

Chapter 6

Loan Amortization and Funding Methods

6.1. General features of loan amortization

We have already seen in Chapter 5 that, given a discount law, the inverse problem of the computation of initial value of an annuity is an amortization, in the sense that *the annuity's installments are the amortization's installment of a debt equal to its initial value.*

Having then clarified the general concept of *loan amortization*, we will consider in this chapter the classification and description of the most common amortization methods of a debt contracted at a given time. We will consider in sections 6.2, 6.3, and 6.4 the *amortization of unshared loans* (i.e. with only one lender, which is the creditor, and only one borrower, which is the debtor) *at fixed rates and at varying rates*. In the loan mortgage contract the borrower guarantees payment to the lender.

The exchange law used will be that of discrete compound interest (*DCI*), because in this context we usually consider operations having pluriennial length and the calculation of interest is performed periodically when the borrower pays.

In sections 6.8 and 6.9 we will consider *shared loans* (i.e. among a number of creditors, in the presence of quite a large amount of debt) at fixed rate. The need for the same conditions among many creditors leads to technical complications and problems in financial evaluations, that we have to consider.

In an unshared loan, assuming a discrete scheme for repayment¹, we can distinguish between the following:

- a) *only one lump-sum repayment of the principal at the end of the term:*
 - a₁) *with only one interest payment at the end of the term,*
 - a₂) *with periodic interest payment;*
- b) *periodic repayments of the principal together with the accrued interests.*

The formulation will consider the scheme of annual payments: for different cases, it is enough to assume the used period as the unit measure, and introduce the equivalent rate. We will then indicate with i the rate per period and with n the number of periods.

In case a₁), operation \hat{O} is simple, consisting of the exchange between the principal C given by the lender in 0 and the amount M paid by the borrower in n as repayment of the debt and payment of all the accrued interests. M is obviously the accumulated value of C after n periods, obtainable using equation (3.24) for a fixed-rate loan and equation (3.23) for loans with varying rates and given times. Therefore, $\hat{O} = (0, -C)U(n, +M)$ from the viewpoint of the lender.

In case a₂), still considering only one final repayment, the interest, which is always calculated on the initial debt, is paid at the end (or beginning) of each period and calculated at rate i (or respectively at the rate d). In the two cases the operations are written as

$$(0, -C)U(1, Ci)U \dots U(n-1, Ci)U(n, C(1+i)) \quad (6.1)$$

$$(0, -C(1-d))U(1, Cd)U \dots U(n-1, Cd)U(n, C) \quad (6.1')$$

Example 6.1

Referring to a debt of €255,000 to pay back with the scheme in a₁), after 5 years at the fixed rate of 6.50%, the final debt amount, together with interest, is

$$D = 255,000 \cdot 1.065^5 = 255,000 \cdot 1.3700867 = \text{€}349,372.10.$$

Using scheme a₂) with delayed annual installments for the interest, for the given debt we have to pay €16,575.00 at the end of each of the first 5 years, adding at the end of the 5th year the repayment of the debt. Adopting, instead, advance annual installments for the interest, we have to pay €15,563.38 at the beginning of each of the first 5 years, with the repayment of the debt after 5 years. Case b) considers in general terms the *gradual amortization*, the form of which at fixed rate will be

¹ An amortization in a continuous scheme, instead, would lead to consideration of a continuous annuity for the debt amortization. Such scheme is possible but has no practical relevance.

described in section 6.2. The periodicity of the installments is usually annual or semi-annual (but sometimes it is quarterly or monthly) and delayed²; less frequent are advance payments.

6.2. Gradual loan amortization at fixed rate

6.2.1. Gradual amortization with varying installments

The gradual amortization, once the initial debt S , the per period rate i and the length (or number of periods) n are given, the installments (also simply named payments) R_h in the delayed case or \ddot{R}_h in the advance case (where h indicates the integer date of payments between 0 and n), must satisfy the constraint of financial closure expressed in the two cases by one of the equations in (5.23) where in the left side, S instead of v_0 or \ddot{V}_0 is used, obtaining

$$S = \sum_{h=1}^n R_h (1+i)^{-h} \tag{6.2}$$

$$S = \sum_{h=0}^{n-1} \ddot{R}_h (1+i)^{-h} \tag{6.2'}$$

The amortization annuities are, in fact, in the two cases: $\bigcup_{h=1}^n (h, R_h)$, $\bigcup_{h=0}^{n-1} (h, \ddot{R}_h)$.

The aforesaid installments, altogether equivalent to the initial debt, are divided into two amounts:

- the “principal repaid”, c_h in the delayed case or \ddot{C}_h in the advance case, that decreases the debt;
- the “interest paid” i_h in the delayed case or \ddot{I}_h in the advance case, which is a gain for the creditor and is proportional to the level of remaining debt (i.e. the outstanding loan balance, defined below)³.

2 In the practice of bank loans there can be a “pre-amortization” phase (see also the following footnote 3), from the day the loan is granted to the end of the first period, in which the debtor pays only the accrued interest and the amortization begins, the times of which are parts of a calendar year.

3 To avoid the remaining debt overcoming the initial loan during the amortization, something that the lender cannot allow (due to the consequent lack of guarantees), the principal repayments must never be negative, i.e. the installment are at least at the level of the amount of the interest paid. It is in particular verified the equality “installments = interest paid” ie the absence of principal repayments, during an interval of “pre-amortization” in the initial phase. The creditor can allow such facility in special cases, for instance when the investment financed by the loan implies a delay in the return and then an initial lack of liquidity for the debtor. In such cases, if the pre-amortization lasts for m periods, then $R_1 = \dots = R_m = Si$ and the real amortization is included in the following $n-m$ periods. Observe that the *amortization with*

By definition, the principal repayments must satisfy the constraints of *elementary closure* expressed by

$$\sum_{h=1}^n C_h = S \quad ; \quad \sum_{h=0}^{n-1} \ddot{C}_h = S \quad (6.3)$$

As time increases during the gradual amortization, it is necessary to account at the internal due dates (= end of periods) the *outstanding loan balance* (or *outstanding balance*, or simply *balance*) D_h and the *discharged debt* $E_h = S - D_h$.

Let us consider the *delayed gradual amortization*. This results in

$$D_0 = S \quad ; \quad D_h = S - \sum_{k=1}^h C_k \quad ; \quad (h=1, \dots, n) \quad (6.4)$$

and then $D_n = 0$ owing to the 1st part of (6.3).

We have already seen that in section 5.4 amortization as the inverse problem of calculating the initial value (*IV*) of an annuity with varying installments, that the solution is not unique, having $n-1$ degrees of freedom: we have only one constraint on the n unknowns R_h . In order that the number of such degrees be zero, so as to have a unique solution, we need to introduce other $n-1$ constraints that are linearly independent. This can be carried out in an infinite number of ways: one of which is the imposition of installments in arithmetic or geometric progression, as was shown in section 5.5. However, in general, we can fix the installments under the constraint in (6.2) or the principal repayments under the constraint in the 1st part of (6.3), taking into account the needs of both parties to the contract.

The solution can be found recursively from the system of $3n$ equations

$$(h=1, \dots, n) \left\{ \begin{array}{l} D_h = D_{h-1} - C_h \\ I_h = i D_{h-1} \\ R_h = C_h + I_h \end{array} \right. \quad (6.4')$$

in the $3n$ unknowns $D_h, I_h, R_h, (h=1, \dots, n)$ with the initial condition $D_0 = S^4$.

only one final repayment of type a_2 can be considered as a total pre-amortization until the end of the loan.

⁴ This means that, taking into account the initial condition $D_0 = S$, if the installments R_h are given, I_1 is found from the 2nd part of (6.5), C_1 from the 3rd part, D_1 from the 1st part, and we repeat such a procedure by increasing h . Instead, if the principal repayments $C_h, (h=1, \dots, n)$, are given, it is found I_1 from the 2nd part of (6.5), D_1 from the 1st part, R_1 from the 3rd part, and we repeat such a procedure by increasing h .

From (6.4') the recursive relation is found

$$D_h = D_{h-1} (1+i) - R_h \tag{6.5}$$

that gives the following alternative for the calculation of R_h :

$$R_h = D_{h-1} (1+i) - D_h = (D_{h-1} - D_h) + i D_{h-1} \tag{6.5'}$$

Both (6.5) and (6.5') have an expressive financial meaning referring to the dynamic of amortization.⁵

It is convenient, for calculation in case of assignments or advance discharge, to give the reserve defined in Chapter 4. At the rate i (which can be the one in the contract or a different one for the evaluation in k) and at the integer due date h the retro-reserve is

$$M_h = S(1+i)^h - \sum_{k=1}^h R_k(1+i)^{h-k} \tag{6.6}$$

while the pro-reserve is

$$W_h = \sum_{k=h+1}^n R_k(1+i)^{-(k-h)} \tag{6.6'}$$

We can easily verify that, due to the fairness of the amortization operation expressed by (6.2),⁶ using for the valuation the loan rate i , $M_h = W_h = D_h$ follows.⁷

5 Let us verify that (6.6) is equivalent to (6.2), i.e. implies the financial closure, and let us give the closed expression for the balances. From (6.6) we find: $D_{h-1} = (R_h + D_h)v$ and when using it for decreasing values of h the following is obtained: $D_{n-1} = R_n v$; $D_{n-2} = (R_{n-1} + D_{n-1})v = R_{n-1} v + R_n v^2$; ... ; $D_{n-h} = (R_{n-h+1} + D_{n-h+1}) v = \sum_{k=1}^h R_{n-h+k} v^k$; ...; $S = D_0 = (R_1 + D_1) v = \sum_{k=1}^n R_k v^k$ and vice versa. We can also write: $D_h = \sum_{k=1}^{n-h} R_{h+k} v^k$, which gives the remaining debt at the h^{th} due date as a function of the installments following R_h .

6 The fairness can be controlled in an alternative way, but which is equivalent, from the final debt. In fact it can soon be seen that if $D_n = 0$ is satisfied, the operation consisting of the loan of S amortized with the sequence $\{R_h\}$ with delayed due dates is fair at rate i . Otherwise, while the pro-reserve is zero due to the absence of remaining obligation, the retro-reserve would not become zero and the final spread D_n , positive or negative, would make the operation favourable respectively for the borrower or for the lender.

7 The retrospective and prospective reserves, evaluated in whichever integer time $h \in [0, n]$, maintain their meaning of evaluation of the net obligation before and after h , even if in h it is

Furthermore, bare ownerships and usufructs, as defined in Chapter 4, at the integer times h and with discontinuous formation of interests, being $I_k = iD_{k-1} = i\sum_{s=k}^n C_s$, are given by

$$(h=1, \dots, n) \left\{ \begin{aligned} P_h &= \sum_{k=h+1}^n C_k (1+i)^{-(k-h)} \\ U_h &= \sum_{k=h+1}^n I_k (1+i)^{-(k-h)} = i \sum_{k=h+1}^n (1+i)^{-(k-h)} \sum_{s=k}^n C_s \end{aligned} \right. \quad (6.7)$$

If we use the CCI regime, it would be possible to define the reserves at each intermediate time between two due dates in succession, to calculate in the exact way the assignment or discharge value for whichever time $t = k+s$ (where $k =$ integer part of t ; $s =$ decimal part of t). We obtain

$$M(t) = M_k (1+i)^s \quad ; \quad W(t) = W_k (1+i)^s$$

Let us consider briefly the variation in an *advance gradual amortization*. Analogously to the delayed case, the solution can be obtained considering recursively the system of $3n$ equations

$$(h=0, \dots, n-1) \left\{ \begin{aligned} D_{h+1} &= D_h - \ddot{C}_h \\ \ddot{I}_h &= d D_{h+1} \\ \ddot{R}_h &= \ddot{C}_h + \ddot{I}_h \end{aligned} \right. \quad (6.4'')$$

in the $3n$ unknowns d_{h+1} (= outstanding balance after h and until $h+1$), \ddot{I}_h , \ddot{R}_h , ($h = 0, \dots, n-1$) with the usual initial condition $D_0 = S$.

adopted for the evaluation of a rate i different from that established at the inception of the loan. However, in such a case we lose the equality between prospective reserve and outstanding balance and also that between prospective reserve and retrospective reserve because, if the sequence of the installments R_h is unchanged, (6.2) does not hold any more and the fairness of the whole operation is lost. The same considerations hold for the case of advance amortization, considered later.

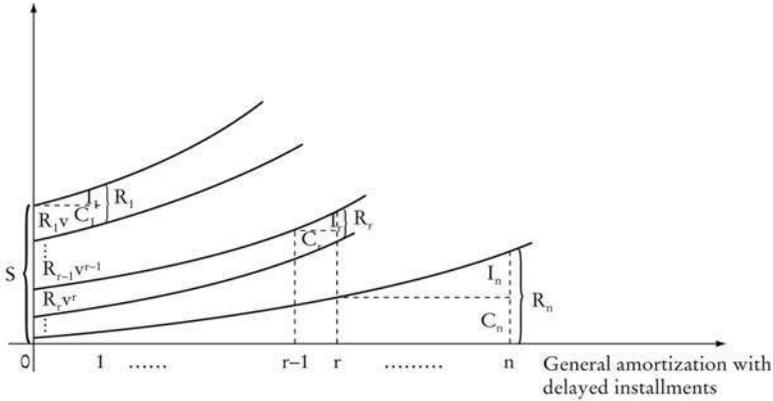


Figure 6.1. Plot of delayed amortization

From (6.4'') we obtain

$$\ddot{R}_h = D_h - D_{h+1} v \tag{6.5''}$$

from which the recursive relation results:

$$D_h = D_{h+1} (1-d) + \ddot{R}_h \tag{6.5'''}$$

being $D_n = 0$ for the 2nd of (6.3).

Furthermore, at the delayed loan interest $i=d/(1-d)$ and at the due integer date h the retro-reserve is

$$M_h = S(1+i)^h - \sum_{k=0}^{h-1} \ddot{R}_k (1+i)^{h-k} \tag{6.6''}$$

while the pro-reserve is

$$W_h = \sum_{k=h}^{n-1} \ddot{R}_k (1+i)^{-(k-h)} \tag{6.6'''}$$

and due to (6.2') we obtain the fairness.

Furthermore, bare ownerships and usufructs at the integer times h and with discontinuous formation of interests, as $\ddot{I}_k = dD_{k+1} = d \sum_{s=k+1}^{n-1} \ddot{C}_s$, are given by

$$(h = 0, \dots, n - 1) \left\{ \begin{array}{l} P_h = \sum_{k=h}^{n-1} \ddot{C}_k (1+i)^{-(k-h)} \\ U_h = d \sum_{k=h}^{n-1} (1+i)^{-(k-h)} \sum_{s=k+1}^{n-1} \ddot{C}_s \end{array} \right. \quad (6.7')$$

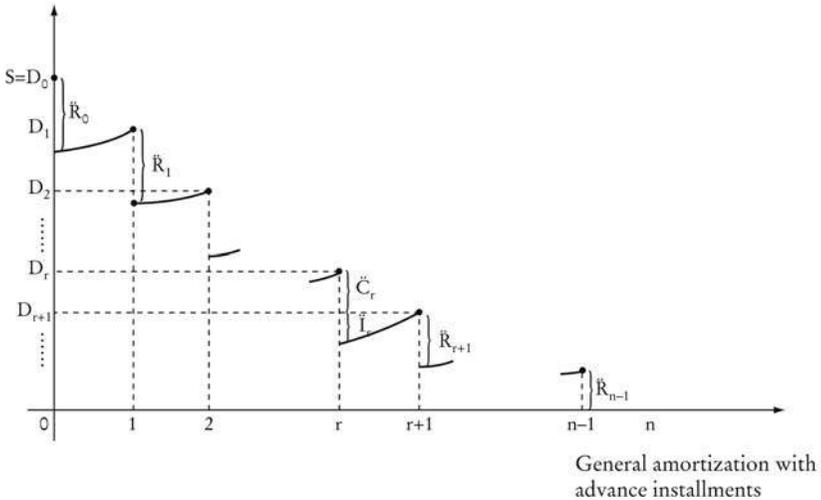


Figure 6.2. Plot of advance amortization

Debt amortization schedules

In the operative practice an amortization schedule is summarized in a table in which for each payment is reported on the same row: 1) period, 2) outstanding balance at the beginning of the period, 3) principal repaid, 4) interest paid, 5) installment, 6) outstanding balance at the end of the period. In the case of *delayed installments* the following table is obtained, for $h=1, \dots, n$.

(1)	(2)	(3)	(4)	(5)	(6)
...
h	D_{h-1}	C_h	$I_h = i D_{h-1}$	$R_h = C_h + I_h$	D_h
...

Table 6.1. Amortization schedule

Here, it is enough to consider only one of the columns (2) or (6), which coincide except for the displacement of one period. This is easy to change in the case of *advance installments*, in which case $I_h = d D_h$.

Example 6.2

What we discussed regarding a gradual amortization with delayed or advance payments is numerically explained here. Consider the delayed or advance amortization of a debt of €90,000 in 5 years at the annual delayed rate $i = 5.50\%$ with principal repayment given by:

$$C_1 = \ddot{C}_0 = C_5 = \ddot{C}_4 = 16,000 \quad ; \quad C_2 = \ddot{C}_1 = C_4 = \ddot{C}_3 = 19,000 \quad ; \quad C_3 = \ddot{C}_2 = 20,000 .$$

The elementary closure is verified, as it can be easily observed.

By applying (6.4') for the delayed case and using $D_0 = €90,000$, the following schedule is recursively obtained, by using a calculator or an *Excel* spreadsheet, as explained below

Debt = 90000		Rate $i = 0.055$		Length = 5	
h	D_{h-1}	C_h	I_h	R_h	D_h
1	90000.00	16000.00	4950.00	20950.00	74000.00
2	74000.00	19000.00	4070.00	23070.00	55000.00
3	55000.00	20000.00	3025.00	23025.00	35000.00
4	35000.00	19000.00	1925.00	20925.00	16000.00
5	16000.00	16000.00	880.00	16880.00	0.00

Table 6.2. Example of gradual amortization with delayed payments

The *Excel* instructions are as follows. B1: 90000; D1: 0.055; F1: 5; using the first two rows for data and column titles, from the 3rd row we have:

- column A (year): A3: 1; A4=A3+1; copy A4, then paste on A5 to A7
- column B (outstanding balance ante): B3: = B1; B4: = F3; copy B4, then paste on B5 to B7;
- column C (principal repaid): from C3 to C7: (insert given data);
- column D (interest paid): D3: = D\$1*B3; copy D3, then paste on D4 to D7;
- column E (installments): E3: = C3+D3; copy E3, then paste on E4 to E7;
- column F (outstanding balance post): F3 = B3-C3; copy F3, then paste on F4 to F7;

For the advance case, being $d = 5.21327\%$, by applying (6.4''), and using $D_0 = €90000$, the following schedule is obtained by using a calculator or an *Excel* spreadsheet, as explained below.

Debt = 90000		Rate $d = 0,052133$		Length = 5	
h	D_h	C^h	I^h	R^h	D_{h+1}
0	90000.00	16000.00	3857.82	19857.82	74000.00
1	74000.00	19000.00	2867.30	21867.30	55000.00
2	55000.00	20000.00	1824.64	21824.64	35000.00
3	35000.00	19000.00	834.12	19834.12	16000.00
4	16000.00	16000.00	0.00	16000.00	0.00

Table 6.3. Example of gradual amortization with advance payments

The *Excel* instructions are as follows. B1: 90000; D1: 0.055; F1: 5. Using the first two rows for data and column titles, from the 3rd row we have:

column A (year): A3: 0; A4=A3+1; copy A4, then paste on A5-A7;
 column B (balance ante): B3: = B1; B4: = F3; copy B4, then paste on B5 to B7;
 column C (principal repaid): from C3 to C7: (insert given data);
 column D (interest paid): D3: = \$D\$1*F3; copy D3, then paste on D4 to D7;
 column E (installment): E3: = C3+D3; copy E3, then paste on E4 to E7;
 column F (balance post): F3: = B3-C3; copy F3, then paste on F4 to F7.

For the manual calculation of the polynomials in v in (6.2) and (6.2') it is enough to alternate multiplications by $v = 0.9478763$ and installment additions:

calculating: $\{[(R_5v + R_4)v + R_3]v + R_2\}v + R_1\}v$ in the delayed case, and $\{[(\ddot{R}_4v + \ddot{R}_3)v + \ddot{R}_2]v + \ddot{R}_1\}v + \ddot{R}_0$ in the advance case, 90,000 is obtained.

When stopping the calculation after k installments from below, the backwards outstanding balances D_{5-k} are obtained, i.e. 16,000; 35,000; 55,000; 74,000.

Exercise 6.1

We have to discharge a loan of 45 million monetary units (MU) for the financing of the building of an industrial plan which, owing to long assembly time, will give net profits only 2 years and 6 months from the inception date of the loan. Furthermore, having the possibility of increasing in time the accumulation of capital for the repayment, the parts agree that, after a pre-amortization with 5 semi-annual installments, the loan is amortized in 7 years with delayed semi-annual installments, increasing in arithmetic progression at the rate of 5% per half-year. Annual rates of 8% for the pre-amortization and 7% for the amortization, are agreed. Calculate the pre-amortization and amortization installments in the two alternatives:

- a) the rates are effective annually,
- b) the rates are nominal annual 2-convertible,

using for the evaluation the effective amortization rate.

