-1,r}) \qquad (7.33'')$$

which extends (7.33) referring to fractional structures. For k=1,2,3,4, the values of i_k are the quarterly spot rates shown in the 4th column of the table, while the values of r_k are shown in the 7th column and are the return rates on the e.h. of the first k quarters, which is also the basis. In particular, for k=4 we obtain (with the data of Example 7.20 to which the table is referred) the rate 0.050310, which is the effective return rate for one year and on an annual basis, better than the nominal rate.

k	Spot price	Fwd price	Spot rate	Fwd rate	Nominal annual rate	Return rate 0-*k*
1	0.9866	0.986600	0.013582	0.013582	0.054328	0.013582
2	0.9788	0.992094	0.010772	0.007969	0.031876	0.021659
3	0.9654	0.986310	0.011807	0.013880	0.055521	0.035840
4	0.9521	0.986223	0.012347	0.013969	0.055876	0.050310

Table 7.8. *Nominal annual rates in the current quarters*

The *Excel* instructions for this table are as follows. After 2nd row for titles, the first 5 columns are the same as those in Example 7.20; in addition:
6th column: (nominal annual rates) F3:= 4*E3; copy F3, then paste on F4 to F6;
7th column: (return rates e.h. 0-k) G3:= (1+D3)^A3-1; copy G3, then paste on G4 to G6.

7.5.3. *Structures with flows "in continuum"*

Let us consider the case in which the flows are continuous (for instance a continuous trading market). Let us first observe that, assuming continuous time, the formulae (7.8) and (7.8") are enough to define the spot rate $i(x,y)$ and the spot intensity $\phi(x,y)$ according to the SP $v(x,y)$.

In addition, with continuous payment flows, the implicit structure, corresponding to the spot structure, we have to consider infinitesimal e.h. $(y, y+dy)$ where the forward prices $s(x;y,y+dy)$ go to 1 and the implicit forward rates $i(x;y,y+dy)$ go to 0, losing any meaning. It is then appropriate to refer directly to the *instantaneous discount intensity time structure*. The spot structure is found from formulae of the type of (2.23) (or inversely (2.24)) reinterpreted in market terms. The term structure is found from the spot structure based on the known constraints. The functions $\delta(x,y)$ are then the starting point of the term structure in continuous time.

With discrete schedules we can build up a term structure starting from an instantaneous intensity $\delta(x,y)$ that gives, always in symmetric hypothesis, an exchange law in continuum, from which are found spot and forward prices, rates and intensities r.m.. We show here the following formulae that are immediately justifiable (referring to a transaction time that is not to restrictive too put in $t = 0$):

$$v_k = e^{-\int_0^k \delta(0,u)du} \tag{7.42}$$

$$s_{h,k} = e^{-\int_h^k \delta(0,u)du} \tag{7.43}$$

$$i_k = e^{(\int_0^k \delta(0,u)du)/k} - 1 \tag{7.44}$$

$$i_{h,k} = e^{(\int_h^k \delta(0,u)du)/(k-h)} - 1 \tag{7.45}$$

$$\phi_k = \frac{1}{k}\int_0^k \delta(0,u)du \tag{7.46}$$

$$\phi_{h,k} = \frac{1}{(k-h)}\int_h^k \delta(0,u)du \tag{7.47}$$

Example 7.22

For investment on the horizon (0;5), the return financial law is ruled by the instantaneous intensity $\delta(0,u)$, according to current time u, $(0 \le u \le 5)$, for operations agreed in 0, defined by $\delta(0,u) = 0.04 + 0.00564\, u - 0.00033\, u^2$, where, for example: $\delta(0;0) = 0.04$; $\delta(0;2) = 0.05$; $\delta(0;5) = 0.06$.

On the horizon (0;5), by means of (7.47), we obtain the intensities r.m.

$\phi_{k\text{-}1,k}$ for annual intervals, that we will indicate with H_k . We obtain:

$$H_k = \int_{k-1}^k \delta(0,u)du = \left| const. + \ 0.04\ \ u + 0.00282\ \ u^2 - 0.00011\ \ u^3 \right|_{k-1}^{k} =$$

$$= 0.04 + 0.000282\ (2k - 1) - 0.00011\ (3k^2 - 3k + 1)$$

Thus:

$H_1 = 0.042710$; $H_2 = 0.047690$; $H_3 = 0.052010$; $H_4 = 0.055670$; $H_5 =$ 0.058670

from which, due to (7.43), the values:

$$s_{0,1} = e^{-H_1} = 0.958189\ ;\ \ s_{1,2} = e^{-H_2} = 0.953429\ ;\ \ s_{2,3} = e^{-H_3} = 0.949319\ ;$$

$$s_{3,4} = e^{-H_4} = 0.945851\ ;\ \ s_{4,5} = e^{-H_5} = 0.943018$$

follow. From the intensities $\phi_{k\text{-}1,k} = H_k$ for annual intervals we find, due to (7.45), the corresponding implicit forward rates $i_{k\text{-}1,k} = e^{-H_k} - 1$, obtaining:

$$i_{1,2} = 0.048845 \ ; \quad i_{2,3} = 0.053386 \ ; \quad i_{3,4} = 0.057249 \ ; \quad i_{4,5} = 0.060425 \ .$$

The spot intensities r.m. for k years are written:

$$\phi_k = \frac{1}{k}\sum_{r=1}^{k} H_r$$

and it follows that:

$$\phi_1 = 0.042710 \ ; \quad \phi_2 = 0.045200 \ ; \quad \phi_3 = 0.047470 \ ; \quad \phi_4 = 0.049520 \ ; \quad \phi_5 = 0.051350 \ .$$

In addition, the forward intensities r.m. are given by:

$$\phi_{h,k} = \frac{1}{k-h}\sum_{r=h+1}^{k} H_r$$

With the given instantaneous intensity we obtain, for example:

$$\phi_{2;4} = (0.05201+0.05567)/2 = 0.05384.$$

The spot rates i_k on an horizon of k years, expressed by (7.44), but which can also be written in the form $e^{\phi(k)} - 1$, are:

$$i_1 = 0.043635 \ ; \quad i_2 = 0.046237 \ ; \quad i_3 = 0.048615 \ ; \quad i_4 = 0.050767 \ ;$$
$$i_5 = 0.052691 \ .$$