

Chapter 17

Portfolio Theory

17.1. Quantitative portfolio management

Let us consider a financial market trading n risky assets called a_1, \dots, a_n ($n \geq 1$). We assume that the absolute returns ξ_1, \dots, ξ_n of these n assets on a fixed time period $[0, T]$ are random variables with means μ_1, \dots, μ_n and variances $\sigma_1^2, \dots, \sigma_n^2$, and moreover as these returns are in general dependent, we have to introduce the following covariances:

$$\sigma_{ij} = E\left[(\xi_i - \mu_i)(\xi_j - \mu_j)\right], \quad i, j = 1, \dots, n. \quad (17.1)$$

The problem of the choice of a portfolio consists of selecting a vector

$$\mathbf{x} = (x_1, \dots, x_n)' \quad (17.2)$$

such that

$$\begin{aligned} \sum_{i=1}^n x_i &= 1, \\ x_i &\geq 0, i = 1, \dots, n \end{aligned} \quad (17.3)$$

under a certain criteria depending on the attitude of the investor against risk. In general, the investors are risk adverse and thus manage their portfolio with a

prudential attitude, but others may be risk lovers, attracted by the expectation of possible high returns.

Mathematically, risk adverse investors having a choice between two portfolios will select the one having a mean turn with the smallest variance provided that the performance of the portfolio is measured with the mean return.

17.2. Notion of efficiency

To find such a portfolio, Markowitz (1959) introduced the concept of efficiency or of efficient, portfolio.

Definition 17.1 A portfolio is efficient if for all the portfolios having the same expectation of return, it is of minimal variance.

Following this definition, there corresponds an efficient portfolio to each fixed expectation of return and so with such a return as variable we obtain a new function which graphs in a plane mean-variance and is called the efficient frontier.

The return of portfolio of vector \mathbf{x}' at time T is given by:

$$R(\mathbf{x}) = \sum_{i=1}^n x_i \xi_i \tag{17.4}$$

Moreover, we have:

$$\begin{aligned} E(r(T)) &= \sum_{i=1}^n x_i \mu_i (= \mu) \\ \text{var } r(T) &= E \left[\left(\sum_{i=1}^n x_i \xi_i - \sum_{i=1}^n x_i \mu_i \right)^2 \right], \\ &= \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_i x_j (= \sigma^2) \end{aligned} \tag{17.5}$$

where

$$\sigma_{jj} = \sigma_j^2, j = 1, \dots, n.$$

The search for an efficient portfolio corresponding to a mean return of value m leads to the following mathematical optimization:

$$\min_{x_1, \dots, x_n} \sigma^2(x_1, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_i x_j, \quad (17.6)$$

under the constraints:

$$(i) \sum_{i=1}^n x_i \mu_i = \mu,$$

$$(ii) \sum_{i=1}^n x_i = 1,$$

$$(iii) x_i \geq 0, i = 1, \dots, n,$$

μ now being a mean return selected by the investor.

Remark 17.1

a) Condition (iii) excludes short sales.

b) The variables $x_i; i = 1, \dots, n$ represent the percentages of the n shares in the portfolio.

To solve this mathematical programming problem with constraints, we must introduce the Lagrangian function L of $n+2$ variables defined by

$$L(x_1, \dots, x_n, \lambda, \nu) = \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_i x_j - 2\lambda \left(\sum_{i=1}^n x_i \mu_i - \mu \right) - 2\nu \left(\sum_{i=1}^n x_i - 1 \right). \quad (17.7)$$

Taking the n partial differentials with respect to $x_i, i = 1, \dots, n$, we obtain the following linear system:

$$\sum_{j=1}^n \sigma_{kj} x_j - \lambda \mu_k - \nu = 0, \quad k = 1, \dots, n$$

or

$$x = \mathbf{V}^{-1}(\lambda \boldsymbol{\mu} + \nu \mathbf{1}) \quad (17.8)$$

where the square matrix \mathbf{V} represents the variance covariance matrix of the vector of returns $\xi = (\xi_1, \dots, \xi_n)'$:

$$\mathbf{V} = [\sigma_{ij}] \quad (17.9)$$

and with the following notations:

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)', \quad \mathbf{1} = (1, \dots, 1)'. \quad (17.10)$$