A. Assuming the half-year as unit measure and using $S = 45,000,000$; R = base of installment ; D = ratio = $0.05 R$; i_1 = semiannual effective pre-amortization rate; i_2 = semiannual effective amortization rate; the equivalence relation must hold (at the amortization rate):

$$S = S i_1 a_{\overline{5}|i_2} + (1+i_2)^{-5}[R a_{\overline{14}|i_2} + D (Ia_{\overline{14}|i_2})]$$

For case a):

$$i_1 = 0.0392305; i_2 = 0.0344080; (1+i_2)^{-5} = 0.8443853; a_{\bar{5}|i_2} = 4.5226323;$$

$$a_{14|\bar{i}_2} = 10.9640169; (Ia_{14|\bar{i}_2}) = 76.2254208; \text{ then the relation becomes}$$

$$45 \cdot 10^6 = 1765372 \cdot 4.5226323 + 0.8443853 R [10.9640169 + 0.05 \cdot 76.2254208]$$

then: $R = 2,966,957.60$; $D = 148,347.90$. Therefore:

– pre-amortization installments: $S i_1 = 1,765,372.50$;

– amortization installments: $R_1 = 3,115,305.50$; $R_2 = 3,263,653.40$; $R_3 = 3,412,001.30$; ...; $R_{14} = 5,043,828.20$.

In case b):

$$i_1 = 0.04; i_2 = 0.035; (1+i_2)^{-5} = 0.8419732; a_{\bar{5}|i_2} = 4.5150524; a_{14|\bar{i}_2} = 10.5691229;$$

$$(Ia_{14|\bar{i}_2}) = 75.8226691; \text{ then the relation becomes}$$

$$45 \cdot 10^6 = 1800000 \cdot 4.5150524 + 0.8419732 R [10.5691229 + 0.05 \cdot 75.8226991]$$

then: $R = 2,966,983.60$; $D = 148,349.20$. Therefore:

pre-amortization installments: $S i_1 = 1,800,000.00$;

amortization installments: $R_1 = 3,115,332.80$; $R_2 = 3,263,682.00$; $R_3 = 3,412,031.20$; ...; $R_{14} = 5,043,872.40$.

6.2.2. Particular case: delayed constant installment amortization

Having developed the general case, it is enough to consider briefly the more diffused cases of the amortization of unshared loans at fixed rates. Let us start from the classical case, in which a loan of amount S is paid back in n periods (annual or shorter) with *constant delayed installments* R calculated on the basis of DCI law at the rate per period i . The equivalence constraint is the particular case of (6.2), as it is given on the basis of the symbols defined in Chapter 5, by:

$$S = R a_{\bar{n}|i} \quad \text{from which} \quad R = S \alpha_{\bar{n}|i} \tag{6.8}$$

The 2nd part of (6.8) gives univocally the amortization installment as a function of S , n , i . We obtain here a particular case of system (6.4') by using $R_h = R$.

An important property of such amortization, also called *French amortization*, that justifies the name of *progressive amortization*, consists of the fact that *the principal repayments increase in geometric progression (GP) with ratio (1+i)*.

Proof. Particularizing (6.6) for consecutive values h and $h+1$, it is found that:

$$R = D_{h-1} (1+i) - D_h ; R = D_h (1+i) - D_{h+1} .$$

Subtracting term by term, we find

$$0 = (D_{h-1} - D_h) (1+i) - (D_h - D_{h+1})$$

from which

$$C_{h+1} = C_h (1+i) , \quad h = 1, \dots, n-1 \tag{6.9}$$

As C_{h+1}/C_h is independent from h , this proves the evolution of C_h in GP.

Starting from the value of the installment given in (6.8) we easily obtain the following French amortization schedule⁸.

Debt (S)	Rate (i)	Length (n)	Installment ($R = S \alpha_{\bar{n} i}$)
Period (h)	Principal (C_h)	Interest (I_h)	Balance (D_h)
1	$R v^n$	$R(1 - v^n)$	$R a_{\overline{n-1} i}$
2	$R v^{n-1}$	$R(1 - v^{n-1})$	$R a_{\overline{n-2} i}$
..
h	$R v^{n-h+1}$	$R(1 - v^{n-h+1})$	$R a_{\overline{n-h} i}$
..
$n-1$	$R v^2$	$R(1 - v^2)$	$R a_{\overline{1} i}$
n	$R v$	$R(1 - v)$	0

Table 6.4. French amortization

In fact, applying (6.4') recursively with $R_h = R$, it results in:

$$I_1 = R i a_{\bar{n}|i} = R(1 - v^n) ; C_1 = R - I_1 = Rv^n ; D_1 = R a_{\bar{n}|i} - Rv^n = R a_{\overline{n-1}|i} ,$$

$I_2 = R i a_{\overline{n-1}|i} = R(1 - v^{n-1}) ; C_2 = R - I_2 = Rv^{n-1} ; D_2 = R a_{\overline{n-1}|i} - Rv^{n-1} = R a_{\overline{n-2}|i} ;$
 etc. The GP behavior of C_h is confirmed.

⁸ It is easy to calculate the principal repaid, the interest paid and the outstanding balance as functions of the debt S . Due to (6.9) and the 1st part of (6.3), we find:

$$S = \sum_{h=1}^n C_h = C_1 \sum_{h=1}^n (1+i)^{h-1} = C_1 s_{\bar{n}|i}$$

i.e.: $C_1 = S \sigma_{\bar{n}|i} , C_h = S \sigma_{\bar{n}|i} (1+i)^{h-1} = S \alpha_{\bar{n}|i} v^{n-h+1} ; D_h = \sum_{k=1}^{n-h} C_{h+k} = S a_{\overline{n-h}|i} / \alpha_{\bar{n}|i} ;$
 $I_h = i D_{h-1} = S (1-v^{n-h+1}) / \alpha_{\bar{n}|i} .$

Exercise 6.2

Make the schedule of a French amortization with annual installments for a debt of €255,000 to pay back in 5 years at the rate $i = 0.065$ (same data as Example 6.1).

A. The constant annual installment of amortization for (6.8) is $R = €61,361.81$ and the schedule can be obtained using $D_0 = 255,000$ and using the recursive system: $I_h = iD_{h-1}$, $C_h = R - I_h$, $D_h = D_{h-1} - C_h$ ($h=1, \dots, 5$). The following amortization schedule with annual due date is obtained, using an *Excel* spreadsheet:

Debt = 255000.00		Rate $i = 0.065$		
Length = 5		Installment = 61361.81		
Year h	Interest I_h	Principal C_h	Installment $I_h + C_h$	Balance D_h
1	16575.00	44786.81	61361.81	210213.19
2	13663.86	47697.95	61361.81	162515.24
3	10563.49	50798.32	61361.81	111716.93
4	7261.60	54100.21	61361.81	57616.72
5	3745.09	57616.72	61361.81	0.00

Table 6.5. Example of French amortization

The *Excel* instructions are as follows: the first 3 rows are used for data and column titles; B1: 255000; E1: 0.065; B2: 5; E2: = B1*E1/(1-(1+E1)^-B2)\$B\$2.

From the 4th row:
 column A (year): A4: 1; A5: = A4+1; copy A5, then paste on A6 to A8.
 column B (interest paid): B4: = E1*B1; B5: = \$E\$1*E4; copy B5, then paste on B6 to B8.
 column C (principal repaid): C4: = \$E\$2-B4; copy C4, then paste on C5 to C8.
 column D (installment): D4: = B4+C4; copy D4, then paste on D5 to D8.
 column E (balance): E4: = B1-C4; E5: = E4-C5; copy E5, then paste on E6 to E8.

Calculation of usufructs and bare ownerships in French amortization

Sometimes it is necessary to distinguish in the attribution to the entitled parties the values due to interest and due to principal transaction, i.e. usufructs and bare ownerships (see Chapter 4). As known, we can make two different hypotheses on the formation of interest that brings about the evaluation of usufruct:

- a) *continuous formation* of interest, with intensity δ ;
- b) *periodic formation* of interest at the end of the period at the per period rate $i = e^\delta - 1$.

Case a) gave rise to the general formule for $\tilde{U}(t)$ and $\tilde{P}(t) = W(t) - \tilde{U}(t)$ developed in section 4.3 and valid in the exponential financial regime.

In case b), given that interest is formed with impulsive flow only at the time of payments, the usufruct $U(t)$ is simply the sum of the discounted interest payments, and the bare ownership $P(t)$ is the sum of the discounted principal repayments.

The distinction between cases a) and b) can be applied to a generic financial operation with discrete distribution of payments and the differences in results have little relevance. To illustrate, let us develop the comparison between $\tilde{U}(t)$ and $U(t)$, evaluated according to the same δ , in the particular case of remaining payments of an annuity with constant periodic installments R , in the case of fair operation: it is enough to consider the French amortization of an amount S , with the constraint (6.8) between S and R . Then:

a) with *continuous* formation of interest, at a generic time $t=h+s \in \mathfrak{R}$, with $0 < s < 1$, the following is obtained:

$$\begin{aligned} \tilde{U}(t) &= \sum_{k=1}^{n-h} \delta R(k-s)e^{-\delta(k-s)} = \delta R e^{\delta s} [(Ia)_{n-\bar{h}|i} - s a_{n-\bar{h}|i}] = \\ &= \frac{\delta R}{id} \left[(1-ds)e^{\delta s} - \{1 + d(n-t)\} e^{-\delta(n-t)} \right] \end{aligned} \tag{6.10}$$

From (6.10), with $s \rightarrow 0$, we find $\tilde{U}(t)$ at integer time h , obtaining

$$\tilde{U}_h = \frac{\delta R}{id} \left[1 - \{1 + d(n-h)\} e^{-\delta(n-h)} \right] \tag{6.10'}$$

The bare ownership $\tilde{P}(t)$ is easily obtained as the difference between $W(t) = W_h e^{\delta s}$ and (6.10);

b) with *periodic* formation of interest, it is meaningful to calculate usufruct and bare ownership only at the integer time h . Using the loan rate, we obtain

$$P_h = \sum_{k=h+1}^n C_k v^{k-h} = (n-h)C_h = (n-h)Rv^{n+1-h} \tag{6.11}$$

As $W(k) = D_k$, the usufruct is obtained as difference:

$$U_h = D_h - P_h = \frac{R}{i} \left[1 - \{1 + d(n-h)\} v^{n-h} \right] \tag{6.12}$$

and from the comparison with (6.10') it results in:

$$\tilde{U}_h = U_h \delta/d \tag{6.13}$$

Then, for the French amortization case, the spread of the usufructs is small, giving a value $\tilde{U}(h)$ slightly bigger than U_h and proportional to the coefficient δ/d .

6.2.3. Particular case: amortization with constant principal repayments

In such a form of amortization with delayed installments⁹, also called *uniform* or *Italian*, given the debt S , the number of periods n and the per period rate i , as a main feature the *principal repayments* $C_h, (h=1, \dots, n)$, are constant in time, and then the outstanding balances D_h linearly decrease. The following recursive relations hold, with the initial condition $D_0 = S$:

$$(h=1, \dots, n) \left\{ \begin{array}{l} C_h = D_{h-1} - D_h = \frac{S}{n} \\ I_h = iD_{h-1} \\ R_h = C_h + I_h \end{array} \right. \tag{6.14}$$

from which we obtain the following closed forms according to S :

$$(h=1, \dots, n) \left\{ \begin{array}{l} C_h = \frac{S}{n} \quad ; \quad D_h = \frac{n-h}{n} S \\ I_h = \frac{n-h+1}{n} S i \quad ; \quad R_h = \frac{1+(n-h+1)i}{n} S \end{array} \right. \tag{6.14'}$$

Equation (6.14) enables us to perform the Italian amortization schedule. Furthermore, with the *periodic* formation of interests, the bare ownership P_h and the usufruct U_h at the loan rate i are:

$$P_h = \frac{S}{n} a_{n-\bar{h}|i} \quad ; \quad U_h = \frac{S}{n} (n-h - a_{n-\bar{h}|i}) \quad ; \quad (h=1, \dots, n) \tag{6.15}$$

⁹ The amortization with constant principal repaid and advance installments, as well as advance interest paid, is seldom used.

Exercise 6.3

Prepare the schedule for the Italian amortization with annual installments for a debt of €255,000 to be paid back in 5 years at the rate $i = 0.065$ (the same data as in Exercise 6.2).

A. By applying (6.14) and using *Excel*, the following amortization schedule with annual due dates is obtained:

	Debt = 255000.00		Rate = 0.065	
	Length = 5			
Year	Principal	Interest	Installment	Balance
1	51000.00	16575.00	67575.00	204000.00
2	51000.00	13260.00	64260.00	153000.00
3	51000.00	9945.00	60945.00	102000.00
4	51000.00	6630.00	57630.00	51000.00
5	51000.00	3315.00	54315.00	0.00

Table 6.6. *Example of Italian amortization*

The *Excel* instructions are as follows: B1: 255000; E1: 0,065; B2: 5; using the first 3 rows for data and column titles, from the 4th row we have:

- column A (year): A4: 1; A5: = A4+1; copy A5, then paste on A6 to A8.
- column B (principal repaid): B4: = B\$1/B\$2; copy B4, then paste on B5 to B8.
- column C (interest paid): C4: = E1*B1; C5: = \$E\$1*E4; copy C5, then paste on C6 to C8.
- column D (installment): D4: = B4+C4; copy D4, then paste on D5 to D8.
- column E (outstanding balance): E4: = B1-B4; E5: = E4-B5; copy E5, then paste on E6 to E8.

6.2.4. Particular case: amortization with advance interests¹⁰

For the general case of advance interest, let J_h be the advance interest paid for the period $(h, h+1)$, C_h be the delayed principal repaid for the period $(h-1, h)$ and R_h^* be the total amount paid in h , $(h=0, 1, \dots, n)$. Comparing with (6.4') and (6.4'') we have:

$$J_h = v I_{h+1} = d D_h \quad (h=0, 1, \dots, n-1) ; J_n = 0 \tag{6.16}$$

$$R_h^* = \begin{cases} J_0 = S d , & \text{if } h=0 \\ J_h + C_h = D_{h-1} - v D_h = \ddot{R}_h , & \text{if } h = 1, \dots, n \end{cases} \tag{6.16'}$$

¹⁰ This is a classic case, even if seldom used, of amortization.

The operation is fair at the rate i^{11} .

In the particular hypothesis in which the advance installments following the first, made up only by interest paid in 0, are equal to the constant R^* , the amortization is called *German*; in such a case the delayed principal repaid increases in geometric progression with ratio $(1+i)$, as in the French amortization. Therefore, the first is $C_1 = S \sigma_{\overline{n}|i}$ and $R^* = S \ddot{\alpha}_{\overline{n}|i}$.

Proof: If $R_h^* = R^*$, ($h=1, \dots, n$), by writing the relation $K_h = K_{h-1}(1+i) - R^*$ for consecutive values of h and subtracting, this results for $h=1, \dots, n-1$:

$$C_{h+1} = D_h - D_{h+1} = (1+i)(K_h - K_{h+1}) = (1+i)^2 (K_{h-1} - K_h) = (1+i)(D_{h-1} - D_h) = (1+i)C_h.$$

Exercise 6.4

With the same data as in Exercise 6.2, apply the German amortization with constant annual installment $R_h^* = R^*$ ($h \geq 1$) to obtain the amortization schedule.

A. By applying (6.16), (6.16'), and using *Excel*, we can obtain the amortization schedule at the annual due date. We find: $C_1 = 44,786.81$; $R^* = 57,616.72$. The following schedule is the result:

Debt = 255,000		Delayed rate = 0.065		
Length = 5		Advance rate = 0.061033	57,616.72	
h	C_h	D_h	J_h	R_h^*
0	0.00	255000.00	15563.38	15563.38
1	4478.81	210213.19	12829.91	57616.72
2	47697.95	162515.24	9918.77	57616.72
3	50798.32	11171.93	6818.40	57616.72
4	54100.21	57616.72	3516.51	57616.72
5	57616.72	0.00	0.00	57616.72

Table 6.7. Example of German amortization

The *Excel* instructions are as the follows. We use the first three rows for titles, data and basic calculations; B1: 255000; E1: 0.065; B2: 5; D2: = E1/(1+E1) (= advance rate) ; E2: = B1*C2/(1-(1+E1)^-B2) (= installment at $h=1, \dots, 5$); from the 4th row, we have:

¹¹ *Proof:* using $K_h = D_h - J_h = vD_h$, ($h=0, 1, \dots, n$), this results in: $K_h = K_{h-1}(1+i) - R_h^*$, ($h = 1, \dots, n$). From here, given that $K_n = 0$, we obtain

$$\sum_{h=1}^n R_h^* (1+i)^{-h} = \sum_{h=1}^n [K_{h-1}(1+i)^{-(h-1)} - K_h(1+i)^{-h}] = K_0 = S - R_0^*, \quad \text{i.e.}$$

$S = \sum_{h=0}^n R_h^* (1+i)^{-h}$. Then the installments R_h^* amortize fairly at the rate i the debt S . For $h \geq 1$ they coincide with the advance installments R_{h-1}^* paid the previous period.

column A (time): A4: 0; A5: = A4+1; copy A5, then paste on A6 to A8.
 column B (principal repaid): B4: 0; B5: = B1*E1/((1+E1)^B2-1); B6:
 =B5*(1+\$E\$1); copy A5, then paste on A6 to A8.
 column C (outstanding balance): C4: = B1; C5: = C4-B5; copy C5, then paste on
 C6 to C8.
 column D (interest paid): D4: = D\$2*C4; copy D4, then paste on D5 to D8.
 column E (installment): E4: = B4+D4; copy E4, then paste on E5 to E8.

6.2.5. Particular case: “American” amortization

To introduce *American* amortization let us consider a variation of the form a_2) of amortization as seen in section 6.1. In such a form the debtor could have difficulties in preparing a large amount as a lump-sum final payment; as guarantee for the creditor, it could be appropriate to agree that the debtor makes constant periodic payments into a bank account so that at the end of the loan the debtor has the amount to be paid back.

In the resulting scheme, the accumulation fund to pay back the debt is called a *sinking fund* (see section 6.4) and such a structure gives rise to American amortization that provides for three economic agents: 1) the *creditor or lender*; 2) the *debtor or borrower*; 3) the *bank* (or other financial institution) managing the *funding*.

For a debt of amount S to be paid back in n periods, we have to fix a *reward rate* i , i.e. the rate of the loan, which rules the periodic interest paid by the borrower, different from (and usually higher than) the *accumulation rate* i^* , which rules the interest earned by the borrower on the funding¹². On the basis of such elements:

- the debtor at the end of each period pays to the creditor the accrued interest Si and pays into the sinking fund *the periodic funding installment* $S \sigma_{\bar{n}|i^*}$ in order to reach at maturity the amount S that the bank, instead of the debtor, will pay to the creditor; then the debtor against the initial cash inflow $(0,+S)$ pays $\bigcup_{h=1}^n (h,-R(i,i^*))$, where

$$R(i,i^*) = S (i + \sigma_{\bar{n}|i^*}) \tag{6.17}$$

- the creditor, due to (6.1), has the cash-flow

$$(0,-S) \cup (1,Si) \cup \dots \cup (n-1,Si) \cup (n,S(1+i))$$

- the bank manages in the interval of n periods the sinking fund at the rate i^* with periodic inflow $+S \sigma_{\bar{n}|i^*}$ and the final outflow $-S$.

¹² In fact, it is well known that, for obvious market reasons, for a private operator against a bank the allowed rates are lower than charged rates.

The role of the bank as a broker allows the different structure of the cash-flow following the inception, for both lender and borrower. For the lender, case a₂), of one lump-sum repayment, holds, while for the borrower we have periodic constant payments, as in progressive amortization. It is then useful to find the *cost rate* z for the debtor of the American amortization in the usual hypothesis: $i > i^*$. z is the solution of the equation

$$\alpha_{\bar{n}|z} = i + \sigma_{\bar{n}|i^*} \tag{6.18}$$

obtained making the constant installment of the French amortization at rate z equal to that of the American amortization and then dividing by S . The problem leads back to the search of the internal rate implied by the cost of a constant annuity (see section 5.2).

Observe that the right side of (6.18) can be written: $(i-i^*) + \alpha_{\bar{n}|i^*}$ (alternative formula for the American installment of the unitary debt) and then (6.18) becomes:

$$\alpha_{\bar{n}|z} = (i-i^*) + \alpha_{\bar{n}|i^*} \tag{6.18'}$$

If $i > i^*$, $\alpha_{\bar{n}|z} > \alpha_{\bar{n}|i^*}$ results, and then, $\alpha_{\bar{n}|z}$ being an increasing function of z , we obtain: $z > i^*$; if instead $i < i^*$, we obtain: $z < i^*$. Furthermore, $\sigma_{\bar{n}|z}$ being a decreasing function of z , if $i > i^*$, recalling (5.9) we obtain $\alpha_{\bar{n}|z} = i + \sigma_{\bar{n}|i^*} > i + \sigma_{\bar{n}|i} = \alpha_{\bar{n}|i}$ and then, due to the behavior of $\alpha_{\bar{n}|z}$, we have: $z > i$; if instead $i < i^*$, using analogous developments we obtain: $z < i$.

In conclusion, z is *external (and not internal mean) to the interval between i and i^** , being the only alternative between $i^* < i < z$ (usual case) and $i^* > i > z$ (exceptional case). In the usual case the American amortization is more expensive for the borrower than the French at rate i , because the borrower must accumulate the amount for the repayment at the earned interest rate $i^* < i$.

American amortization with equality of rates

It is appropriate to consider the case $i=i^*$ in the American amortization¹³. With regard to the cost rate z for the debtor, $i^*=i=z$ results. In addition, for (5.9), the

13 We have has to mention that the “geographical” terms for the different amortization that are usually used to differentiate are not always unique; it can be preferred to use technical adjectives (progressive and uniform instead of French and Italian). de Finetti (1969) (cited for deeper investigation, together with Volpe di Prignano (1985)) uses “English” amortization for the scheme, which we here call “American”, with two different rates and “American” when the rates are the same.

periodic payment $S(i + \sigma_{\bar{n}|i})$ of the debtor coincides with $S\alpha_{\bar{n}|i}$, payment that he would have on the basis of the French amortization at the rate i . However, this is the same as the total payment that he would have if he pays the interests Si to the lender and accumulates the repayment capital S at the same rate i agreed for the payment. This situation can be realized if the bank¹⁴ that manages the sinking fund at the rate $i^* < i$ is not present. It could be the same lender, if a financial institution gives loans at the rate i , to operate at the reciprocal rate i with the borrower for a deposit operation as guarantee for the loan. In such a case the American amortization with sinking fund is managed by the lender, because (5.9) is substantially reduced to the progressive amortization at the rate i .

However, this is not the case for the formal aspects. If the sinking fund at rate i is not managed by the lender, or it is but with separate accounting until maturity, then the periodic payment $S\sigma_{\bar{n}|i}$ for the sinking fund is not “principal repaid”, because it does not reduce the debt that always remains at the level S , but “accumulation amount”; in the same way Si is not French “interest paid” but is constant “reward amount”. However, if the accumulation payments are accounted periodically at the rate i to the lender to reduce the debt, so that at the due date h it becomes $Sa_{\overline{n-h}|i} / a_{\bar{n}|i}$, a sinking fund does not arise and we lead back to installment $S\alpha_{\bar{n}|i}$ decomposition in principal repaid and interest paid of a progressive amortization,¹⁵ varying with h and given by

$$C_h = S\sigma_{\bar{n}|i}(1+i)^{h-1}, \quad I_h = S\alpha_{\bar{n}|i}(1-v^{n-h+1}).$$

In such cases, the American amortization does not hold.

Exercise 6.5

We have to amortize the amount $S = 35000$ in $n = 10$ years with the sinking fund method with two different rates; $i = 7.2\%$ for debt repayment, $i^*_a = 3.6\%$ in case a) and $i^*_b = 4.7\%$ in case b) for accumulation. Calculate the delayed annual payment $R(i, i^*)$ for the borrower and the constant rate z solution of (6.18) in the two cases.

A. On the basis of (6.17), the annual payment is given:

- in case a) by $R(0.072; 0.036) = 2520.00 + 2969.69 = 5489.69$
- in case b) by $R(0.072; 0.047) = 2520.00 + 2821.86 = 5341.86$

then being the amount in the sinking fund equal to 2969.69 and 2821.86 in the two cases. For the calculation of the cost rate z , we can use the numerical methods

¹⁴ See de Finetti (1969).

¹⁵ Recall that in such amortization the principal repaid is $S\sigma_{\bar{n}|i}$ only for the 1st period, after that it increases in GP while the interest paid decreases proportionally to the outstanding balance.

described in section 5.2 to obtain exact results through iterative methods until convergence, or with lower approximation if we use the linear interpolation;

– in case a), using equation: $\alpha_{1\bar{0}|z} = 5489.69/35000 = 0.1568483$ and a financial calculator with: [n] = 10; [pv] = -35000; [pmt] = 5489.69; [fv]=0; comp[i]; we obtain $\hat{z} = 9.14969\%$. With the linear interpolation on (9%; 9.25%) the results are: $\alpha_{1\bar{0}|9\%} = 0.1558201$; $\alpha_{1\bar{0}|9.25\%} = 0.1575389$; then

$$\bar{z} = 0,09 + \frac{10282}{17188} 0.0025 = 0.0914955$$

We can also apply the classic iteration method, using *Excel* starting from $z_0 = \bar{z} = 0.0914955$. Since the equation $\alpha_{1\bar{0}|z} = 0.1568483$ has the form $g(z) = g_0$, with $g(z) = \alpha_{1\bar{0}|z}$, $g_0 = 0.1568483$, we can go to the canonical form $f(z) = z$ using $f(z) = z \alpha_{1\bar{0}|z} / g_0$. However, the iteration process on f diverges, resulting in: $|f'(\bar{z})| = 1.4400328 > 1$. We then have to apply the transformation, analogous to that seen in case B of Example 4.3: $h(z) = [f(z) - mz] / [1 - m]$, where $h(z) = z$ is equivalent to $f(z) = z$, using $m = |f'(\bar{z})|$. Starting from $z_0 = \bar{z}$, the following expansion, rapidly converging to \hat{z} , is obtained.

	$[g_0, z_0, f(z_0)] =$	0.15684830	0.09149550	1.44003280
k	z_k	$G(z_k)$	$f(z_k)$	$h(z_k)$
0	0.09149550	0.15684730	0.09149492	0.09149683
1	0.09149683	0.15684821	0.09149678	0.09149694
2	0.09149694	0.15684829	0.09149694	0.09149695
3	0.09149695	0.15684830	0.09149695	0.09149695
4	0.09149695	0.15684830	0.09149695	0.09149695

Table 6.8. Calculation of cost rate by iteration

The *Excel* instructions are as follows. The first two rows are used for data and titles; A1: 10 (= length); C1: 0.1568483 (= $\alpha_{1\bar{0}|z}$); D1: 0.0914955 (= \bar{z}); E1: 1.44003280 (= $|f'(\bar{z})|$); from the 3rd row:

column A (step k); A3: 0; A4: = A3+1; copy A4, then paste on A5 to A7;
 column B (approximate rate z_h); B3: = D1; B4: = E3; copy B4, then paste on B5 to B7;

column C ($g(z_k)$); C3: = B3/(1-(1+B3^A-A\$1)); copy C3, then paste on C4 to C7;

column D ($f(z_k)$); D3: = B3*C3/C\$1; copy D3, then paste on D4 to D7;

column E ($h(z_k)$); E3: = (D3-E\$1*B3)/(1-E\$1); copy E3, then paste on E4 to E7;

– in case *b*), the solving equation is: $\alpha_{10|z} = 5341.86/35000 = 0.1526246$; using the financial calculator with: [n]=10; [pv]=-35000; [pmt]=5341.86; [fv]=0; comp [i], we obtain $z = 0.0853193$. With the linear interpolation on (8.5%; 8.75%) the results are: $\alpha_{10|8.5\%} = 0.1524077$; $\alpha_{10|8.75\%} = 0.1541097$; then

$$z = 0.085 + \frac{2169}{17020} 0.0025 = 0.0853186$$

By using the *Excel* spreadsheet, it is sufficient to change the rate i^* .

6.2.6. Amortization in the continuous scheme

A gradual amortization scheme that is widely used for theoretical aims is produced using a continuous annuity.

Let us consider briefly such a case, assuming a continuous flow $\alpha(t)$ of payments covering interest, used to amortize in a temporal interval $I(t_1)$ from time 0 to t_1 the amount S . It is not restrictive to assume for simplicity that $S=1$ (otherwise it is enough to multiply the results by S). In addition, let us assume a financial law strongly decomposable with intensity $\delta(t)$, that, as known, is a function only of the varying time t (in particular $\delta(t)=\delta$ if the exponential law is assumed). Using:

$$\varphi(t) = \int_0^t \delta(z) dz \quad , \quad t \in I(t_1) \quad (6.19)$$

$\varphi(t)$ is the natural logarithm of the accumulation factor from 0 to t . With such positions, the flow $\alpha(t)$ can be fixed varying in the interval $I(t_1)$, but must satisfy the constraint of financial closure:

$$\int_0^{t_1} \alpha(t) e^{-\varphi(t)} dt = 1 \quad (6.20)$$

If $\alpha(t) = \alpha$ constant and $\delta(t) = \delta$ constant, due to (5.16) and (6.20),

$$\alpha = \frac{1}{a_{t_1|i}^{(\infty)}} \quad (6.20')$$

holds, with extension of the meaning of the symbol $a_{t_1|i}^{(\infty)}$ if t_1 is not integer.

In addition, let us define:

- $c(t)$ = amortization flow (for the principal repayment) at time t ;
- $j(t)$ = interest flow (allowed for the borrower) at time t ;
- $B(t, t_1)$ = discharged debt at time t ;
- $D(t, t_1) = 1 - B(t, t_1)$ = outstanding balance at time t ;
- $A(t, t_1)$ = initial value of the payments of the borrower made from 0 to t .

Such quantities are linked by the following relations, of trivial interpretation, that determine them completely:

$$\alpha(t) = c(t) + j(t) \quad , \quad t \in I(t_1) \tag{6.21}$$

$$\int_0^{t_1} c(t) dt = 1 \tag{6.22}$$

$$B(t, t_1) = \int_0^t c(z) dz \quad , \quad t \in I(t_1) \tag{6.23}$$

$$D(t, t_1) = \int_t^{t_1} c(z) dz \quad , \quad t \in I(t_1) \tag{6.24}$$

$$j(t) = [1 - B(t, t_1)] \delta(t) \quad , \quad t \in I(t_1) \tag{6.25}$$

$$A(t, t_1) = \int_0^t \alpha(z) e^{-\varphi(z)} dz \quad , \quad t \in I(t_1) \tag{6.26}$$

The value $M(t, t_1) = [1 - A(t, t_1)] e^{\varphi(t)}$ represents the *retrospective reserve* (or *retro-reserve*, at credit for the lender) at time t with the meaning defined in Chapter 4. In addition, $W(t, t_1) = \int_t^{t_1} \alpha(z) e^{-\int_t^z \delta(\tau) d\tau} dz$ expresses the *prospective reserve* (or *pro-reserve*). Maintaining in $t \in I(t_1)$ the decomposable financial $\delta(t)$ initially adopted that assures the validity of (6.20), i.e. the fairness of the amortization operation, the following equalities hold:

$$M(t, t_1) = W(t, t_1) = D(t, t_1) \quad ; \quad t \in I(t_1) \tag{6.27}$$

then from (6.20) follows

$$[1 - A(t, t_1)] e^{\varphi(t)} = \int_t^{t_1} \alpha(z) e^{[\varphi(z) - \varphi(t)]} dz \quad , \quad t \in I(t_1)$$

and $D(t, t_1)$ is also the amount which can be fairly cashed in t instead of the payments with flow $\{\alpha(t)\}$ in the interval (t, t_1) ¹⁶.

¹⁶ In the continuous scheme the observations in footnote 7 on the lack of inequality between the prospective reserve, the outstanding balance and the retrospective reserve calculated in

6.3. Life amortization

6.3.1. Periodic advance payments

So far we have analyzed the debt amortization methods in case of certainty, then through annuities certain, not considering randomness in the repayment of the loan or on the interest payment. This is assuming that, in the event of the borrower dying, his heirs or other people must enter into and fulfill his obligations.

We can take into account the risk of death of the borrower and the difficulty for his heirs to pay back the loan, excluding, due to the contract, the continuation of the repayment in case of death of the borrower and then establishing for the borrower the debt amortization through a temporary life annuity of n years (= life of the loan). In such a way the debt is discharged by means of a *life amortization*¹⁷ and the contract becomes stochastic, as with an insurance contract: the financial equivalence is obtained only as average, i.e. it has an actuarial nature.

We have to take into account the uncertainty on the borrower's survival, the probability of which is considered to depend only on his age x at the inception date of the amortization. This is obtained by replacing the financial discount factors $(1+i)^{-h}$ by the demographic-financial ones ${}_hE_x$ ¹⁸.

$t \in I(t_1)$ remain valid, when at such a time an interest intensity $\delta^*(\tau)$ in $\tau \in I(t_1)$ different from the intensity $\delta(\tau)$ initially fixed is used, maintaining unchanged in $I(t_1)$ the flow $\alpha(\tau)$. In fact, in such case, (6.20) does not hold.

17 For a better understanding of life amortization see Boggio, Giaccardi (1969) and also Volpe di Prignano (1985).

18 We recall here that – with the symbols used in actuarial mathematics and assuming the discrete time scheme, starting from a demographic table of *survival* $\{l_x\}$ as a function of age (integer) x of a generic member of the community – the *survival probability* for h years of a person aged x is introduced and it is indicated with ${}_h p_x$ resulting in ${}_h p_x = l_{x+h} / l_x$; in particular for surviving one year we put $p_x = {}_1 p_x$. In addition, considering the *financial discount factor* $(1+i)^{-h}$ for h years at the annual rate i , we introduce the value ${}_h E_x = {}_h p_x (1+i)^{-h}$ which is called *demographic-financial (or actuarial) discount factor* and is the mean present value of the unitary amount payable within h years only in case of the survival of a person aged x , i.e. the amount that it is fair to pay with certainty today, at age x , to receive the unitary amount within h years only in case of survival. It is obvious that ${}_0 E_x = 1$ and we use ${}_1 E_x = E_x$. Also, we introduce, for a person aged x , the mean present value of a unitary life perpetuity-due or -immediate, denoted respectively by \ddot{a}_x or a_x , and also of a unitary life annuity-due or -immediate for n years, denoted respectively by ${}_{/n} \ddot{a}_x$ or ${}_{/n} a_x$. Such perpetuities or annuities give the unitary annual amount until death or at most for n years. This is, obviously:

Let us describe the operation with integer length n at fixed rate for the initial debt $D_0 = S$, incepting at time 0, with the borrower aged x (integer).

To discharge the loan the borrower pays a periodic life annuity-immediate, in particular annual, n -temporary with varying installments ${}_z\ddot{\alpha}_{x,n,S}$, payable at times $z = 0, 1, n-1$, referring to the periods $(z, z+1)$. For the congruity of the amortization, the constraint of actuarial equivalence

$$\sum_{z=0}^{n-1} {}_z\ddot{\alpha}_{x,n,S} {}_zE_x = S \tag{6.28}$$

has to be satisfied. Equation (6.28) generalizes (5.23) of Chapter 5. Therefore the sequence $\{ {}_z\ddot{\alpha}_{x,n,S} \}$ can be chosen with $n-1$ degrees of freedom¹⁹.

We can immediately verify that by the installment ${}_z\ddot{\alpha}_{x,n,S}$ (or briefly: $\ddot{\alpha}_z$, omitting x,n,S) the borrower pays:

- 1) the advance *principal repaid* ${}_z\ddot{c}_{x,n,S}$ (or briefly: \ddot{c}_z);
- 2) the advance *financial interest paid* dD_{z+1} on the outstanding balance D_{z+1} in $z+1$;
- 3) and also – and here is the difference of the life amortization compared to the certain amortization – the *insurance natural premium* for the year $(z,z+1)$. Recalling that $v=(1+i)^{-1}=1-d$ and using: $q_y = 1-p_y = 1-l(y+1)/l(y)$ (= death probability between ages $y,y+1$), such a premium is given by $vq_{x+z}D_{z+1}$, proportional to the outstanding balance D_{z+1} that the borrower will not discharge in case of his death at the year $(z,z+1)$, leaving such duty to the lender, which in this aspect acts as insurer.²⁰

The three installment's components make it possible to understand how the life amortization can be interpreted as a normal gradual amortization together with an insurance policy in case of the death of the borrower, which lasts for the length of the loan, and with varying capital given by the current outstanding balance, the premium of which is an addition of the financial installment.

$${}_nE_x = \prod_{k=0}^{n-1} E_{x+k} \quad (n > 0); \quad {}_n\ddot{a}_x = \sum_{h=0}^{n-1} {}_hE_x; \quad {}_n a_x = \sum_{h=1}^n {}_hE_x; \quad \ddot{a}_x = \sum_{h \geq 0} {}_hE_x; \quad a_x = \sum_{h \geq 1} {}_hE_x$$

and for $k < n$ results in: ${}_n\ddot{a}_x = {}_k\ddot{a}_x + {}_kE_x {}_{n-k}\ddot{a}_{x+k}$.

19 The inequality constraints that can be introduced for the non-negativity of the principal repayments do not reduce the number of degrees of freedom, because such a decrease holds only by the equality constraints.

20 If the lender does not manage the insurance himself, he can transfer the premium to an insurance company that accept the same technical bases to cover the risk.

We can define as *actuarial interest paid* for the year $(z, z+1)$, indicating it with ${}_z\ddot{j}_{x,n,S}$ (or briefly: \ddot{j}_z), the amount $[dD_{z+1} + vq_{x+z}D_{z+1}] = (1 - E_{x+z})D_{z+1}$, the sum of the amounts defined in points 2) and 3) above. Therefore, as in the certain case, the installment is divided into principal repaid and interest paid, but the interest paid is actuarial.

A more precise argument leads to the conclusion that the two components in the expression for $\ddot{j}_z^{i',i''} = \ddot{j}_z$, i.e. dD_{z+1} and $vq_{x+z}D_{z+1}$, are antithetic with respect to the rate: in the first, the rate is at debt for the borrower; in the second, it is at credit. If due to market law we keep them separate, indicating them with i' and i'' , ($i' > i''$), then the actuarial interest amount is

$$\ddot{j}_z^{i',i''} = \left\{ \left[1 - (1+i')^{-1} \right] + (1+i'')^{-1} q_{x+z} \right\} D_{z+1}$$

We obtain $\ddot{j}_z^{i',i''} = \ddot{j}_z$ if $i' = i''$. Therefore, analogously to what happens for the American amortization, indicating with x the cost rate for the borrower, the result is: $x > i' > i''$. This scheme, which leads to further complications, is not discussed further.

To better clarify, let us consider the dynamic aspect of the life operation, assuming as already fixed the principal repayments \ddot{c}_z , which are under the *elementary closure* constraint

$$\sum_{z=0}^{n-1} \ddot{c}_z = S \tag{6.29}$$

- in the first year the actuarial interest paid is $\ddot{j}_0 = (1 - E_x)D_1$, where $D_1 = D_0 - \ddot{c}_0$, and $\ddot{\alpha}_0 = \ddot{j}_0 + \ddot{c}_0 = \dots = D_0 - E_x D_1$;
 - in the second year the development, starting from the debt D_1 , is repeated;
- we obtain: $\ddot{j}_1 = (1 - E_{x+1})D_2$, where

$$D_2 = D_1 - \ddot{c}_1, \text{ and } \ddot{\alpha}_1 = \ddot{j}_1 + \ddot{c}_1 = \dots = D_1 - E_{x+1}D_2;$$

- and in general, due to: $D_{z+1} = D_z - \ddot{c}_z$ and $\ddot{\alpha}_z = \ddot{c}_z + \ddot{j}_z$, we obtain for the year $(z, z+1)$, where $z+1 \leq n$,

$$\ddot{j}_z = (1 - E_{x+z})D_{z+1} \quad \ddot{\alpha}_z = D_z - E_{x+z}D_{z+1}.^{21} \tag{6.30}$$

If instead the installments $\ddot{\alpha}_z$ under constraint (6.28) are fixed in advance, then for the actuarial equivalence $\forall z = 0, 1, \dots, n-1$, (6.28) is generalized in

21 These formulae generalize (6.5) and (6.6'''), which hold in the absence of death. In fact, if $l_z = \text{constant}$, $p_z = 1$, $\forall z$, results.

$$D_z = \sum_{k=z}^{n-1} \ddot{\alpha}_k \cdot {}_{k-z}E_{x+z} \tag{6.31}$$

By obtaining D_z and D_{z+1} from (6.31) it is immediately verified that (6.30) is satisfied for $\ddot{\alpha}_z$, and also for \ddot{j}_z , given by definition $\ddot{c}_z = D_z - D_{z+1}$. Therefore the components of $\ddot{\alpha}_z$ are obtained from

$$\ddot{c}_z = D_z - D_{z+1} \quad ; \quad \ddot{j}_z = (1 - E_{x+z})D_{z+1} \tag{6.32}$$

The *life amortization with advance installments* schedule has in the row relative to the period $(z, z+1)$, $(z=0, \dots, n-1)$, the following elements

- payment time: z
- principal repaid: \ddot{c}_z
- discharged debt (after payment in z): $B_{z+1} = \sum_{k=0}^z \ddot{c}_k$
- outstanding balance (after payment in z): $D_{z+1} = \sum_{k=z+1}^{n-1} \ddot{c}_k$
- actuarial interest paid $\ddot{j}_z = (1 - E_{x+z})D_{z+1}$
- installment $\ddot{\alpha}_z = \ddot{j}_z + \ddot{c}_z = D_z - E_{x+z}D_{z+1}$

Making successive substitutions on D_{z+1} in the expression for $\ddot{\alpha}_z$ for $z=0, \dots, n-1$ and taking into account ${}_zE_x = \prod_{k=0}^{z-1} E_{x+k}$ we obtain:

$$D_0 = S = \sum_0^{k-1} \ddot{\alpha}_z \cdot {}_zE_x + {}_kE_x D_k,$$

from which, due to $k=n$, (6.28) follows.

The expression

$$M_k = \frac{S - \sum_{z=0}^{k-1} \ddot{\alpha}_z \cdot {}_zE_x}{{}_kE_x} \tag{6.33}$$

can be interpreted as *retro-reserve* at time k of the life amortization operation from 0 to n , extending what is seen in Chapter 4 to the mean values obtaining, in the

22 Note that: $\ddot{\alpha}_z = D_z - E_{x+z}(D_z - \ddot{c}_z) = E_{x+z}\ddot{c}_z + (1 - E_{x+z})D_z$. Therefore, $\ddot{\alpha}_z$ is the weighted mean of \ddot{c}_z and D_z . In addition, in the particular case where $z=n-1$, as $D_n = 0$, $\ddot{\alpha}_{n-1} = D_{n-1} = \ddot{c}_{n-1}$ results, and then $\ddot{j}_{n-1} = 0$, in accordance with the fact that for the period $(n-1, n)$ both the financial interest and the insurance premium are zero.

actuarial sense an *insurance retro-reserve*. In fact, such a formula gives the difference between the expected supplies of the lender and the borrower between the dates 0 and k , evaluated actuarially at time k . Instead, the *insurance pro-reserve*, considered as the difference between the expected obligations of the borrower and the lender (the latter are absent, because the lender's supply occurs only at the inception of the loan) between the dates k and n , evaluated actuarially at time k , is given by

$$W_k = \sum_{z=k}^{n-1} \ddot{\alpha}_{z-k} E_{x+k} \tag{6.33'}$$

Maintaining in $k \in (0, n)$ the actuarial base $\{i, l_x\}$ initially adopted that ensures the validity of (6.28), i.e. the actuarial fairness of the operation of life amortization, $M_k = W_k = D_k, \forall k$, hold, taking into account (6.28) and the fact that D_k is the amount in k that finds a fair counterpart in the payments of $\ddot{\alpha}_z (z = k, \dots, n-1)$.²³

If the life amortization is carried out with constant installment $\ddot{\alpha}_z$, for (6.28) it must be $\ddot{\alpha}_z = S / {}_z\ddot{a}_x$ and the outstanding balances are given by

$$D_z = \frac{S}{{}_z\ddot{a}_x} \frac{{}_z\ddot{a}_x - {}_z\ddot{a}_x}{z E_x} = S \frac{{}_z\ddot{a}_x - {}_z\ddot{a}_x}{{}_z\ddot{a}_x z E_x} \tag{6.31'}$$

Equation (6.32) is still applied for the calculation of the principal repayments and interest payments.