It is possible to show that the unique solution of this algebraic system (see, for example, Poncet, Portait and Hayat (1996)) is given by

$$\mathbf{x}^* = \mathbf{g} + \mu \mathbf{h} \quad (17.11)$$

where

$$\begin{aligned} \mathbf{g} &= \frac{1}{d} (b\mathbf{V}^{-1}\mathbf{1} - a\mathbf{V}^{-1}\boldsymbol{\mu}) \\ \mathbf{h} &= \frac{1}{d} (c\mathbf{V}^{-1}\boldsymbol{\mu} - a\mathbf{V}^{-1}\mathbf{1}) \end{aligned} \quad (17.12)$$

with

$$\begin{aligned} d &= bc - a^2 \\ a &= \mathbf{1}'\mathbf{V}^{-1}\boldsymbol{\mu} \\ b &= \boldsymbol{\mu}'\mathbf{V}^{-1}\boldsymbol{\mu} \\ c &= \mathbf{1}'\mathbf{V}^{-1}\mathbf{1} \end{aligned} \quad (17.13)$$

The two Lagrange parameters are given by:

$$\begin{aligned} \lambda &= \frac{c\mu - a}{d}, \\ \nu &= \frac{b - a\mu}{d}. \end{aligned} \quad (17.14)$$

As it is also possible to prove that the second conditions order to obtain a maximum are satisfied, we now have the following proposition.

Proposition 17.1 In the plane (σ, μ) , the efficient frontier of the considered portfolio is represented by a hyperbola of equation

$$\frac{\sigma^2}{A^2} - \frac{(\mu - C)^2}{B^2} = 1,$$

where:

$$A^2 = \frac{1}{c}, B^2 = \frac{d}{c^2}, C = \frac{a}{c}. \quad (17.15)$$

coordinates of the vertex:

$$\left(\sqrt{\frac{1}{c}}, \frac{a}{c} \right);$$

coordinates of the center:

$$\left(0, \frac{a}{c} \right);$$

asymptotes:

$$\mu = \frac{a}{c} \pm \sqrt{\frac{d}{c}} \sigma.$$

Proposition 17.2 In the plane (σ^2, μ) , the equation of the efficient frontier takes the form of a parabola of equation:

$$\sigma^2 = \frac{1}{d}(c\mu^2 - 2a\mu + b) \quad (17.16)$$

having as vertex $(1/c, a/c)$.

Proof If σ_{\min}^2 represents the minimum value under constraints of the function defined by relation (17.6), we can write:

$$\sigma_{\min}^2 = \mathbf{x}^{*'} \mathbf{V} \mathbf{x}^*. \quad (17.17)$$

Replacing x from its value given by relation (17.8), we obtain:

$$\begin{aligned}
 \sigma_{\min}^2 &= \mathbf{x}^* \mathbf{V} \mathbf{V}^{-1} (\lambda \boldsymbol{\mu} + \nu \mathbf{1}) \\
 &= \mathbf{x}^* (\lambda \boldsymbol{\mu} + \nu \mathbf{1}) \\
 &= \lambda \mathbf{x}^* \boldsymbol{\mu} + \nu \mathbf{x}^* \mathbf{1} \\
 &= \lambda \mu + \nu.
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{\min}^2 &= x^* V V^{-1} (\lambda \bar{\mu} + \nu \bar{1}), \\
 &= x^* (\lambda \bar{\mu} + \nu \bar{1}), \\
 &= (\lambda x^* \bar{\mu}) + \nu x^* \bar{1}, \\
 &= \lambda \mu + \nu.
 \end{aligned} \tag{17.18}$$

From relations (17.14), we finally obtain:

$$\sigma_{\min}^2 = \frac{1}{d} (c \mu^2 - 2a \mu + b), \tag{17.19}$$

which is relation (17.16). □

Remark 17.2 It is clear that we must only use the upper branch of hyperbola (17.15).

Let us now introduce a non-risky asset of unitary return r on the considered time period $[0, T]$ so that the portfolio may also contain a proportion of this new asset.