Exercise 6.6

We have to make a life amortization of €95,000 with advance annual installments for 10 years at rate $i = 4.50\%$ on a borrower aged 42 years. Calculate the amortization schedule on the basis of principal repayments assigned.

A. The survival probability is found on suitable tables for an age $x = 42$. We can apply the formulations on footnote 18 and in (6.30), using a calculator or an *Excel* spreadsheet. The values E_{42+z} are calculated and the principal repayments for $z = 0, 1, \dots, 9$, the sum of which is 95000, are assigned. Then we find the discharged debts and the outstanding balances for 10 years, and also the actuarial interest payments and the advance installments. With both procedures the following schedule is found, with obvious meaning:

23 Due to its decomposability the simplification effects of the actuarial law that leads to the discount ${}_k E_x$ and accumulation $1/{}_k E_x$ factors are obvious. Extended for the retro-reserve and pro-reserve in actuarial sense, the considerations of footnote 16 if at time k are adopted the technical bases $\{i, l(x)\}$ different from the ones initially used to prepare the life amortization .

z	Debt = 95000			Rate = 0.045			Length = 10	
	l_{42+z}	E_{42+z}	C_z	B_z	D_z	j_z	Installment C_z+j_z	Installment (6.30)
0	96400	0.955230	8700	0	95000	3863.62	12563.62	12563.62
1	96228	0.955128	8650	8700	86300	3484.32	12134.32	12134.32
2	96046	0.954975	9800	17350	77650	3054.94	12854.94	12854.94
3	95849	0.954761	9600	27150	67850	2635.15	12235.15	12235.15
4	95631	0.954486	9300	36750	58250	2227.90	11527.90	11527.90
5	95386	0.954189	10100	46050	48950	1779.76	11879.76	11879.76
6	95112	0.953879	9700	56150	38850	1344.42	11044.42	11044.42
7	94808	0.953647	9750	65850	29150	899.24	10649.24	10649.24
8	94482	0.953302	9200	75600	19400	476.32	9676.32	9676.32
9	94123	0.953105	10200	84800	10200	0.00	10200.00	10200.00
10	93746			95000	0			
total			95000					

Table 6.9. Example of general life amortization

The *Excel* instructions are as follows. The 1st row contains data: C1: 95000; F1: 0.045; I1: 10. The 2nd row is for column titles. The values for the year z are in the row $z+3$ and are as follows:

- column A (time). A3: 0 ; A4: = A3+1; copy A4, then paste on A5 to A13;
- column B (l_{42+z}). from B3 to B13: demographic data l_{42}, \dots, l_{52}
- column C (E_{42+z}). C3: = B4*(1/(1+F\$1))/B3; copy C3, then paste on C4 to C12;
- column D (\ddot{c}_z). from D3 to D12; principal repayments; D14:= SUM(D3:D12) (= C1 to control);
- column E (B_z). E3: 0; E4: = E3+D3; copy E4, then paste on E5 to E13;
- column F (D_z). F3: = C1; F4: = F3-D3; copy F4, then paste on F5 to F13;
- column G (\ddot{j}_z). G3: = (1-C3)*F4; copy G3, then paste on G4 to G12;
- column H ($\ddot{\alpha}_z = \ddot{c}_z + \ddot{j}_z$). H3: = D3+G3; copy H3, then paste on H4 to H12;
- column I ($\ddot{\alpha}_z$ from (6.30)). I3: = F3 -F4*C3; copy I3, then paste on I4 to I12.

Exercise 6.7

Calculate an advance life amortization with the same data as in Exercise 6.6 for the debt amount, length, rate and the borrower data, but with constant installments. Calculate the installment amount and make the amortization schedule.

A. The survival probability is found on suitable tables for an age of $x=42$. Applying the formulae in footnote 17 and in (6.31) and (6.32), and using an *Excel*

spreadsheet, where the debt is in C1 and the rate is in E1, the following schedule, divided into two parts, is set up.

	Debt = 95000		Rate = 0.045		Length = 10
z	A_{2+z}	E_{42+z}	zE_{42}	zA_{42}	$10-zA_{42}z$
0	96400	0.955230	1.000000	0.000000	8.191301
1	96228	0.955128	0.955230	1.000000	7.528342
2	96046	0.954975	0.912367	1.955230	6.835045
3	95849	0.954761	0.871288	2.867598	6.110155
4	95631	0.954486	0.831872	3.738886	5.352285
5	95386	0.954189	0.794010	4.570757	4.559820
6	95112	0.953879	0.757636	5.364768	3.730728
7	94808	0.953647	0.722693	6.122404	2.862761
8	94482	0.953302	0.689194	6.845097	1.953302
9	94123	0.953105	0.657010	7.534291	1.000000
10	93746		0.626200	8.191301	0.000000
	Installment =	11597.67			

z	D_z	c_z	j_z	Installment control = $c_z + j_z$
0	95000.00	7688.78	3908.89	11597.67
1	87311.22	8040.63	3557.04	11597.67
2	79270.59	8407.04	3190.63	11597.67
3	70863.55	8789.52	2808.15	11597.67
4	62074.03	9190.75	2406.92	11597.67
5	52883.28	9615.53	1982.14	11597.67
6	43267.75	10066.40	1531.27	11597.67
7	33201.35	10547.61	1050.06	11597.67
8	22653.75	11056.08	541.59	11597.67
9	11597.67	11597.67	0.00	11597.67
10	0.00			

Table 6.10. Example of life amortization with constant installments

The Excel spreadsheet is set up in two parts.

In the top part, taking into account that the first two rows are for data and column titles, the values for the year z are in row $z+3$. The instructions are as follows:

- column A (year). A3: 0 ; A4: = A3+1; copy A4, then paste on A5 to A13;
- column B (l_{42+z}). demographic data l_{42}, \dots, l_{52} ;

column C (E_{42+z}). C3: = B4*(1/(1+\$E\$1))/B3; copy C3, then paste on C4 to C13;
 column D (${}_zE_{42}$). D3: 1 ; D4: = C3*D3 ; copy D4, then paste on D5 to D13;
 column E (${}_z\ddot{a}_{42}$). E3: 0 ; E4: = E3+D3 ; copy E4, then paste on E5 to E13;
 column F (${}_{/10-z}\ddot{a}_{42+z}$). F3: = (\$E\$13-E3)/D3 ; copy F3, then paste on F4 to F13.

In C14 the installment is calculated according to: $\ddot{a}_z = S / {}_{/n}\ddot{a}_x$, then C14: = C1/E13.

In the bottom part, row 16 is for column titles and the values for year z are in row $z+17$ with the following instructions:

column A(year). A17: 0 ; A18: = A17+1; copy A18, then paste on A19 to A27;
 column B (D_z). B17: = \$C\$1*\$F3/\$E\$13 ; copy B17, then paste on B18 to B27;
 column C (\ddot{c}_z). C17: = B17-B18 ; copy C17, then paste on C18 to C26;
 C28: = SUM(C17:C26) (= C1 to control);
 column D (\ddot{j}_z). D17: = (1-C3)*B18 ; copy D17, then paste on D18 to D26;
 column F ($\ddot{\alpha}_z = \ddot{c}_z + \ddot{j}_z$). F17: = C17+D17 ; copy F17, then paste on F18 to F26.

6.3.2. Periodic payments with delayed principal amounts

The life amortization, still with advance actuarial interest payments, can also be made with *delayed principal repayments* c_z . We then have an actuarial generalization of the scheme seen in section 6.2.4, in particular of the German scheme if the installment invariance is imposed.

Easy calculations lead to the conclusion that, when we have chosen the principal repayments c_z so that their sum is equal to the initial debt $D_0 = S$, the installments that realize the equivalence have the following values:

$$\begin{aligned} - \hat{\alpha}_0 &= \ddot{j}_0 = (1 - E_x)D_0 \\ - \hat{\alpha}_1 &= c_1 + \ddot{j}_1 = D_0 - D_1 + (d + vq_{x+1})D_1 = D_0 - E_{x+1}D_1 \\ - \hat{\alpha}_z &= c_z + \ddot{j}_z = D_{z-1} - D_z + (d + vq_{x+z})D_z = D_{z-1} - E_{x+z}D_z; (z = 2, \dots, n-1) \end{aligned} \quad 24$$

24 It is soon seen that the values $\hat{\alpha}_z$ paid in z , if introduced in (6.28), satisfy it, therefore realizing the actuarial congruity of this life amortization form. In fact, considering that $\sum_{z=1}^n c_z = S$ and $D_n = 0$, and that ${}_hE_x E_{x+h} = {}_{h+1}E_x, \forall h \geq 0$, the following formulae:

$$\sum_{z=0}^n \hat{\alpha}_{z,z} E_x = (1 - E_x)D_0 + (D_0 - D_1 E_{x+1})E_x + \sum_{z=2}^{n-1} (D_{z-1} - D_z E_{x+z})_z E_x + D_{n-1,z} E_x =$$

$$= D_0 - D_0 E_x + D_0 E_x - D_1 {}_2E_x + D_1 {}_2E_x - \dots - D_{n-1} {}_nE_x + D_{n-1} {}_nE_x = D_0 = S$$
 are obtained.

$$- \hat{\alpha}_n = c_n = D_{n-1}$$

The calculation of the retro-reserves and pro-reserves in z can be undertaken immediately, analogously to what was seen in section 6.3.1.

6.3.3. Continuous payment flow

In sections 6.3.1 and 6.3.2 we considered life amortization in the discrete scheme of periodic payments, in particular annual, for the loan. However, theoretically, for a limit case or as an approximation of a scheme with fractional payments with high frequency, for such a operation we can adopt a continuous payment flow, generalizing to the stochastic case the scheme considered in section 6.2.6.

Using the time origin in the inception date of the loan and assuming for the life amortization a length t^* , thus the time interval of the corresponding annuity is $I(t^*) = [0, t^*]$, we indicate with $\alpha(t; x, t^*)$, or more easily $\alpha(t)$, the payment flow²⁵ from the borrower for the loan and assuming a demographic technical base in the continuum²⁶. It is then obvious that the actuarial equivalence constraint, i.e. the congruity of $\alpha(t)$ in order to realize the life amortization of the debt, that we assume unitary, in the interval $I(t^*)$, is expressed by

$$\int_0^{t^*} \alpha(t) {}_t\bar{E}_x dt = 1 \tag{6.28^*}$$

25 The symbol α used for the payment flow is chosen in analogy with $\ddot{\alpha}$ and $\hat{\alpha}$ for the discrete case, stressing therefore the dimensional difference.

26 With such an aim we consider a survival law $\{l(x)\}$ as a function of the age $x \in \mathfrak{R}$ where

$$l(x) = l(a) e^{-\int_a^x \mu(y) dy} \text{ and } \mu(y) = -\frac{l'(y)}{l(y)}$$

is the mortality intensity in y . Thus, the continuous

actuarial discount factor, which is also dependent on the intensity $\delta(t)$ of the financial exchange law, that is assumed as strongly decomposable (in particular, $\delta(t) = \delta$ constant in the exponential case) it is written:

$${}_h\bar{E}_x = e^{-\delta h} l(x+h) / l(x) = e^{-\int_0^h [\delta + \mu(x+t)] dt}$$

while its reciprocal is the continuous actuarial accumulation factor. Furthermore the IV of a unitary life annuity paid in the interval $I(t^*)$ is expressed by ${}_{|t^*}\bar{a}_x = \int_0^{t^*} e^{-\int_0^h [\delta + \mu(x+t)] dt} dh$.

which can be written

$$\int_0^{t^*} \alpha(t)e^{-\Psi(t)} dt = 1 \tag{6.28''}$$

where $\Psi(t) = \int_0^t [\delta + \mu(x + \tau)] d\tau$ is the natural logarithm of the actuarial accumulation factor, δ being the intensity of the exponential financial law and resulting, obviously, in $\Psi(0) = 0$.

Proceeding as in section 6.2.6, let us define the following quantities:

- $c(t)$ = amortization flow (for principal repayment) at time t ;
- $j(t)$ = actuarial interest flow at time t ;
- $B(t, t^*)$ = mean discharged debt at time t ;
- $D(t, t^*) = 1 - B(t, t^*)$ = mean outstanding balance at time t ;
- $A(t, t^*)$ = mean initial value of the borrower payments made from 0 to t .

The following constraints are valid:

$$\alpha(t) = c(t) + j(t) , \quad t \in I(t^*) \tag{6.21'}$$

$$\int_0^{t^*} c(t) dt = 1 \tag{6.22'}$$

$$B(t, t^*) = \int_0^t c(z) dz , \quad t \in I(t^*) \tag{6.23'}$$

$$D(t, t^*) = \int_t^{t^*} c(z) dz , \quad t \in I(t^*) \tag{6.24'}$$

$$j(t) = D(t, t^*) [\delta + \mu(x+t)] , \quad t \in I(t^*) \tag{6.25'}$$

$$A(t, t^*) = \int_0^t \alpha(z) e^{-\Psi(z)} dz , \quad t \in I(t^*) \tag{6.26'}$$

Evaluating at time t in the actuarial sense (i.e. acting on the mean values), the value

$$M(t, t^*) = [1 - A(t, t^*)] e^{\Psi(t)}$$

expresses the *retro-reserve*, while

$$W(t, t^*) = \int_t^{t^*} \alpha(z) e^{-\int_t^z [\delta + \mu(\tau)] d\tau} dz$$

expresses the *pro-reserve*. Maintaining in $t \in I(t^*)$ the bases $\{\delta, \mu(x)\}$ fixed at the inception date, we obtain

$$M(t, t^*) = W(t, t^*) = D(t, t^*) , \quad t \in I(t^*)$$

given that, as $\psi(z) - \psi(t) = \int_t^z [\delta + \mu(x + \tau)] d\tau$ from (6.28"), it follows that

$$[1 - A(t, t^*)] e^{\psi(t)} = \int_t^{t^*} \alpha(z) e^{-\int_t^z [\delta + \mu(\tau)] d\tau} dz$$

and, furthermore, that $D(t, t^*)$ is a fair actuarial counterpart for payments in the interval (t, t^*) with flow $\alpha(z)$.²⁷

The previous formulations show that with a continuous payment flow we move from the certain amortization to the life one, substituting the purely financial intensity δ with the actuarial one $\delta + \mu(x + t)$ and then the function $\varphi(t)$, defined in (6.19), with $\psi(t)$.

We obtain easy generalizations by assuming, instead of the intensity δ of the exponential financial law, the intensity $\delta(t)$ of any decomposable financial law.

6.4. Periodic funding at fixed rate

6.4.1. Delayed payments

We saw in section 5.1 that the final value of an annuity on the basis of a given law can be considered as the final result of a funding operation on a saving account with such a law. Let us develop here in detail such an operation considering how it is done in the most important cases, starting from that of *delayed payments*.

Let us consider a generic operation of *funding* in n periods (years) of a capital S by means of accumulation on a saving account at the per period rate i of the set of payments of amount R_h at the end of the h^{th} period ($h = 1, \dots, n$). Then in such an account a *sinking fund* is increasing.

The following constraint

$$S = \sum_{h=1}^n R_h (1+i)^{n-h} \quad (6.34)$$

must then be satisfied. It implies the financial equivalence between the set of supplies (h, R_h) of the investor and the dated amount (n, S) that is the result of the investment operation.

Different from the discrete amortization schemes described previously, the *principal amount* C_h is the increase of the fund at time h and then is obtained

²⁷ In the continuous case the considerations in footnote 23 also hold if at time z the technical bases $\{\delta, l(x)\}$ different from those initially assumed for the continuous life amortization are adopted.

adding, and not subtracting, to the installment R_h the *interest amounts* I_h earned by the investor on the sinking fund in the period $(h-1, h)$ and proportional to the amount accumulated in $h-1$. Indicating with G_h the level of the *sinking fund* at the integer time h on the saving account due to the operation (then $G_0 = 0$) and having fixed in advance the principal amount C_h non-negative and satisfactory, as for the amortization, the constraint of the 1st of (6.3), the following recursive relations hold:

$$\left. \begin{aligned} G_h &= G_{h-1} + C_h \\ I_h &= i G_{h-1} \\ R_h &= C_h - I_h \end{aligned} \right\} (h = 1, \dots, n) \tag{6.35}$$

Starting from the initial condition $G_0 = 0$, all the values $\{I_h\}, \{R_h\}, \{G_h\}$ are obtained and in particular, due to the 1st part of (6.3): $G_n = S$, i.e. the requested funding. In the dynamics of the operation, the fundamental recursive relation holds

$$G_h = G_{h-1}(1+i) + R_h, \quad (h = 1, \dots, n) \tag{6.36}$$

and can be written as

$$R_h = (G_h - G_{h-1}) - i G_{h-1}, \quad (h = 1, \dots, n) \tag{6.36'}$$

The *retro-reserve* $M(h; i)$ and the *pro-reserve* $W(h; i)$ (at credit for the investor) of the operation, at time h and at rate i (the rate chosen at the beginning or adjusted in h) are given by the expression

$$\begin{aligned} M(h; i) &= \sum_{s=1}^h R_s (1+i)^{h-s} \\ W(h; i) &= S(1+i)^{-(n-h)} - \sum_{s=h+1}^n R_s (1+i)^{-(s-h)} \end{aligned} \tag{6.37}$$

and, if i is the rate of the law initially adopted for the funding, the result is

$$M(h; i) = W(h; i) = G_h \tag{6.38}$$

As in gradual amortization, if the CCI regime is adopted, the reserves in each intermediate time between consecutive payments can be defined (for example to calculate exactly the assignment value of the credit) at each time $t=k+s$ (where k = integer part of t ; s = decimal part of t). We find

$$M(t; i) = M(k; i) (1+i)^s \quad ; \quad W(t; i) = W(k; i) (1+i)^s \tag{6.39}$$

28 Also for the funding, the considerations for retro-reserve and pro-reserve found in footnote 16 are extended if at time k different rates are adopted from the one initially chosen.

By varying t in the real numbers between 0 and n , in $[0,n]$ we obtain two functions, M and W , coincident if the funding operation between 0 and n is fair, discontinuous (right-continuous) at integer time k .

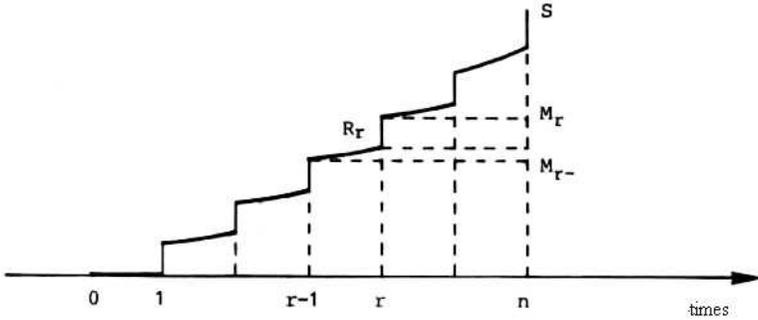


Figure 6.3. Plot of delayed funding

If the funding is made with constant delayed payments $R_h = R^{29}$, all the relations are adopted with this position. In particular, the equivalence constraint between S and R is given by

$$S = R s_{\bar{n}|i} \quad \text{or} \quad R = S \sigma_{\bar{n}|i} \tag{6.40}$$

By adopting (6.36), and using it for consecutive values of h and subtracting, it is verified that, as in French amortization, the principal amount changes in geometric progression with ratio $(1+i)$, resulting in

$$C_h = R(1+i)^{h-1} ; G_h = R s_{\bar{h}|i} = S \frac{s_{\bar{h}|i}}{s_{\bar{n}|i}} \tag{6.41}$$

The retro-reserve and the pro-reserve in h at rate i are expressed by

$$M(h;i) = R s_{\bar{h}|i} ; V(h;i) = S(1+i)^{-(n-h)} - R s_{\overline{n-h}|i} \tag{6.37'}$$

29 An example of term funding by means of delayed constant periodic payments has been encountered in the American amortization considered in section 6.2.5.

Exercise 6.8

We have to form a capital at maturity of €25,500 in 5 years on a saving account at the annual delayed rate of 5.25%, with annual delayed payments corresponding to the following sequence of principal repayments, the sum of which is 25,500:

$$C_1 = 4,500 ; C_2 = 5,300 ; C_3 = 5,600 ; C_4 = 6,000 ; C_5 = 4,100.$$

Calculate the funding schedule.

A. Applying (6.35) on an Excel spreadsheet, from the given value $\{C_h\}$ in the 2nd column are found the end of year balances $\{G_h\}$; from here we find the earned interest $\{I_h\}$ and the installments $\{R_h\}$ to be paid by the investor. The following schedule is obtained.

DELAYED FUNDING WITH GIVEN PRINCIPAL AMOUNTS				
Capital = 25,500		Rate = 0.0525		
h	C_h	G_h	I_h	R_h
1	4,500.00	4,500.00	0.00	4,500.00
2	5,300.00	9,800.00	236.25	5,063.75
3	5,600.00	15,400.00	514.50	5,085.50
4	6,000.00	21,400.00	808,50	5,191.50
5	4,100.00	25,500.00	1123,50	2,976.50

Table 6.11. Example of delayed funding

The *Excel* instructions are as follows: the 1st, 2nd and 4th rows are for data and titles: B2: 25500; D2: 0.0525; the 3rd and 5th rows are empty. Starting from the 6th row:

- column A (year): A6: 1; A7:= A6+1; copy A7, then paste on A8 to A10;
- column B (principal amount): insert data on B6 to B10; sum in: B2);
- column C (accumulated amount): C6:= C5+B6; copy C6, then paste on C7 to C10;
- column D (interest amount): D6:= C5*D\$2; copy D6, then paste on D7 to D10;
- column E (installment): E6:= B6-D6; copy E6, then paste on E7 to E10.

Exercise 6.9

With the same data as exercise 6.8 for the amount at maturity, for the length and the rate, calculate the funding schedule imposing the installments invariance.

A. By applying (6.35) and the 2nd part of (6.40), and by using an *Excel* spreadsheet the following schedule is found.

DELAYED FUNDING WITH CONSTANT INSTALLMENT			
Capital = 25,500			Rate = 0.0525
Years = 5			Installment = 4,591.87
h	C_h	I_h	G_h
1	4,591.87	0.00	4,591.87
2	4,832.94	241.07	9,424.81
3	5,086.67	494.80	14,511.48
4	5,353.72	761.85	19,865.21
5	5,634.79	1,042.92	25,500.00

Table 6.12. Example of delayed funding

The *Excel* instructions are as follows. Rows 1, 2, 3 and 5 are for data, titles and one calculation: B2: 25,500; D2: 0.0525; B3: 5; D3:= B2*D2/((1+D2)^B3-1); rows 4 and 6 are empty. From row 7:

- column A (year): A7:= A6+1; copy A7, then paste on A8-A11;
- column B (principal amounts): B7:= D\$3*(1+D\$2)^A6; copy B7, then paste on B8 to B11;
- column C (interest amounts): C7:= B7-D\$3; copy C7, then paste on C8 to C11;
- column D (sinking fund): D7:= D6+B7; copy D7, then paste on D8 to D11.

6.4.2. Advance payments

Let us consider briefly the variations in relation to section 6.4.1 when the payments, indicated using \ddot{R}_h , are made at integer time h referring to the period $(h, h+1)$, ($h = 0, 1, \dots, n-1$), and therefore are called *advance payments*. The closure constraint with the amount S to be formed at time n becomes

$$S = \sum_{h=0}^{n-1} \ddot{R}_h (1+i)^{n-h} \tag{6.42}$$

The recursive relations regarding the accumulated capitals G_h at time h , the principal amounts \dot{C}_h subject, as for the amortization, to the 2nd of (6.3), the interest amounts \ddot{I}_h and the installments \ddot{R}_h , starting from the initial condition $G_0 = 0$, are:

$$\left. \begin{aligned} G_{h+1} &= G_h + \ddot{C}_h \\ \ddot{I}_h &= d G_{h+1} \\ \ddot{R}_h &= \ddot{C}_h - \ddot{I}_h \end{aligned} \right\} (h = 0, \dots, n-1) \tag{6.43}$$

where $d=i/(1+i)$ (obtaining, in particular, $G_n = S$) and the recursive relation on the accumulated amount is found to be

$$G_{h+1} = (G_h + \ddot{R}_h)(1+i) \quad , \quad (h = 0, \dots, n-1) \tag{6.44}$$

and the decomposition is found to be

$$\ddot{R}_h = (G_{h+1} - G_h) - d G_{h+1} \quad , \quad (h=0, \dots, n-1) \tag{6.44'}$$

For the retro-reserve $M(h;i)$ and the pro-reserve $W(h;i)$ at time h the following expressions hold:

$$\begin{aligned} M(h;i) &= \sum_{s=0}^{h-1} \ddot{R}_s (1+i)^{h-s} \\ W(h;i) &= S(1+i)^{-(n-h)} - \sum_{s=h}^{n-1} \ddot{R}_s (1+i)^{-(s-h)} \end{aligned} \tag{6.45}$$

which are equal to each other and to G_h if i is the rate initially adopted for the funding.

If the CCI regime is adopted, in the advance case we can also define the reserves in whichever non-integer time $t=k+s$ (where k = integer part of t ; s = decimal part of t), resulting in

$$M(t;i) = M(k+1;i) (1+i)^{-(1-s)} \quad ; \quad W(t;i) = W(k+1;i) (1+i)^{-(1-s)} \tag{6.39'}$$

By varying t in the real numbers between 0 and n we obtain in $(0,n)$ two functions, M and W , coincident if the funding operation between 0 and n is fair, discontinuous (continuous to left) at the integer time k .

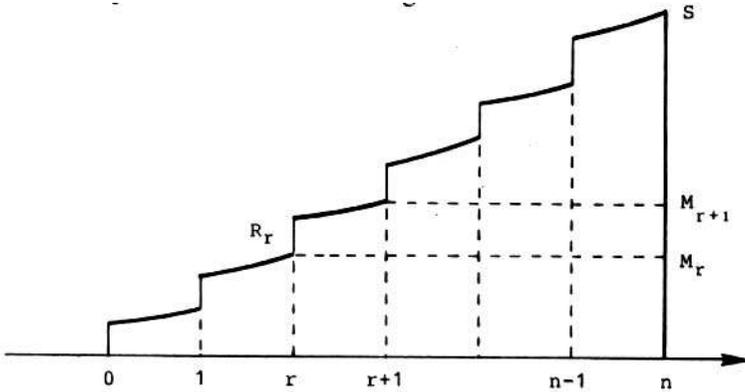


Figure 6.4. Plot of advance funding

In the case of constant advance payments it is enough to put $\ddot{R}_h = \ddot{R}$ constant in the previous formulation. The following is then obtained:

$$\ddot{S} = \ddot{R} \ddot{s}_{\overline{n}|i} \quad \text{or} \quad \ddot{R} = S \ddot{o}_{\overline{n}|i} \tag{6.40'}$$

and, with $G_0=0$:

$$G_{h+1} = (G_h + \ddot{R})(1+i) \quad , \quad (h=0, \dots, n-1) \tag{6.46}$$

from which

$$\ddot{R} = (G_{h+1} - G_h) - d G_{h+1}, \quad (h=0, \dots, n-1) \tag{6.46'}$$

Also in this case the principal amount varies in geometric progression with ratio $(1+i)$, resulting in:

$$\ddot{C}_h = \ddot{R}(1+i)^{h+1} \quad ; \quad G_h = \ddot{R} \ddot{s}_{\overline{h}|i} = S \frac{\ddot{s}_{\overline{h}|i}}{\ddot{s}_{\overline{n}|i}} \tag{6.47}$$

The retro-reserve and pro-reserve in h are

$$M(h;i) = \ddot{R} \ddot{s}_{\overline{n}|i} \quad ; \quad W(h;i) = S(1+i)^{-(n-h)} - \ddot{R} s_{\overline{n-h}|i} \tag{6.45'}$$

which are equal to each other and to G_h if i is the initially adopted rate for the funding.

Exercise 6.10

With the same data as in Exercise 6.8 for the capital at maturity, for the length and the rate, calculate the advance funding schedule imposing the invariance of installments.

A. Applying (6.43) and the 2nd of (6.40') we obtain on an *Excel* spreadsheet the following schedule.

ADVANCE FUNDING WITH CONSTANT INSTALLMENT			
Capital = 25,500		Delayed rate = 0.0525	
Years = 5		Installment = 4,362.82	
<i>h</i>	<i>C_h</i>	<i>I_h</i>	<i>G_h</i>
0	4,591.87	229.05	0.00
1	4,832.94	470.12	4,591.87
2	5,086.67	723.85	9,424.81
3	5,353.72	990.90	14,511.48
4	5,634.79	1,271.97	19,865.21
5	0.00	0.00	25,500.00

Table 6.13. *Example of advance funding*

The *Excel* instructions are as follows: the first 3 row and the 5th row are for data, titles and one calculation:

B2: 25500; D2: 0.0525; B3: 5; D3:= B2*D2/(1+D2)/((1+D2)^B3-1);

the 4th row is empty; from the 6th row:

column A (year): A6: 0; A7:= A6+1; copy A7, then paste on A8 to A11;

column B (principal amount): B6:= D\$3*(1+D\$2)^A7; copy B6, then paste on B to B10; B11: 0;

column C (interest amount): C6:= B6-D\$3; copy C6, then paste on C7 to C10 C11: 0;

column D (sinking fund): D6: 0; D7:= D6+B6; copy D7, then paste on D8 to D11.

6.4.3. Continuous payments

Analogous to the continuous amortization scheme (see section 6.2.6) is that of the certain funding³⁰ of a capital *S* by means of an continuous annuity with flow $\sigma(t)$ in the time interval $I(t_1)$ from 0 to t_1 .

³⁰ Together with the classification of amortizations and for reasons of completeness we should briefly mention the funding by means of payments that are conditioned to an investor's survival, i.e. by a life annuity. However, it is evident such a scheme coincides with that of life

This issue has been already discussed in general terms in Chapter 4, where the value of the accumulated amount in (4.14) has been found as the solution of the differential equation (4.13) in which the flow that leads to the variation of the accumulated amount is the sum of the interest flow and of the increasing flow for the net payment $-\varphi(t)$ (negative from the viewpoint of the cash) to the fund to be formed. It will be enough to mention it briefly in order to highlight conditions by which a payment flow $\sigma(t) = -\varphi(t)$ is used to form in t_1 a capital S . Let us assume for simplicity $S=1$ and a financial low strongly decomposable with intensity $\delta(t)$ ($\delta(t) = \delta$ constant if the low is exponential). Using:

$$\chi(t) = \int_t^{t_1} \delta(z) dz, \quad t \in I(t_1) \tag{6.48}$$

to form the unitary capital at time t_1 the flow $\sigma(t)$ varying in $I(t_1)$ must satisfy the constraint of financial closure:

$$\int_0^{t_1} \sigma(z) e^{\chi(z)} dz = 1 \tag{6.49}$$

If $\sigma(t) = \sigma$ constant and $\delta(t) = \delta$ constant, due to (5.16) and (6.49),

$$\sigma = 1 / s_{\overline{t_1}|i}^{(\infty)} \tag{6.49'}$$

must hold, extending the meaning of the symbol $s_{\overline{t_1}|i}^{(\infty)}$ if t_1 is not an integer.

Using:

- $G(t)$ = sinking fund formed in t ;
- $c(t)$ = flow in t of variation of the sinking fund;
- $j(t)$ = flow in t of interest (received for the investor);

the following recursive relations hold, starting from $G(0)=0$

$$\left. \begin{aligned} j(t) &= \delta(t)G(t) \\ c(t) &= j(t) + \sigma(t) \\ \int_0^t c(z) dz &= G(t) \end{aligned} \right\} \forall t \in I(t_1) \tag{6.50}$$

In the further hypothesis of constant payment flows, it is possible to extend (5.9) to the continuous scheme. Considering (6.20') and (6.49') and also the relation $1 / s_{\overline{t_1}|i}^{(\infty)} + \delta = 1 / a_{\overline{t_1}|i}^{(\infty)}$, that can be immediately verified, we find:

$$\sigma + \delta = \alpha \tag{6.51}$$

insurances with endowment (temporary or perpetual) policies. Then for life funding it is enough to refer to a treatise on life insurances.

This relationship links the constant flows of unitary amortization and funding (then intensities from the dimensional point of view, having divided the flows by the amount S) in operations of the same length in an exponential regime.

6.5. Amortizations with adjustment of rates and values

6.5.1. Amortizations with adjustable rate

For the reasons explained in Chapter 1, the quantifications discussed so far consider monetary amounts. This is not only for homogenization of values, but it can be used to settle obligations because money is the legal measure of wealth.

The phenomenon of monetary inflation or other causes that lead to variations (more often a decrement) of the purchasing power of money, which is now no longer linked to gold or any other assets with stable and intrinsic value, is more and more widespread in the presence of macroeconomic imbalances.

Due to this phenomenon, loan operations and the following amortization, fair in monetary terms at a given rate, are not fair in real terms, i.e. considering the purchasing power of the traded sums. Then the receiver of the sums with future maturity is substantially damaged if the variation of the purchasing power is a decrement. Therefore, in recent times, which are characterized by permanent inflation, financial schemes for amortization have been developed which are used to correct its distorting effects by means of opportune variations in the aforementioned methods. Such schemes are not only useful to neutralize these negative effects for the investor, of monetary depreciation, but more generally are used to reduce the risk of oscillation of the financial market in both directions.

The *first variation* consists of making the rate fluctuate up and down, adjusting it to the current rate for new operations in the financial market, without changing the outstanding loan balance. With this procedure the interest amount of one period is calculated by multiplying per period the updated rate by the outstanding balance at the beginning of the period.

Limiting ourselves to the delayed installment case, let us consider two forms of amortization with adjustable rate, highlighting that the rate variations are not known at the beginning but are fixed in the h^{th} period in relation with the aforementioned phenomena, regarding the inflation and the following depreciation of money. Therefore, it is not possible to fix at the inception date of the loan the effective amortization plan that will be adopted.

a) French amortization with adjustable rate

In this form, we proceed initially with the progressive method described in section 6.2.2, calculating the installments by means of (6.8). The installments remain unchanged for the following periods if the rate is not adjusted, but, in the case of variations, new installments are calculated, using the adjusted rate, the outstanding loan balance and the remaining time, on the basis of (6.8).

In formulae, indicating with $i^{(1)}, \dots, i^{(n)}$ the rates (not necessarily different) that will be applied in the subsequent periods $1, \dots, n$, the installments and the outstanding balances of each period are obtained recursively from the following equation system (where $D_0=S$)

$$(h = 1, \dots, n) \begin{cases} R_h = D_{h-1} \alpha_{n-h+1} \bar{\mid} i^{(h)} \\ D_h = R_h a_{n-h} \bar{\mid} i^{(h)} \end{cases} \quad (6.52)$$

Obviously the interest payments and the principal repayments are calculated using

$$I_h = D_{h-1} i^{(h)} ; C_h = R_h - I_h = D_{h-1} - D_h \quad (6.52')$$

From (6.52) it follows that the installments remain unchanged between two subsequent rate variations; furthermore the installment variations are concordant to the rate variation, if it changes. To prove this statement, we can observe that the recursive relation

$$R_{h+1} = R_h \frac{a_{n-h} \bar{\mid} i^{(h)}}{a_{n-h} \bar{\mid} i^{(h+1)}}, \quad h = 1, \dots, n-1 \quad (6.52'')$$

on the installments follows from (6.52), and that $a_{\bar{\mid} i}$ is a decreasing function of rate i . In addition, the principal repaid in $h+1$ is

$$C_{h+1} = D_h \sigma_{n-h} \bar{\mid} i^{(h+1)} = R_{h+1} (1 + i^{(h+1)})^{-(n-h)} \quad (6.53)$$

and $|\sigma_{\bar{\mid} i}|$ decreases with the rate. Therefore, the variation of the principal repayment due to the rate variation is discordant to it; the result is that a rate increment slows down the amortization, giving rise to higher outstanding balances and higher installments than those in the absence of adjustments, even if the “closure” remains unchanged, i.e. the debt becomes zero at the end of the loan.

b) Amortizations with adjustable rate and prefixed principal amount

In the previous form of amortization, a) in the case of rate adjustments there is a novation of the contract on the outstanding loan balance and remaining length, such that with respect to the progressive scheme at fixed rate not only are the sequences

of interest payments subject to variations, but also those of principal repayments and then of the outstanding balances.

Or it can be agreed that the principal repaid remains unchanged in case of adjustment of the contractual rate, so as to eliminate the uncertainty of the principal repayments and to reduce that of the interest payments, obtained multiplying the rate, varying with h in a way not previously foreseen, for the prefixed outstanding loan balances. Thus we lead back to the recursive system (6.4'), modified to take into account the rate variability in the period h , i.e.

$$(h = 1, \dots, n) \left\{ \begin{array}{l} D_h = D_{h-1} - C_h \\ I_h = i^{(h)} D_{h-1} \\ R_h = C_h + I_h \end{array} \right. \quad (6.4''')$$

which, using $D_0=S$, enables the calculation of the interest payments, the installments and the outstanding loan balances in the following periods.

Example 6.3

This example clarifies the comparison, set out in the following table, between the amortizations in 5 years of the amount $S = \text{€}100,000$ in the three different forms:

- 1) "French" at rate $i = 0.05$ that gives the delayed constant installment $R = 23,097.48$;
- 2) form a) with rates $i^{(h)}$ specified in the table;
- 3) form b) with the same $i^{(h)}$ and constant principal amount $C_h = 20,000.00$.

h	French		form a)			form b)	
	D_h	$i^{(h)}$	R_h	D_h	I_h	R_h	D_h
1	81,902.52	0.05	23,097.48	81,902.52	5,000.00	25,000.00	80,000.00
2	62,900.16	0.07	24,179.93	63,455.77	5,600.00	25,600.00	60,000.00
3	42,947.69	0.07	24,179.93	43,717.75	4,200.00	24,200.00	40,000.00
4	21,997.60	0.05	23,511.62	22,392.02	2,000.00	22,000.00	20,000.00
5	0.00	0.05	23,511.62	0.00	1000.00	21,000.00	0.00

Table 6.14. Comparison of different amortization rules

It can be seen that in form a) the installment of the 1st year coincides with the installment R of the French amortization at rate 5% but in the 4th year, after two years of increasing rates, even if the rate returned back to the initial level, due to the higher outstanding balance, $R_4 > R$ results. Thus, with $i^{(5)} = i^{(4)}$ we have $R_5 = R_4$.

6.5.2. Amortizations with adjustment of the outstanding loan balance

The rate adjustment considered in section 6.5.1 solves, in an approximated manner, the problem of money depreciation (or, more generally, of the variation of purchasing power of money) because it acts only by an additive variation of rate which does not exactly reflects Fisher’s equation. A procedure to fully solve this problem is that of *indexation* of the prefixed outstanding loan balances, obtained by multiplying such balances by coefficients derived from a series of statistical indices measuring the mean prices varying with the same periodicity as the redemption payments.

In such a way, the installments and their components for interest and amortization, that are proportional to the outstanding balances, will be modified multiplicatively according to the same coefficients, where the constraint of elementary closure, which assumes the non-modifiability of the principal payments and thus of the outstanding balances, is not satisfied.

Let us formalize the procedure, limiting ourselves to the adjustment of the French amortization³¹ of the landed amount $S = D_b$ at time b in n periods at the per period rate i by means of installments that, if the index remains constant, would all assume the value $R = S i / (1 - (1 + i)^{-n})$.

Let $\{Z_h\}$, ($h = b, b + 1, \dots, b + n - 1$) be, the series of statistical indices needed for the adjustment in n periods, with the same periodicity of payments. The updating coefficient between time h and $h + 1$ is $K_{h+1} = Z_{h+1} / Z_h = 1 + p_{h+1}$, where p_{h+1} is the corresponding per period updating rate; therefore

$$\pi_b = 1 ; \pi_h = \prod_{j=b+1}^h K_j = Z_h / Z_b ; (h = b + 1, \dots, b + n - 1) \tag{6.54}$$

are the global updating factors for $h - b$ periods to be used in the calculations. In the absence of adjustments the outstanding loan balances at time r would be

$$D_h = R a_{b+n-h|i} \tag{6.55}$$

while, due to what has been said, the updating modifies the sequence $\{D_h\}$ in $\{D'_h\}$ defined by

$$D'_h = D_h \pi_h \tag{6.56}$$

31 The same conclusions hold with different amortization schemes that give rise to any development of the outstanding balances before the updating.

It is clear that $b+n-h$ delayed payments of constant amount R'_{h+1} (= updated installment of the period $h+1$) would amortize D'_h in the absence of further updating. Thus $D'_h = R'_{h+1} a_{b+n-h|i}$ and then due to (6.22): $D'_h/D_h = R'_{h+1}/R = \pi_h$, which can be written

$$R'_{h+1} = R \pi_h \tag{6.56'}$$

Proceeding analogously, the updated interest paid is

$$I'_{h+1} = i D'_h = i D_h \pi_h = I_{h+1} \pi_h \tag{6.57}$$

and subtracting (6.24) from (6.23') it is obtained for the updated principal repaid

$$C'_{h+1} = (R - I_h) \pi_h = C_{h+1} \pi_h \tag{6.57'}$$

Briefly, the outstanding loan balance after $h-b$ periods from the inception and also the installment paid at the end of the period, i.e. at time $h+1$, and its principal and interest components are updated by means of the factor π_h given by (6.21).

Exercise 6.11

Amortize in 5 years the amount €80,000 loaned at time 6 at the annual rate of 4.5% with value adjustments according to the index $\{Z_h\}$, ($h = 6, 7, 8, 9, 10$), of the “cost of life” on the basis of the observed values, specified in Table 6.15.

A. On the basis of the data and using: $D_h = D_{h-1} - C_h$, the following amortization schedule is obtained, that compares the non-updated values of the French amortization and the updated values in the outstanding loan balances on the basis of $\{Z_h\}$. By using $S=€80,000$; $n=5$; $i=0.045$; $R=€18,223.33$, the following data is obtained (rounding off €amounts to no decimal-digit).

$S = 80000$		Start = 6		Length = 5		Rate = 0.045		Payment = 18223	
h	Z_h	π_h	l_h	l'_h	C_h	C'_h	R'_h	D_h	D'_h
6	120.0	1.0000						80,000	80,000
7	122.5	1.0208	3,600	3,600	14,623	14,623	18,223	65,377	66,739
8	125.7	1.0475	2,942	3,003	15,281	15,600	18,603	50,095	52,475
9	129.6	1.0800	2,254	2,361	15,969	16,728	19,089	34,126	36,856
10	133.2	1.1100	1,536	1,659	16,688	18,023	19,681	17,439	19,357
11			785	871	17,439	19,357	20,228	0	0

Table 6.15. Amortizations with adjustment of the outstanding loan balance

The *Excel* instructions are as follows. The first 2 rows are for data, titles and one calculation: B1: 80000; D1: 6; F1: 5; H1: 0.045; J1:= (B1*H1)/(1-(1+H1)^-F1). From the 3rd row:

column A (time): A3:= D1; A4:= A3+1; copy A4, then paste on A5 to A(3+F1);
 column B (Z_h): from B3 to B7 insert periodic index numbers for F1 periods;
 column C (π_h): C3: 1; C4:= B4/B\$3; copy C4, then paste on C5 to C7;
 column D (I_h): D4:= I3*H\$1; copy D4, then paste on D5 to D8;
 column E (I'_h): E4:= D4*C3; copy E4, then paste on E5 to E8;
 column F (C_h): F4:= J\$1-D4; copy F4, then paste on F5 to F8;
 column G (C'_h): G4:= F4*C3; copy G4, then paste on G5 to G8;
 column H (R'_h): H4:= E4+G4 (or := J\$1*C3); copy H4, then paste on H5 to H8;
 column I (D_h): I3:= B1; I4:= I3-F4; copy I4, then paste on I5 to I8;
 column J (D'_h): J3:= I3*C3; copy J3-paste on J4 to J8.

6.6. Valuation of reserves in unshared loans

6.6.1. General aspects

The valuation of the pro-reserve $W(t, i^*)$ at a given time t of a financial operation, obtained by discounting the supplies after t on the basis of a prefixed law, in particular the exponential one at a valuation rate i^* generally different from the contractual rate i originally agreed for the calculation of interest (because can be different the valuation time, the evaluating subject, the aims and the market conditions), is often interesting. We have such valuations when a company balance is prepared for internal or external/official use, or for the assignment of credits or for the carrying of debts regarding the operation.

We will consider the calculation of the pro-reserve and its components in relation to the gradual amortization of a debt during its development (or sometimes at the inception date). Using periodic then discrete payments, we can assume the conjugate of a DCI. law. We will complete this consideration with the development of the so-called *Makeham's formula* and the calculation of the usufruct in the discrete scheme, using any valuation rate i^* , for the most important amortization methods.

In a gradual amortization with n periodic installments R_k delayed and varying, of the type seen in section 6.2.1 (with simple variations for the advance case) assuming a unitary period, the pro-reserve $W(t, i^*)$ is the current value in t of the installments R_k with due dates $k \geq t$; it is equal to the outstanding loan balance $D_h = \sum_{k=h+1}^n R_k (1+i)^{-(k-h)}$ if $i^* = i$ and $t = h$. If there is a need to distinguish

between the value of the principal repayments C_k and that of the interest payments I_k (e.g. because the creditor of interest is different from that of the principal), we will have to evaluate separately at the rate i^* , the usufruct $U(t, i^*)$ and the bare ownership $P(t, i^*)$, the sum of which is $W(t, i^*)$ ³².

Let us consider the position $t = h$ evaluating at integer time h the pro-reserve and its components *usufruct and bare ownership in the discrete*, then the present value of the interest payments and the principal repayments, at an evaluation rate i^* ³³.

In formulae with already defined symbols,

$$(h = 1, \dots, n) \begin{cases} W_h^* = W(h, i^*) = \sum_{k=h+1}^n R_k (1+i^*)^{-(k-h)} \\ U_h^* = U(h, i^*) = \sum_{k=h+1}^n I_k (1+i^*)^{-(k-h)} \\ P_h^* = P(h, i^*) = \sum_{k=h+1}^n C_k (1+i^*)^{-(k-h)} \end{cases} \quad 34 \quad (6.58)$$

obtaining $W_h = D_h, U_h, P_h$ as particular values when $i^* = i$.

6.6.2. Makeham's formula

The *additivity* expressed by

$$W_h^* = U_h^* + P_h^* ; W_h = U_h + P_h ; (h = 1, \dots, n) \quad (6.59)$$

is obvious (and it has already been found).

The following *Makeham's formula*, which links values at rate i^* to those at the contractual rate i , also holds:

$$W_h^* = P_h^* + \frac{i}{i^*} (D_h - P_h^*) \quad (6.60)$$

32 The examined valuation is apparently an operation with two rates, i and i^* , but looking at it more closely, the only rate i^* is used as a variable with the meaning of discount rate of the amounts – principal repayments, interest payments, installments, etc. – that at the valuation time are already fixed as a function of the original data, between which there is the repayment rate i .

33 They will be initial values if $h=0$, residual values if $h = 1, \dots, n$.

34 The values for non-integer time t in exponential regime, using $t=h+s$ ($0 < s < 1$), are obtained from those in (6.25) multiplying by $(1+i)^s$.

and from which, due to (6.59), the following expression to evaluate U_h^* as a function of P_h^* is found:³⁵

$$U_h^* = \frac{i}{i^*} (D_h - P_h^*) \tag{6.60'}$$

Proofs of Makeham's formula

1) A brief proof of Makeham's formula based on the equivalence at the rate i^* can be given. It is enough to observe that at the rate i the debt D_h is amortized with installments $R_s = C_s + I_s$ ($s = h+1, \dots, n$), i.e. it is fair to exchange D_h with the installments R_s , while at rate i^* , if the principal amounts C_s and then the outstanding loan balances D_s remain unchanged, to preserve the equivalence the interest payments must be: $I_s^* = i^* D_{s-1} = I_s i^*/i$, i.e. at rate i^* it is fair to exchange D_h with the installments $R_s^* = C_s + (i^*/i)I_s$. It then follows that:

$$D_h = \sum_{s=h+1}^n R_s^* (1+i^*)^{-(s-h)} = \sum_{s=h+1}^n C_s (1+i^*)^{-(s-h)} + \frac{i^*}{i} \sum_{s=h+1}^n I_s (1+i^*)^{-(s-h)}$$

or, due to (6.58),

$$D_h = P_h^* + \frac{i^*}{i} U_h^* \tag{6.61}$$

from which we obtain (6.60') and (6.60). ☼

2) Due to the closure equation, it follows that $D_h = \sum_{k=1}^{n-h} C_{h+k}$, i.e. the outstanding loan balance D_h at time h is decomposed in subsequent principal repayments C_{h+k} , ($k=1, \dots, n-h$), each of which leads to its refund after k years and the payment of interest iC_{h+k} for k years. The overall valuation in h of these obligations at rate i^* is W_h^* . Therefore, using $v_*^k = (1+i^*)^{-k}$ the following is obtained

35 By adding and subtracting D_h in the 2nd part of (6.27), Makeham's formula becomes:

$W_h^* = P_h^* + \frac{i}{i^*} (D_h - P_h^*)$, which highlights the decreasing of W_h^* with respect to i^* (then the

convenience for the debtor, that assigns the debt during the amortization, to evaluate it at the highest possible rate) and gives a measure of the spread between the valuation at rate i^* and that at rate i of the future obligation of the debtor, as $(W_h^* - D_h)$ has the sign of $(i-i^*)$. If, in particular, $h=0$, it is sufficient to put in the formulae $D_h = S =$ landed capital, to evaluate the obligations at any rate since from inception. Given the biunivocity of the relations, we can exchange the role between i^* and W_h^* , assuming the value W_h^* fixed by the market and obtaining i^* that takes the meaning of internal rate of return (IRR).

$$W_h^* = \sum_{k=1}^{n-h} C_{h+k} (v_*^k + i a_{k|i^*}) = \sum_{k=1}^{n-h} C_{h+k} \left[v_*^k + \frac{i}{i^*} (1 - v_*^k) \right] = P_h^* + \frac{i}{i^*} (D_h - P_h^*)$$

i.e. (6.61), from which we obtain (6.60') and (6.60).

3) A purely analytical proof of Makeham's formula is obtained by applying Dirichlet's formula, i.e. summing by columns instead of by rows the elements $m_{s,k} = C_k (1 + i^*)^{-(s-h)}$ of a triangular matrix. It follows that

$$\begin{aligned} U_h^* &= \sum_{s=h+1}^n I_s (1 + i^*)^{-(s-h)} = i \sum_{s=h+1}^n D_{s-1} (1 + i^*)^{-(s-h)} = \\ &= i \sum_{s=h+1}^n \sum_{k=s}^n C_k (1 + i^*)^{-(s-h)} = i \sum_{k=h+1}^n C_k \sum_{s=h+1}^n (1 + i^*)^{-(s-h)} = \\ &= \frac{i}{i^*} \sum_{k=h+1}^n C_k \left[1 - (1 + i^*)^{-(k-h)} \right] = \frac{i}{i^*} (D_h - P_h^*) \end{aligned}$$

i.e. we obtain (6.60') and (6.60). ✧

Observations

1) By adding and subtracting D_h on the right side of (6.60) Makeham's formula becomes:

$$W_h^* = D_h - \frac{i^* - i}{i^*} (D_h - P_h^*) \tag{6.60''}$$

which, as $P_h^* < D_h$, highlights the increasing of W_h^* with respect to i^* . Thus, W_h^* is the assignment value of the residual credit of the lender at the integer time h (then the debtor that assigns the debt during the amortization has the convenience of evaluating at the highest possible rate) and gives a measure of the spread between the valuation W_h^* of the outstanding loan balance and its *nominal value* D_h if $i^* \neq i$, because we obtain $W_h^* < D_h$ or $W_h^* > D_h$ if $i^* > i$ or $i^* < i$ respectively. If $h=0$, it is sufficient to use $D_h = S$ in (6.60'').

2) Given the biunivocity of the relations, we can exchange in (6.60) or in its transforms the roles of i^* and W_h^* , assuming the latter as the value given exogenously by the market laws and obtaining i^* that assumes the meaning of return rate for the investor lender or cost rate for the financed borrower (see section 4.4.1).

3) A recurrent relation analogous to (6.6) also holds for W_h^* . In fact, as it is easily verifiable, it results in:

$$W_h^* = W_{h-1}^* (1 + i^*) - R_h \tag{6.62}$$

4) New expressions of U_h^* and P_h^* are obtained by considering the variation of W_h^* due to that of the rate i^* and finding U_h^* and P_h^* from the system of equations (6.59) and (6.60') with $W_h = D_h$. This results in:

$$U_h^* = -i \frac{W_h^* - W_h}{i^* - i}, \quad P_h^* = \frac{i^* W_h^* - i W_h}{i^* - i} \tag{6.63}$$

and therefore U_h^* is the partial difference quotient of W_h^* in the variation from i to i^* multiplied by $-i$, while P_h^* is the partial difference quotient of $i^* W_h^*$ in the same variation. Taking the limit for $i^* \rightarrow i$ on the differentiable functions W_h^* and $i^* W_h^*$, we obtain the following result

$$U_h = \lim_{i^* \rightarrow i} U_h^* = -i \left(\frac{\partial W_h^*}{\partial i^*} \right)_{i^*=i}, \quad P_h = \lim_{i^* \rightarrow i} P_h^* = \left(\frac{\partial (i^* W_h^*)}{\partial i^*} \right)_{i^*=i} \tag{6.64}$$

6.6.3. Usufructs and bare ownership valuation for some amortization forms

In the concrete case of amortization, we are also interested in the valuation of the residual installments and their components for interest and for amortization at any rate i^* and at any time $t=h+s \in \mathfrak{R}$, with $0 < s < 1$, to which the additivity, expressed by (6.59), is extended. Let us note that, given the delayed or advance payments at integer times h , we obtain (see footnote 34):

$$\begin{cases} W(t, i^*) = W(h, i^*) (1 + i^*)^s, & \text{with delayed payments} \\ W(t, i^*) = W(h + 1, i^*) (1 + i^*)^{-(1-s)}, & \text{with advance payments} \end{cases} \tag{6.65}$$

using analogous formulae for $U(t, i^*)$ and $P(t, i^*)$.

We can then limit ourselves to the calculation for integer time h , making explicit the valuations of usufruct and bare ownership (from which summing we find the pro-reserves) in the following usual forms of amortization. As a function of parameters S, n, i , and evaluating at the rate i^* we easily obtain, using (6.63):

a) Amortization with one final lump-sum refund and periodic delayed interest

$$U(h, i^*) = S i a_{n-h} \rfloor_{i^*}; \quad P(h, i^*) = S (1 + i^*)^{-(n-h)} \tag{6.66}$$

b) Delayed amortization with constant principal repayments

$$\left\{ \begin{aligned} U(h, i^*) &= \frac{S i}{n} \left[(n-h+1)a_{n-h|i^*} - (Ia)_{n-h|i^*} \right] \\ P(h, i^*) &= \frac{S}{n} a_{n-h|i^*} \end{aligned} \right. \quad (6.67)$$

c) Amortization with constant delayed installments

$$\left\{ \begin{aligned} U(h, i^*) &= \frac{R i}{i-i^*} \left[a_{n-h|i^*} - a_{n-h|i} \right] \\ P(h, i^*) &= \frac{R}{i-i^*} \left[(1+i^*)^{-(n-h)} - (1+i)^{-(n-h)} \right] \end{aligned} \right. \quad (6.68)$$

where $R = S\alpha_{\bar{n}|i}$.

Example 6.4: application of Makeham’s formula and comparisons

Let us apply in this example Makeham’s formula for the calculation of usufruct, starting from that of bare ownership and using any valuation rate, in the customary amortization forms for unshared loans, comparing the results with those obtainable using the closed formulae (6.29), (6.30) and (6.31):

a) Amortization with one final lump-sum refund and annual delayed interest.

Let us use:

$S = \text{€}2,000$ (debt); $n = 10$ year; $i = 5.5\%$ (annual contractual rate);
 $i^* = 6.2\%$ (annual valuation rate).

With formula (6.29), the initial valuation ($h=0$) is obtained:

$$U_0^* = 110 a_{\bar{10}|0.062} ; P_0^* = 2,000 (1.062)^{-10} = 1,095.94 ; W_0^* = 1,897.93$$

at time $h=5$ the result is:

$$U_5^* = 110 a_{\bar{5}|0.062} ; P_5^* = 2,000(1.062)^{-5} = 1,095.94 ; W_5^* = 1,941.35$$

By applying Makeham’s formula in $h=0$ and $h=5$, with the values for bare ownership previously found, we obtain the same values for the usufruct:

$$U_0^* = \frac{0.055}{0.062} (2000.00 - 1095.94) = 801.99 ;$$

$$U_5^* = \frac{0.055}{0.062} (2000.00 - 1480.50) = 460.85 .$$

b) Annual amortization with constant principal repayments

Let us use:

$$S = \text{€}1,500 \text{ (debt); } n = 4 \text{ years; } i = 6\% \text{ (annual contractual rate);}$$

$$i^* = 5.2\% \text{ (annual valuation rate).}$$

We then obtain the following amortization schedule.

Year	Principal repaid	Interest paid	Installment	Balance
1	375.00	90.00	465.00	1,125.00
2	375.00	67.50	442.50	750.00
3	375.00	45.00	420.00	375.00
4	375.00	22.50	397.50	0.00

Table 6.16. Example of amortization with constant principal repayments

For the initial valuation ($h=0$) with a direct calculus for U_0^* and using formula (6.30) for P_0^* we obtain:

$$U_0^* = 90.00 \cdot 1.052^{-1} + 67.50 \cdot 1.052^{-2} + 45.00 \cdot 1.052^{-3} + 22.50 \cdot 1.052^{-4} = 203.57$$

$$P_0^* = 375 a_{\overline{4}|0.052} = 1,323.58; W_0^* = 1,527.15$$

For $h=2$ we find

$$U_2^* = 45.00 \cdot 1.052^{-1} + 22.50 \cdot 1.052^{-2} = 63.11; P_2^* = 375 a_{\overline{2}|5.2\%} = 695.31;$$

$$W_2^* = 758.42$$

Applying Makeham's formula for $h=0$ and $h=2$, with the values for bare ownership previously found, we obtain the same values for the usufruct:

$$U_0^* = \frac{0.06}{0.052} (1500.00 - 1323.58) = 203.57; U_2^* = \frac{0.06}{0.052} (750.00 - 695.31) = 63.11$$

c) Annual amortization with constant installments

Let us use, as in b):

$$S = \text{€}1,500 \text{ (debt); } n = 4 \text{ years; } i = 6\% \text{ (annual contractual rate);}$$

$$i^* = 5.2\% \text{ (annual valuation rate); then } R = 432.89.$$

We obtain the following amortization schedule.

<i>Year</i>	<i>Principal repaid</i>	<i>Interest paid</i>	<i>Outstanding balance</i>
1	342.89	90.00	1,157.11
2	363.46	69.43	793.65
3	385.27	47.62	408.38
4	408.38	24.51	0.00

Table 6.17. Example of amortization with constant installments

Using (6.31), for the initial valuation ($h=0$) we obtain

$$U_0^* = \frac{432.89 \cdot 0.06}{0.008} [3.529538 - 3.465106] = 209.19;$$

$$P_0^* = \frac{432.89}{0.08} [1.052^{-4} - 1.06^{-4}] = 1,318.71; W_0^* = 1,530.80$$

For $h=2$ we find

$$U_2^* = \frac{25.9734}{0.008} [1.854154 - 1.833393] = 67.41;$$

$$P_2^* = \frac{432.89}{0.08} (1.052^{-2} - 1.06^{-2}) = 735.23; W_2^* = 802.64.$$

By applying Makeham's formula for $h=0$ and $h=2$, with the values for bare ownership previously found, we obtain the same values for the usufruct:

$$U_0^* = \frac{0.06}{0.052} (1500.00 - 1318.71) = 209.19; U_2^* = \frac{0.06}{0.052} (793.65 - 735.24) = 67.41$$

6.7. Leasing operation

6.7.1. Ordinary leasing

It is appropriate, for completeness, to mention briefly an operation which can be a convenient investment for a financial company and at the same time a form of financing, often preferred by firms to other forms considered in this chapter.

Let us summarize this operation as follows. A company working in leasing is a broker between the owner of an asset or real estate and the lessee firm, in the sense that it gives the financial means for the purchase and, maintaining the property of the asset, grants its use against payment. For this company the costs are those related to the purchase of the asset, while the returns are the payments for the leasing, which are called rent and form a periodic annuity.

On the opposite side, the lessee company, against the use of the asset, pays a periodic rent for the whole length of the contract and also pays an *earnest payment*, usually a multiple of the rent; furthermore the possibility of *redemption*, i.e. the purchase by the lessee of the leased asset, usually at a price strongly reduced and prefixed at the beginning of the lease, it is often provided at expiry³⁶.

There is then the issue of comparing it with the loan operation for purchasing the asset. From this comparison follows a problem of choosing between alternative loans. In fact, we have to compare, on one side, the purchase of the property of the asset using his own means and loaned capital, with the resulting lost profit for the self-financing part that was invested at a return rate i_1 and the emerging cost for the loaned part, at a cost rate i_2 ; and on the other side, the leasing operation that implies the payment of advance, periodic rents and the possible final redemption. The maintenance expenses, in both cases, are paid by the company that uses the asset.

The leasing rent cannot be limited only to remuneration, at the contractual per period rate i , of the amount S used by the lessor for the purchase, at net for the advance and the discounted redemption, because being assets with a limited economic life (due to wear, obsolescence, etc), it must take into account an amount for the funding of the used capital for the renewal. There is then a situation analogous to the American amortization with two coincident rates, where on the basis of (5.9) the rent C is given by $S(i + \sigma_{\overline{n}|i}) = S\alpha_{\overline{n}|i}$, where S is the net amount already specified. Therefore, the rent, if constant and not indexed, is *calculated as the progressive amortization installment of a loaned principal equal to the aforementioned net amount*.

In formulae, if the operation, with a length of n periods, is not indexed and it is provided for a value F , an advance A and also a redemption at expiry R , the delayed per period rent C if constant³⁷ is obtained from the following relation, justified on the basis of the equivalence principle:

$$F = A + C a_{\overline{n}|i} + R(1+i)^{-n} \quad (6.69)$$

36 It is suitable to mention briefly the "real estate leasing". The length is usually long and the redemption value has to take into account that the real estate is not subject to the same depreciation that other assets or industrial equipment are subject to. In addition, there are the taxation problems particular to such leasing arrangements.

37 A financial calculator with the keys (n), (i), (pv), (pmt) and (fv) allows for the immediate automatic calculation of one of the quantities n , i , $(F-A)$, C , R , given the others, because (6.27) can be written: $-(F-A) + C a_{\overline{n}|i} + R(1+i)^{-n} = 0$. In addition, if α is known, we obtain: $F = (F-A)/(1-\alpha)$; $A = F - (F-A)$.

Using $\alpha = A/F$ (= advance quota) and $\rho = R/F$ (= redemption quota), from (6.69) we find the expression for the periodic rent C ³⁸

$$C = F [1 - \alpha - \rho (1+i)^{-n}] / a_{\overline{n}|i} \quad (6.69')$$

If the first m rents are paid at the beginning, they form the advance, then C is found from (6.69) using $A = mC$ and $a_{\overline{n-m}|i}$ instead of $a_{\overline{n}|i}$. Therefore

$$C = \frac{F [1 - \rho (1+i)^{-n}]}{m + a_{\overline{n-m}|i}} \quad (6.69'')$$

Exercise 6.12

1) The lessor gives a plant, the total cost of which is €24,000, for leasing with delayed monthly rents for 5 years and with a redemption equal to the 5% of the cost and

- a) an advance of 8% of the cost; or
- b) an advance equal to 3 rents.

Calculate the rent for the two cases in the hypothesis that an annual remuneration rate 12-convertible of 9.5% is applied.

A. In case a) used in (6.28'): $F=24000$, $n=60$, $\alpha=0.08$, $\rho=0.05$ and using months as the unit measure for time, the monthly rate is $i_{1/12} = 0.007917$ and the rent C (that can be found with a financial calculator as in footnote 37) is

$$C = 24,000 (1 - 0.08 - 0.05 \cdot 1.007917^{-60}) / a_{\overline{60}|0.007917} = 448.02$$

In case b), used in (6.27'') the previous data and $m=3$, we obtain³⁹

$$C = 24000 (1 - 0.05 \cdot 1.007917^{-60}) / (3 + a_{\overline{57}|0.007917}) = 477.16$$

2) The lessor gives a plant for 3 years, with advance monthly rent, without earnest, providing the redemption as 2% of the price and with a clause for a decrement of 40% of the rent after 20 month. Calculate the corresponding rents, considering that the price of the plant is €16,500 and the nominal rate 12-convertible is 11.20%.

A. The equivalent monthly rate is 0.009333, the equation to find the rent C for the first 20 months is given by

38 Footnote 37 also holds for (6.69').

39 It has been agreed that the redemption is paid in the month of the last rent; the length is then reduced to 57 months. In this case the rent is:

$$C = 24,000 (1 - 0.05 \cdot 1.007917^{-57}) / (3 + a_{\overline{57}|0.007917}) = 476.80$$

$$-16,500.00 + C (\ddot{a}_{\overline{36}|0.009333} - 0.40 {}_{20/\overline{16}|0.009333}) + 330.00 \cdot 1.009333^{-36} = 0$$

from which: $C = 610.95$. Therefore, the first 20 rents are €610.95 and the following 16 are €366.57.

6.7.2. *The monetary adjustment in leasing*

In section 6.5, which was dedicated to the adjustment and indexation in the amortization of an unshared loan, we considered the remedies to cover the creditor from monetary depreciation in a long-term operation. As the leasing can also be considered as a pluriennial loan, for this problem the same remedies can be applied, then we refer to those, limiting ourselves here to a brief discussion.

For the phenomenon of the purchase power variation, and in particular of depreciation, two remedies are used:

1) *line interest compensation*, through a procedure of varying rates that are the sum of a fixed real remuneration share i_h and a varying share Δi_h of compensation nature if it is adjusted to the level of the monetary depreciation rate;

2) *line value compensation*, if the same plant value (which is under a real financial amortization given the criteria for the calculation of the rent) is indexed proportionally to a statistical series of prices representing the interested phenomenon.

6.8. Amortizations of loans shared in securities

6.8.1. *An introduction on the securities*

In the previous chapter we examined methods to manage the remuneration and repayments of loans with two contractual parts: lender and borrower. However, loans of a large amount to relevant companies frequently occur. Then it is practically impossible to realize such operations by only one lender, and therefore many lenders will share the debt.

Such operations are then realized in the following ways:

1) *many private lenders*, which give the money against an obligation of repayment and a credit security;

2) *brokerage by third party*, in the sense that a bank or a group of banks formalize the obligations and securities, collect the money in the “stock market” of the subscribers of the credit securities (using its own organization through a Stock Exchange and its own branches), and give the debt sum in one or more “slices”;

3) *public guarantees*, in the case of loans for public enterprise.

The stock market offers many possibilities for financial investments, typical or not. We will consider here only credit securities for which the principal to be paid back is well determined (even if interest can be paid according to varying rates). We will then not consider:

- equity shares, that from the juridical viewpoint are joint ownership stocks;
- “investment funds” which are prevailingly formed by mixtures of shares and bonds, that have risky elements and are sometimes linked to an insurance component;
- values due to rights linked to share exchanges, that have their own specificity and autonomy and are traded in the “derivative market”.

The description of most of these financial products can be found in the second part of this book. However, for further information the interested reader can refer to specific books.

A fundamental distinction between credit instruments placed against a shared loan between many creditors is that between:

- a) *Treasury Bonds* (placed by the State) with one maturity;
- b) *bonds*, which can have different type of redemption. For these, we must make a further distinction:
 - b₁) bonds with redemption at only one maturity for all creditors; and
 - b₂) bonds with redemption at different maturities amongst the creditors.

If the length of the operation is not longer than one year, the return for the investor is obtained through a purchase cost discounted with respect to the redemption amount. This cost can depend on the dynamics of the negotiation during the “auction” in which the bonds are placed. The financial regime that follows is that of the rational discount⁴⁰ (see Chapter 3).

If the length of the operation is pluriennial, and n is the number of years, the interest (through coupons) with delayed semiannual or annual due date on the basis of nominal rate – also termed *coupon rate*, constant or varying according to a prefixed rule – is usually paid. In this case the interest is a form of “*detached return*” of the security. We must distinguish for each security between the *issue value* p and the *redemption value* c , which we assume coincident with the *nominal*

40 If 100 is the redemption value of the bond, not considering taxes, the purchase price A is linked to the annual rate i and to the days of investment g by the relation: $A = 100/(1+ig/360)$.

value on the security (if the last two are different, for financial purposes, only the redemption value must be considered):

- if $p < c$, we talk about *issue at a discount* or *below par*;
- if $p = c$, we talk about *issue at par*;
- if $p > c$, we talk about *issue at a premium* or *above par*.

6.8.2. Amortization from the viewpoint of the debtor

The debtor (issuer) must plan an *amortization schedule* for the whole debt with one of the methods considered before for the unshared loan. The presence of many creditors is irrelevant from a financial point of view; there are only the practical complications of dividing amongst them the payments for redemptions and interest, called *coupons*. Let us assume 0 as the issue time of the loan and suppose the absence of adjustment. Furthermore, let N be the *number* of issued bonds, each with an *issue value* p and *redemption value* c , and j the annual coupon rate for the computation of delayed interest, that is *nominal 2-convertible* if the coupons are semiannual.

Given that, in case b_1 we can apply the scheme, seen in section 6.1, of one final lump-sum at maturity n and periodic payment of interest, dividing both of them amongst the issued bonds. Therefore, in this case we can immediately verify that for the issuer against the income supply $(0, +Np)$, the amortization consists of the outflow supplies:

- $(1, -Ncj) \cup (2, -Ncj) \cup \dots \cup (n-1, -Ncj) \cup (n, -Nc(1+j))$, for annual coupons;
- $(1/2, -Ncj/2) \cup (1, -Ncj/2) \cup \dots \cup (n-1/2, -Ncj/2) \cup (n, -Nc(1+j/2))$, for semiannual coupons.

In case b_2) we can apply, for the issuer, the general scheme of gradual delayed amortization seen in section 6.2, fixing the redemption plan, i.e. the number N_h of securities to redeem completely at the end of each year h , with the obvious constraint: $N = \sum_{h=1}^n N_h$. In fact, a gradual amortization for each bond is inconvenient. We can then calculate the numbers

$$L_h = N - \sum_{k=1}^h N_k ; h = 1, \dots, n \quad (6.70)$$

which identify the numbers of bonds “alive” soon after the h^{th} gradual redemption, i.e. not redeemed at times $k \leq h$. For L_h the recursive relation holds

$$L_h = L_{h-1} - N_h \quad ; \quad L_0 = N \quad ; \quad \text{then } L_n = 0 \tag{6.70'}$$

It is clear that the issuer must also pay the annual interest cj or semiannual $cj/2$ on each of the alive bond. Therefore, against the income supply $(0,+Np)$ the amortization consists of the outflow supplies:

- $\bigcup_{h=1}^n (h, -N_h c - L_{h-1} c j)$, (annual coupons);
- $\left[\bigcup_{h=1}^n (h, -N_h c - L_{h-1} c j / 2) \right] \cup \left[\bigcup_{h=1}^n (h - 1/2, -L_{h-1} c j / 2) \right]$, (semiannual coupons).

To summarize, with annual coupons the installment to be paid by the issuer is

$$R_h = N_h c + L_{h-1} c j \quad , \quad (h = 1, \dots, n) \tag{6.71}$$

while for semiannual coupons, the interest is divided into two equal amounts.

The one lump-sum redemption of all securities implies a large financial need for the issuer at time n , that – if not covered by a previous new bonds issue – can be very difficult to realize for a private company without adequate means and guarantees, which can also be used to become trusted by the creditor; therefore form b_1 is more adequate for *Treasury Bonds* or public securities. On the contrary, form b_2 allows for a gradual repayment, by choosing in a suitable way the sequence $\{N_h\}$ in relation to the incomes following the investments financed by such loan, and it is suitable for loans to companies with private structure.

6.8.3. Amortization from the point of view of the bondholder

Referring to the bondholders-creditors, we need to distinguish case b_1 from case b_2 and the following considerations hold.

In case b_1 the number of creditors does not change the amortization procedure, in the sense that for the bondholder of each of the N bonds the amortization is with one final lump-sum at maturity n , the same for all bondholders, with periodic payment of interest on the basis of the same parameters. The financial operation is obtained from the one described in section 6.6.2 for the issuer dividing it into N equal parts (with administrative complications due to the large number of counterpart⁴¹) and

41 Such complications disappear when the bond loan is entirely subscribed by a large company, public or private. In such cases, the operation is equivalent to an unshared loan,

changing the sign, i.e. dividing by $-N$. Then each bond at the issue date⁴², against the payment of the amount p , must receive the supplies:

- $(1,cj) \cup (2,cj) \cup \dots \cup (n-1,cj) \cup (n,c(1+j))$, for annual coupon;
- $(1/2,cj/2) \cup (1,cj/2) \cup \dots \cup (n-1/2,cj/2) \cup (n,c(1+j/2))$, from semiannual coupon.

The case b_2 , usual for pluriennial bonds of large amount, which are sold at the inception to a great number of private investors, implies for the bondholder of each of the N bonds an amortization with one final lump-sum redemption, but *with staggered redemption dates*. The rate evolution on the stock market is the cause of a continuous and varying spread between the current rate, for reinvestment after redemption, and the nominal one on the current loan. Therefore, at time h , according to the sign of the spread, all (if the spread is positive) or none (if the spread is negative) of the L_{h-1} residual bondholder are interested to be included amongst the N_h redeemed. To avoid complications and to obtain the fairness among the creditors with a symmetric situation between the residual bondholders, the system of *amortization by drawing* is common, in the sense that the *repayment schedule* becomes a *drawing schedule* to concretely find at time h the N_h bonds (simple, i.e. not considering possible grouping in multiple bonds). The bonds subject to this type of management are termed *drawing bonds*.

In this form, while from the viewpoint of the issuer the financial operation is certain, from the point of view of the bondholder for each security we have a *stochastic maturity*, then *the amortization cash-flow is stochastic in length, with one lump-sum redemption and periodic (annual or semiannual) inflow of interest*.

6.8.4. Drawing probability and mean life

Proceeding with the consideration of hypothesis b_2 that implies for the bondholder the randomness due to the drawable bond system for redemptions, it is appropriate to find the drawing probability at a given integer time $h \leq n$. For reasons of symmetry the probability, valued at issue date, of drawing a bond at time h (i.e. of a life of h years from the issue) can be assumed equal to N_h/N , ratio of bonds issued that are redeemed after h years, while the probability that a bond still not

where bonds are only used for tax advantages and the possibility of placing the bonds in the exchange market. Another form that simplifies the loan amortization is that, which is widely applied in mature economies, of the purchase of their own bonds on the exchange market, which is convenient when the current cost rate is lower than the loan rate.

⁴² If the bondholder is *incoming*, buying the security at integer time r (simplifying hypothesis which ignores here the “day-by-day interest”) at price p_r and if the bondholder waits for the maturity without selling, the inflow operation is for him:

- $(r+1, cj) \cup (r+2, cj) \cup \dots \cup (n-1, cj) \cup (n, c(1+j))$, with annual coupons;
- $(r+1/2, cj/2) \cup (r+1, cj/2) \cup \dots \cup (n-1/2, cj/2) \cup (n, c(1+j/2))$, with semiannual coupons.

drawn at time r has a residual life of h years can be assumed equal to N_{r+h}/L_r , the ratio of residual bonds at time r that are redeemed after other h years.

It is also interesting to consider, in order to summarize with just one number the length of the investment for the bondholder as it occurs in the case of certain maturity, the *mean life* for the generic bond of a given loan⁴³.

We can calculate the *mean life at issue date* as a weighted arithmetic average of the lengths, expressed by the formula

$$e_0 = \sum_{h=1}^n h \frac{N_h}{N} \tag{6.72}$$

It is also useful to evaluate, in the case of purchase or assignment r years after the issue date, the variation of *residual mean life* of a bond still not drawn at time r , expressed by

$$e_r = \sum_{h=1}^{n-r} h \frac{N_{r+h}}{L_r} \tag{6.72'}$$

Example 6.5

Let us consider an amortization for a bond loan, gradual for the issuer and then with a drawing plan for the bondholder, issued *at a discount*. Let us take, with amounts in €:

- $p = 1,760$ = issue value;
- $c = 2,000$ = nominal and redemption value;
- $j = 6.2\%$ = annual coupon rate;
- $N = 10,000$ = number of issued bonds;
- $n = 5$ = length of the loan;
- $\{N_h\} = \{1,500, 1,800, 2,500, 1,600, 2,600\}$ = draws plan.

It follows that the number of residual bonds after each draw is: $L_1 = 8,500$, $L_2 = 6,700$, $L_3 = 4,200$, $L_4 = 2,600$ and $L_5 = 0$. The inflow for the issuer at 0 is 17,600,000 gross of inflow costs, while the whole debt is €20million, not considering the management costs.

With an annual coupon, their value is €124.00 and the annual installments for the payment to the creditor are:

43 The bond mean life is a concept analogous to the mean life of a person, valued at his birthday, for which mortality is measured by means of a demographic table. In probabilistic terms, the bond mean life is the expected value of its random length. In fact, (6.72) expresses it as the ratio between the whole life length of all bonds, according to the redemption schedule, and the numbers of issued bonds.

$$R_1 = 2,000 (1,500 + 0.062 \cdot 10,000) = 4,240,000.00;$$

$$R_2 = 2,000 (1,800 + 0.062 \cdot 8,500) = 4,654,000.00;$$

$$R_3 = 2,000 (2,500 + 0.062 \cdot 6,700) = 5,830,000.00;$$

$$R_4 = 2,000 (1,600 + 0.062 \cdot 4,200) = 3,720,000.00;$$

$$R_5 = 2,000 (2,600 + 0.062 \cdot 2,600) = 4,240,000.00.$$

With a semiannual coupon, their value is €62.00 and the semiannual installments for the payment to the creditor are:

$$R_{1/2} = 2,000 (0.031 \cdot 10,000) = 620,000.00;$$

$$R_1 = 2,000 (0.031 \cdot 10,000 + 1,500) = 3,620,000.00;$$

$$R_{3/2} = 2,000 (0.031 \cdot 8,500) = 527,000.00;$$

$$R_2 = 2,000 (0.031 \cdot 8,500 + 1,800) = 4,127,000.00;$$

$$R_{5/2} = 2,000 (0.031 \cdot 6,700) = 415,400.00;$$

$$R_3 = 2,000 (0.031 \cdot 6,700 + 2,500) = 5,015,400.00;$$

$$R_{7/2} = 2,000 (0.031 \cdot 4,200) = 260,400.00;$$

$$R_4 = 2,000 (0.031 \cdot 4,200 + 1,600) = 3,460,400.00;$$

$$R_{9/2} = 2,000 (0.031 \cdot 2,600) = 161,200.00;$$

$$R_5 = 2,000 (0.031 \cdot 2,600 + 2,600) = 5,361,200.00$$

The mean life at the issue date, due to (6.72), is 3.2 years = $2y+2m+12d$, while the residual mean life at time 3, due to (6.72)', is 1.619 years = $1y+7m+13d$.

6.8.5. Adjustable rate bonds, indexed bonds and convertible bonds

Introduction

Modern capitalistic economies are characterized by a strong dynamism, by a wide variety of technical schemes for investments also by monetary systems, which are subject to variations of the purchasing power from which the investors must protect himself. Thus, even the management of shared loans, as that of unshared loans, considered in sections 6.2 and 6.3, is subject to adjustments and variations that make them more interesting for investors.

The listing and description of such investments would be too long if we wanted to consider all the modalities that sometimes have a very short life, because due to needs which are not valid any more.

It is then sufficient to briefly consider a few types, general and consolidated, which are widely applied.

Bonds with adjustable rate

As for the unshared loans, the issue of bonds can provide, as safeguard against inflation or better adjustment of the investment to the evolving market conditions, for an adjustable nominal rate, according to an appropriate linking rule to external parameters that allow not only the recovering of inflation and/or the adjustment due to the market measured at the issue date, but also during the time towards maturity.

We can then provide in the previous formulae for the substitution of the fixed rate i with a varying rate $i^{(h)}$ with current year h , and then adapt all the results.

Indexed bonds

Due to the requirement of protection against inflation, we can prefer, for a better recovery both on interest and principal, to leave the bond rate to a real return level and make the nominal value c varying and adjustable substituting it in the previous developments, both for redemption and for the calculation of semiannual or annual interest, with an amount $c^{(h)}$ varying with the current year h , indexed proportionally to an appropriate statistical series, for example to the consumer price index.

A more detailed formulation on the valuation of updated rates and indexed bonds will be given in section 6.9.4.

Convertible bonds

Convertible bonds are more complex and require a more in-depth discussion.

We can limit ourselves, here, by saying that a firm that wants to increase its capital, can initially collect money in the loan market as *credit capital* leaving the possibility to the subscribers – with appropriate limits and according to prefixed exchange ratios – to convert, in a given temporal interval, the *credit capital* into *risky capital*. In this way they become shareholders, then co-owners and partners in the enterprise. This is due to a number of reasons of convenience, also tax reasons, that allows the redemption of the debt by means of compensation with capital increasing.

6.8.6. Rule variations in bond loans

Bond loans often provide for variations that modify the cost and return parameters and that must then be taken into account. Leaving to the reader the easy calculation of the financial effect of such variations, we limit ourselves here to

listing the most frequently used variations, warning that it is almost impossible to give a complete view of this topic.

1) *Redemption value higher than nominal value*

This is an additional premium and higher cost for the debtor. To calculate it, it is enough to take into account this redemption price, no longer the nominal value.

2) *“All inclusive” bonds with premium*

For such bonds, there is no payment of interest in the drawn year. Our formulae are adapted to this case decreasing the redemption value by the amount of the coupon.

3) *Bonds with premium*

A total premium amount $P^{(h)}$ can be provided for bonds drawn at year h . The debtor must take them into account adding C to $N_h c$ while for the bondholder the redemption value c is on average increased by $P^{(h)}/N_h$.

4) *Bonds with incorporated interest (= full accumulation)*

The loan can provide for the absence of coupons and a redemption value increasing with time, together with interest. It is obvious that the return for the different length h is found by considering the redemption value as an accumulated value after h years of the purchase price.

5) *Bonds with pre-amortization*

It can happen that there are no redemptions for the first h years, i.e. $N_1 = N_2 = \dots = N_h = 0$. In this case, not having redemptions, the cost of interest for the debtor concerns all the issued bonds for the whole length of the pre-amortization.

6.9. Valuation in shared loans

6.9.1. Introduction

In section 6.8 we examined, from an *objective* point of view, the problem of management and amortization of loans shared in bonds. In this section 6.9 we will consider the problem of *subjective* valuation, made at inception or during the loan life, of the residual rights connected to owning the bonds, from the point of view of the creditor bondholder, the debtor (issuer) and a potential buyer. The logic is then that behind Makeham’s formula and the more convenient choice between alternative investments on the basis of comparison rates fixed by the decision-maker.

We limit ourselves to the case of gradual amortization of a bond loan that, as we have seen, implies a pluriennial repayment plan by means of draws and randomness for the bondholder (but not for the issuer) of the values, and also of the usufructs and bare ownerships. Also for the valuation this is the more interesting case that gives rise to higher complexity.

Indeed, if the *issue is not at par*, only *ex-post*, after the draw, it is possible for the investor to calculate the effective return exactly⁴⁴. In fact, the difference (positive or negative) $c-p$ between return at redemption and purchasing cost is a “capital gain” or “capital loss”, i.e. an “incorporated” return component that, from a previous point of view, is gained (or lost) in a random number of years T , where $(c - p)/(pT)$, i.e. the intensity, is also random. It follows that the IRR of the given bond investment is random. We will consider, for valuations and choices, an appropriate functional average, called the *ex-ante mean rate of return* for the bond, coinciding with j for at par issue.

In the not at par issues we can highlight the *immediate rate of return* or *current yield*, given by cj/p , that measures the return of the investment p given by the coupon, but not considering the capital gain or loss⁴⁵.

Everything will be clarified in what follows, starting from the case of a bond with a given maturity.

6.9.2. Valuation of bonds with given maturity

Let us consider first the model that follows from the hypothesis of certainty of the length, i.e. assuming that the bond has a given maturity. This can occur:

- a) if all the bonds have a common maturity. This is case b_1 of amortization with one lump-sum redemption, where for both parties length and returns are certain;
- b) only for the bonds that will be redeemed at a given maturity, in the drawing bond case.

Valuing from the bondholder point of view, let us consider the bonds that will be called after s years from the issue date, i.e. all in case a) with common maturity s , or only the N_s , ($1 \leq s \leq n$), defined in case b).

With the usual symbols, in the case of annual coupons, with i being the effective delayed annual evaluation rate (subjectively chosen according to the market

44 We highlight that if the issue is at par, the randomness of the length does not imply the randomness of the IRR, that coincides with the coupon rate, as it is obvious for the financial equivalence principle. Analytically we can deduce that the *issue at par is a necessary and sufficient condition such that IRR = j*. *Proof*: necessity: if $IRR = j$, let T be the random length, b the issue price and c j being the coupon, must be: $-b + c j [1 - (1+j)^{-T}]/j + c (1+j)^{-T} = 0 \forall T$, then $b = c$. *Proof*: sufficiency. If the issue is at par, $-c + c j [1 - (1+x)^{-T}]/x + c(1+x)^{-T} = 0$, where $x = IRR$, then: $cj[1 - (1+x)^{-T}]/x = c[1 - (1+x)^{-T}] ; j/x = 1$, i.e. $x = j$.

45 In the not at par issues the immediate rate is obviously always between the coupon rate and the mean rate (or certain rate) of effective return. In at par issues, all the aforesaid rates coincide.

behavior and to the returns of alternative investments), indicating with $W_0(i)$ the valuation at issue date (in $r=0$) on the basis of the expected encashment of the bondholder, dependent on the return rate i , results in

$$W_0(i) = c j a_{\overline{s}|i} + c (1 + i)^{-s} \tag{6.73}$$

(independent of the issue date because we adopted a uniform financial law). The symbol W means that this value coincides with the pro-reserve evaluated just after the purchase.

Assuming a market logic (a topic which will be more fully developed in Chapter 7), we indicate with $z^{(s)}$ the purchase price of the bond at issue date ($z^{(s)} < c$ if at a discount, $z^{(s)} = c$ if at par, $z^{(s)} > c$ if at a premium). Thus, solution x , existing and unique, of the equation in i

$$c j a_{\overline{s}|i} + c (1 + i)^{-s} = z^{(s)} \tag{6.74}$$

(that, due to (6.73), expresses the equality between the value $V_0(i)$ and the price $z^{(s)}$ at time 0) is the IRR, the rate to which the mean return rate is taken back, given that in the bond investment with certain length the return rate is not random but certain, even with not at par issues.

Given that $W_0(i)$ is a decreasing function of i and that $i=j$ if the bond is issued at par, it is obvious that in the *at discount* case the solution for i in (6.74) is $x > j$, while in the *at premium* case the solution for i in (6.74) is $x < j$.

Constraint (6.74) between price $z^{(s)}$ and IRR in case of a given maturity acts biunivocally: given the wanted IRR, we obtain the corresponding issue price; and conversely, given the price $z^{(s)}$, we find the IRR as rate x that makes fair the operation to pay $z^{(s)}$ and to cash s annual coupon cj and the redemption c after s years. Clearly, at fixed c and j , the IRR is a decreasing function of $z^{(s)}$.

In (6.74), using the solution value x instead of i , then $z^{(s)}$ is obviously also the *value* $W_0(x)$ of the bond at rate x , while the two addenda at the left side form, respectively, the *usufruct* and *bare ownership* of the bond at rate x .

If, instead, at issue date the valuation is made at an intermediate time (integer) $r > 0$, then $z^{(s)}$, to be written $z^{(s-r)}$, becomes the “forward” in r of the security on the

exchange market and it is enough to substitute the residual life $s-r$ instead of maturity s ⁴⁶.

With a semiannual coupon, it is enough to consider in (6.74) the fractional annuity $a_{\overline{s}|x}^{(2)}$ and make the appropriate changes.

Example 6.6

For valuations connected to the return, it is enough to consider a single bond, even better a virtual share, putting the nominal value (that we suppose equal to the redemption value) equal to 100. Let us consider a security with certain maturity, which pays semiannual coupons at the nominal rate 7% (semiannual convertible) and is redeemed after 8 years. Let us assume the time unit is a half-year and let us put the time origin 0 at purchase (at issue date or a following one in the market of issued securities) of this share; then put in 16 the redemption time.

The purchase price P that assures an annual effective return of 6% (being $i_2 = \sqrt{1.06} - 1 = 0.029563$ the semiannual rate equivalent to 6% annually) is given by

$$P = 3.5 (1 - 1.029563^{-16})/0.029563 + 100 \cdot 1.029563^{-16} = 106.85$$

then the purchase is “at premium”, given that the annual effective return rate of 6% below the coupon annual effective rate, equal to 7.1225% corresponding to a nominal rate of 7%. The usufruct is the first addend of the right side, whose value is 44.11. The bare ownership is the second addend, whose value is 62.74.

Example 6.7

Let us consider a bond with certain maturity and the following data: nominal value and also redemption value at 9 years after the purchase = 100; annual coupons at rate of 6%; purchase price = 94.65, then the bond is “at discount”. The current yield is by definition: $6/94.65 = 6.3391\%$. The IRR, that measures the effective return with the “capital gain”, is solution x of the equation in i

$$-94.65 + 6 [1 - (1 + i)^{-9}]/i + 100 (1 + i)^{-9} = 0$$

46 Precisely the pro-reserve W_r in $r > 0$, dependent on i , is obtained from the right side of (6.73), using $s-r$ instead of s . A simple calculation shows that the following recursive between subsequent values of W_r , dependent on the IRR of the security: $W_r = (1+i)^{-1}(cj+W_{r+1})$ with $W_s = C$ (thus putting the redemption soon after time s). In fact, in $r < s$ the bond with value W_r gives right after one year, accumulating at rate i , to the coupon cj and to further rights valued W_{r+1} at time $r+1$. Such a simple formula is useful to calculate, using *Excel*, the sequence of residual values at integer times between 0 and s . From another point of view W_r is in $r = 0$ the *spot price* at issue date and in $r > 0$ the *forward price*, which are found from the right side of (6.73), in biunivocal correspondence with the value $I = x = \text{IRR}$.

It is found with appropriate methods (it is sufficient a financial calculator) to be:

$$\text{IRR} = 6.8147\% > 6.3391\% (= \text{current yield}) > 6\% (= \text{coupon rate})$$

6.9.3. Valuation of drawing bonds

inlet us consider case b_2 with repayments in n years randomly, according to a draw of N_1, \dots, N_n drawing bonds in the years $1, \dots, n$. Thus after the h^{th} draw the number of bonds L_h is given by (6.70). Therefore, extending the considerations of section 6.9.2, given the symmetry between the securities, the issue price z is found to equal the price Nz of the whole of the bond issue to the sum of the present values, calculated according to the prefixed IRR x , of the number of bonds which have to redeem at different maturities s , the number of which N_s is previously known.

Thus, the following relation holds

$$Nz = \sum_{s=1}^n N_s c j a_{\overline{s}|x} + \sum_{s=1}^n N_s c (1+x)^{-s} \quad (6.75)$$

i.e.

$$z = \sum_{s=1}^n N_s z^{(s)} / N \quad (6.75')$$

that expresses z as the weighted mean of $z^{(s)}$ with weights N_s/N , which express the probabilities, valued at issue date, of draw after s years.

In (6.75) the 1st addendum of the right side expresses the usufruct and the 2nd addendum the bare ownership, referred to the whole of the N bond issue. Therefore, we find, dividing by N , the *mean usufruct* u_0 and the *mean bare ownership* np_0 of a single bond at time 0.

Equation (6.75), with given z and unknown x , is also the equation that gives (univocally for the algebraic properties of (6.75)) the IRR as the *mean effective yield rate*⁴⁷ of the investment at price z . Instead, the *ex-post yield rate*, in the case of a draw after s years, is found by solving (6.75) with respect to the unknown rate x , with the value of s corresponding to the verified time of draw.

47 We must highlight that the *mean effective yield rate* is not the real profit rate for the investor in a bond, taking into account incorporated revenues and costs; this is the *ex-post* rate, valuable only after the bond call. In fact, the *mean effective yield rate* is a suitable functional mean of feasible *ex-post* rates owing to drawing.

If the valuation is performed at time $h \geq 0$, in (6.75') it is enough to add from 1 to $n-h$ and substitute N_{h+s}/L_h to N_s/N . In this way the mean values z_h , the mean usufructs u_h and the bare ownerships np_h for each bond still alive at time h are obtained, resulting in

$$\left\{ \begin{aligned} u_h &= \sum_{s=1}^{n-h} \frac{N_{h+s}}{L_h} c j a_{\overline{s}|x} ; & np_h &= \sum_{s=1}^{n-h} \frac{N_{h+s}}{L_h} c(1+x)^{-s} \\ z_h &= \sum_{s=1}^{n-r} \frac{N_{h+s}}{L_r} (c j a_{\overline{s}|x} + c(1+x)^{-s}) = u_h + np_h \end{aligned} \right. \quad (6.76)$$

Example 6.8

Let us value the prices, the mean usufructs and bare ownerships, at issue and after 2 years, of the drawing bonds loan, the data of which are:

$$n = 5; N = 1,000; c = \text{€}5,000; j = 5.60\%; x = 6.14\%; \\ N_1 = 150; N_2 = 170; N_3 = 200; N_4 = 230; N_5 = 250.$$

To apply the resolving formulae we build the following table.

s	$a_{\overline{s} x}$	$(1+x)^{-s}$	$z^{(s)}$	N_s/N	L_s
	(1)	(2)	(3)	(4)	(5)
1	0.942152	0.942152	4,974.56	0.15	850
2	1.829802	0.887650	4,950.59	0.17	680
3	2.666103	0.836301	4,928.01	0.20	480
4	3.454026	0.787923	4,906.74	0.23	250
5	4.196369	0.742343	4,886.70	0.25	0

Table 6.18. Elements for calculating values, usufructs and bare-ownerships

– The price z_0 at issue, corresponding to IRR 6.14%, is the arithmetic weighted mean of values $z^{(s)}$, obtainable as a scalar product of vectors (= component product sum) given by columns 3 and 4: $z_0 = \text{€}4,923.61$;

– We obtain the mean usufruct at issue by scalar product of column vectors 1 and (4), then multiplying by $c j = 280$: $u_0 = \text{€}792.16$;

– We obtain the mean bare ownership at issue by the scalar product of column vectors 2 and 4, then multiplying by $c = 5,000$: $np_0 = \text{€}4,131.45$.

$u_0 + np_0 = \text{€}4,923.61$ gives the value z_0 in another way.

Valuing after 2nd refund ($r=2$), with residual time length of the loan = 3, we have to repeat the procedures of calculation already shown, but limit ourselves to the averages of the first three elements of columns 1 and 2, taking as weights the

redemption percentages $N_3/L_2 = 200/680 = 0.294118$; $N_4/L_2 = 230/680 = 0.338235$; $N_5/L_2 = 250/680 = 0.367647$. We obtain:

$$u_2 = 525.33 ; np_2 = 4,424.01 ; z_2 = u_2 + np_2 = 4,949.34.$$

Particular case: constant principal repayments

If N is a multiple of n , we can choose $N_r = \text{const.} = N/n$. By introducing this value in the 3rd equation into (6.76), we find

$$\frac{z_h}{c} = \frac{a_{\overline{n-h}|x}}{n-h} + \left(1 - \frac{a_{\overline{n-h}|x}}{n-h}\right) \frac{j}{x} \quad 48 \quad (6.76')$$

Exercise 6.13

Let us consider a bond loan of €750,000 shared into 750 bonds redeemable at nominal value according to draw, with constant principal repayments and annual coupons, with the following parameters:

- length in years $n = 10$
- coupon rate $j = 5.5\%$
- mean effective yield rate $x = 6\%$

calculate for one bond the issue price and the forward price after the 3rd draw, which realize the assigned yield at 6%.

A. The unitary result does not depend on the number of issued bonds. Applying (6.76'), the following is obtained:

$$\text{at issue date } (h=0): \quad \frac{z_0}{1000} = \frac{7.3600871}{10} + \left(1 - \frac{7.3600871}{10}\right) \frac{5.5}{6.0}$$

$$z_0 = 1000 (0.7360087 + 0.2639913 \cdot 0.9166667) = 978.0007$$

$$\text{after 3 years } (h=3): \quad \frac{z_3}{1000} = \frac{5.5823814}{7} + \left(1 - \frac{5.5823814}{7}\right) \frac{5.5}{6.0}$$

$$z_3 = 1000 (0.7974831 + 0.2025169 \cdot 0.9166667) = 983.1236$$

48 Equation (6.76') shows that z_h/c is a weighted mean between 1 and j/x with weights varying with h . Therefore $z_h < c$ iff $j < x$ (at discount) while $z_h > c$ iff $j > x$ (at premium).

Particular case: debtor installments (almost) constant

The redemption of a bond loan can also be made with constant annual delayed payments for the issuer on the model of the French amortization. Therefore a constant installment for the loan of the type $R = Nc/a_{\overline{n}|j}$ is to be valued. Then, the value of redemption of each security being constant c , both the total redemption amount at time r and the numbers N_r of redeemed bonds have to increase in geometric progression with ratio $(1+j)$. Thus, it must be $N_r = k(1+j)^r$ and from

$\sum_{r=1}^n N_r = N$ follows:

$$k = N \ddot{\sigma}_{\overline{n}|j} \quad ; \quad N_r = N \ddot{\sigma}_{\overline{n}|j} (1+j)^r$$

to substitute in (6.76) for $h=0$. An easy calculation leads to the formula

$$\frac{z_0}{c} = \frac{j}{x} + (1 - \frac{j}{x})np_0 \tag{6.76''}$$

where in this case the total redemption shares discounted are constant and then: $np_0 = nc \ddot{\sigma}_{\overline{n}|j}$, highlighting that z_0/c is a weighted mean between 1 and np_0 . The changes to make the calculation for z_h with $h>0$ are obvious.

In addition, we have to observe that the values N_r previously obtained are always integer. Therefore, this scheme must be corrected by approximating for each year the theoretical number N_r , by its floor and transferring to the following year in acc/repayments the accumulated value of the not amount used, then valuing the new number of bonds to redeem, always rounding off at integer, and carrying on this way till the term.⁴⁹

Exercise 6.14

Let us consider the bond loan with data of Exercise 6.13 but ruled by constant installments. Not considering the rounding off to obtain integer numbers, calculate such theoretically drawn numbers and also the issue price of one bond.

A. Using the formulae discussed above, as $\ddot{\sigma}_{10|0.055} = 0.073619$, we find:

$$\begin{aligned} N_1 &= 750 \cdot 0.073619 \cdot 1.055 = 58.250827; \quad N_2 = 1.055 \cdot N_1 = 61.454622; \\ N_3 &= 1.055 N_2 = 64.834626; \quad N_4 = 1.055 N_3 = 68.400531; \\ N_5 &= 1.055 N_4 = 72.162560; \quad N_6 = 1.055 N_5 = 76.131501; \\ N_7 &= 1.055 \cdot N_6 = 80.318733; \quad N_8 = 1.055 \cdot N_7 = 84.736263; \\ N_9 &= 1.055 \cdot N_8 = 89.396758; \quad N_{10} = 1.055 \cdot N_9 = 94.313580. \end{aligned}$$

⁴⁹ It is obvious that this rounding operation changes the mean yield rate and the ex-post rates very little with respect to the calculated ones according to the theoretical redemption with exactly constant installments.

To check, adding the numbers above, we obtain 750.

As $np_0 = nc\ddot{\alpha}_{10|0.055} = 10 \cdot 1000 \cdot 0.073619 = 736.19$, the issue value is

$$z_0 = 1000 (0.916667 + 0.083333 \cdot 0.73619) = 978.02.$$

The approach for management years

Equation (6.76) is obtained using a total direct valuation but it is also possible to use the *management years approach*, that offers the advantage of analyzing the temporal development and easily enables a generalization for the hypothesis of varying rate and adjustment of the values.

Proceeding for management years, we find for the year $h+s$ the total amount for the paid coupon by the debtor as interest and for redemptions as principal. This amount originates from the L_h bonds circulating at time h (or from the N bonds issued, if $h=0$). If it is divided by L_h we obtain for symmetry reasons the mean amount s years after h for the generic purchased bond. Then, the value z_h assigned to each bond, on the basis of an appropriate valuation rate x , is given by

$$z_h = \frac{1}{L_h} \sum_{s=1}^{n-h} (L_{h+s-1}ci + N_{h+s}c)(1+x)^{-s} \tag{6.77}$$

It is easy to show algebraically the equivalence between the last equation in (6.76) and (6.77). Furthermore in (6.77) L_{h+s-1}/L_h and N_{h+s}/L_h are respectively the probabilities to be drawn for bonds not drawn till h , of no drawing for another $s-1$ years and to be drawn in the following year. Given that $L_{h+s-1} = L_{h+s} + N_{h+s}$, the total amount of year s can be written as $L_{h+s-1}ci + N_{h+s}c(1+i)$, distinguishing for the circulating bonds at the beginning of the year $h+s$ the amount for interest for the bonds not drawn in the year and the amount for interest and redemptions for the drawn bonds⁵⁰.

50 Let us find here a relevant property for z_h . Indicating with $c^* = ci/x$ the capital that reproduces the annual coupon given at rate x and resulting for equivalence

$$\sum_{s=1}^{n-h} (L_{h+s-1}c^*x + N_{h+s}c^*)(1+x)^{-s} = c^*L_h,$$

(6.77) can be written as

$$z_h = c^* + (c - c^*) \sum_{s=1}^{n-h} N_{h+s} (1+x)^{-s} / L_h,$$

where, with an obvious financial interpretation, the result of Σ is: $0 < \Sigma < 1$. Therefore, z_h is always between c^* and c .

If we use the delayed semiannual interest coupon, the same correction factor $x/2(\sqrt{1+x}-1)$, that transforms the value of the delayed constant annual annuity in to that of the semiannual fractional annuity (see Chapter 5), must be introduced in the valuation.

By introducing a direct argument, if we are using a semiannual coupon we have to replace ci by the accumulated value at the year's end at rate x of the two semiannual coupon $ci/2$, i.e. the value $cix[\sqrt{1+x}-1]/2$.

In practice, with a semiannual coupon it is enough to substitute in (6.77) the annual coupon rate i for its transformed one $i' = ix/2(\sqrt{1+x}-1)$.

Recursive relation of a bond value at fixed coupon rate

In addition, for the valuation of bond loans with drawing redemption at any rate x^* we can consider the dynamic aspect on the basis of the management years approach. Using the symbols already defined, the relation⁵¹ between subsequent values z_h valued at rate x is as follows:

$$L_h z_h = (1+x)^{-1} (cN_{h+1} + cjL_h + z_{h+1}L_{h+1}), \quad (h = 0, \dots, n-1) \quad (6.78)$$

Equation (6.78) extends, to the loans shared in bonds, the recursive relation examined in section 6.2 for the unshared loan and is based on a principle of preserving the value in equilibrium conditions, expressing the equality between the valuation of residual securities at time h , and the sum of the differently used amount of such securities in $h+1$, soon after the $(h+1)^{th}$ draw, valued in h . In fact, considering that $N_{h+1} + L_{h+1} = L_h$, at the right side of (6.78) are added for the total loan: 1) the payment in principal for the redemption of drawing bonds in $h+1$; 2) the payments of interest for the living bonds between h and $h+1$; 3) the valuation of residual bonds in $h+1$.

Mathematical life and Achard's formula

Let us define *mathematical life* at time r and rate x the exponential mean of the possible residual life length of a bond still not drawn in r , on the basis of the repayment plan; this indicated by em_r , is implicitly defined by

$$(1+x)^{-em_r} = \sum_{s=1}^{n-r} \frac{N_{r+s}}{L_r} (1+x)^{-s} \quad (6.79)$$

51 The considered recursive relation, concerning random values due to the call, shows an analogy with the known *Fouret's equation* about life insurance theory.

Defining $a_{\overline{s}|x} = (1 - (1+x)^{-s})/x$ as well for non-integer times, the mean value z_r taken from (6.76), given (6.77), can be transformed in

$$z_r = c j a_{\overline{em_r}|x} + c (1 + x)^{-em_r} \tag{6.80}$$

The right side of (6.80) can be split into mean usufruct and mean bare ownership, i.e. $u_r = c j a_{\overline{em_r}|x}$; $np_r = c (1 + x)^{-em_r}$. Therefore, the mean valuation of usufruct and bare ownership, in uncertainty conditions following the repayment plan, are equivalent to the certain ones with length em_r . In other words, the mean financial valuation for random maturity is equivalent to the one that would be obtained with a maturity certain at time $r+em_r$, i.e. after a time equal to the mathematical life.

For the expression of u_r and np_r taken from (6.76), the mean usufruct of a bond with nominal and redemption value c can be expressed according to the mean bare ownership in the form:

$$u_r = \frac{j}{x} [c - np_r] \tag{6.81}$$

that is *Achard's formula*⁵². It particularizes the Maheham's formula on a single bond, given that, as the amortization with one lump-sum redemption at maturity, the intermediate outstanding balances remains always equal to the redemption value c .

6.9.4. Bond loan with varying rate or values adjusted in time

It is known that, to face monetary variations or to adjust pluriennial operations to the changing of market conditions, it is possible to adopt in the management of loans, varying coupon interest rates or indexed outstanding loan balance.

Sometimes such schemes are also adopted in bond loans. In particular, for the valuation considered in this chapter, it is possible to formalize such a scheme if we use the approach for management years described in section 6.9.3.

Let us refer to formula (6.77) and observe that if, due to the varying rates and/or to indexing of values, we assume a sequence of coupon interest rates $i^{(s)}$ and/or a

52 The proof follows from:

$$\begin{aligned} \sum_{s=1}^{n-r} \frac{N_{r+s}}{L_r} c j a_{\overline{s}|x} &= c j \sum_{s=1}^{n-r} \frac{N_{r+s}}{L_r} [1 - (1+x)^{-s}] / x = \\ &= \frac{c j}{x} \sum_{s=1}^{n-r} \frac{N_{r+s}}{L_r} - \frac{c j}{x} \sum_{s=1}^{n-r} \frac{N_{r+s}}{L_r} (1+x)^{-s} = \frac{j}{x} \left[c - c \sum_{s=1}^{n-r} \frac{N_{r+s}}{L_r} (1+x)^{-s} \right] \end{aligned}$$

sequence of redemption values $c^{(s)}$ corresponding to years $s = 1, \dots, n-h$ starting from year $h \geq 0$, then it is enough to replace in (6.77) for each time $h+s$ the coupon constant rate i by the varying rate $i^{(s)}$ and/or the constant unitary debt c by the indexed debt $c^{(s)}$. Considering that usually the indexing of debt is used as an alternative to the variation of coupon rate, the following formulae, that at an appropriate valuation rate x give the pro-reserve of the total outstanding balance at time $h \geq 0$, hold. In the case of *varying coupon rate* the pro-reserve is

$$W_h = \sum_{s=1}^{n-h} (L_{h+s-1} c i^{(s)} + N_{h+s} c) (1+x)^{-s} \quad (6.77')$$

while in the case of *indexing of the outstanding balance* the pro-reserve is

$$W_h = \sum_{s=1}^{n-h} (L_{h+s-1} c^{(s)} i + N_{h+s} c^{(s)}) (1+x)^{-s} \quad (6.77'')$$