If y represents the proportion of this asset in the portfolio, $(1-x)$ will represent the part of the risky efficient portfolio. The mean and standard deviation of this new portfolio are:

$$\begin{aligned}
 \mu_P &= xr + (1-x)\mu, \\
 \sigma_P &= (1-x)\sigma,
 \end{aligned} \tag{17.20}$$

By elimination of x between these two equations, we obtain:

$$\mu_P = r + \frac{\mu - r}{\sigma} \sigma_P \tag{17.21}$$

In the plane (σ_P, μ_P) , this equation represents the tangent from the point $(0, r)$ to the Markowitz hyperbola of equation

$$\frac{\sigma_P^2}{A^2} - \frac{(\mu_P - C)^2}{B^2} = 1. \quad (17.22)$$

Thus, the introduction of a non-risky asset modifies the structure of the curve of optimal portfolios called the efficacious frontier, composed with the tangent up to the efficient frontier and after of the part of the efficient frontier for portfolios without a risky asset.

When the tangent is above the efficient portfolio, the corresponding portfolio no longer satisfies result (17.6) condition (iii) as $(1-x)$ is strictly greater than 1 or $x < 0$. This means that the investor borrows from the bank at rate r to buy the risky asset part of his portfolio, increasing his mean return but also his risk!

Such an investor is clearly a risk lover attracted by high return expectation.

17.3. Exercises

1) Prove that every linear convex combination of two efficient portfolios is still an efficient portfolio.

2) Let us consider two efficient portfolios of mean returns $\mu_1, \mu_2 (\mu_1 \neq \mu_2)$; for a given mean return, show that the corresponding efficient portfolio can be given as a linear combination of the two given portfolios.

Answer

The following reasoning solves the two exercises.

Let us consider three efficient portfolios having different mean returns μ_1, μ_2, μ_3 respectively.

From result (17.11), we obtain for the constitution of these three portfolios:

$$\begin{aligned} \mathbf{x}^{(1)} &= \mathbf{g} + \mu_1 \mathbf{h}, \\ \mathbf{x}^{(2)} &= \mathbf{g} + \mu_2 \mathbf{h}, \\ \mathbf{x}^{(3)} &= \mathbf{g} + \mu_3 \mathbf{h}. \end{aligned} \quad (17.23)$$

Let k be the real number such that:

$$\mu_3 = k\mu_1 + (1-k)\mu_2. \quad (17.24)$$

Let us now form a portfolio as follows:

$$\mathbf{x} = k\mathbf{x}^{(1)} + (1-k)\mathbf{x}^{(2)}. \quad (17.25)$$

By unicity of linear convex combinations, we have:

$$\mathbf{x} = \mathbf{x}^{(3)}. \quad (17.26)$$

17.4. Markowitz theory for two assets

Here, x will represent the proportion invested in asset A and of course $(1-x)$ that invested in asset B always with $0 \leq x \leq 1$; moreover, without loss of generality, we assume that:

$$\begin{aligned} \mu_1 &< \mu_2, \\ \sigma_1 &< \sigma_2. \end{aligned} \quad (17.27)$$

In this case, the general results of section 17.2 become:

1) for the return on $[0, T]$:

$$R(x) = (1-x)\xi_1 + x\xi_2, \quad (17.28)$$

2) for the mean return on $[0, T]$:

$$\begin{aligned} E(R(x)) &= (1-x)\mu_1 + x\mu_2, \\ &= x(\mu_2 - \mu_1) + \mu_1, \end{aligned} \quad (17.29)$$

3) for the variance to be minimized:

$$\sigma^2 = (1-x)^2 \sigma_1^2 + x^2 \sigma_2^2 + 2x(1-x)\rho\sigma_1\sigma_2 \quad (17.30)$$

where ρ is the correlation coefficient between the two assets.

We will now discuss the different possibilities with respect to the value of ρ .

Case 1: $\rho = 1$

Intuitively, this means that the two assets vary in the same sense and so we will not have a portfolio of risk less than that of asset A.

Indeed, from relation (17.30), we obtain:

$$\begin{aligned}\sigma^2 &= (1-x)^2\sigma_1^2 + x^2\sigma_2^2 + 2x(1-x)\sigma_1\sigma_2, \\ &= (x\sigma_2 + (1-x)\sigma_1)^2,\end{aligned}\tag{17.31}$$

so, from assumption (17.27):

$$\sigma = x(\sigma_2 - \sigma_1) + \sigma_1.\tag{17.32}$$

With relation (17.29), we obtain:

$$\begin{cases} \mu = x(\mu_2 - \mu_1) + \mu_1, \\ \sigma = x(\sigma_2 - \sigma_1) + \sigma_1 \end{cases}\tag{17.33}$$

representing the parametric equations of a straight line in the Markowitz plane (σ, μ) having Cartesian equation:

$$\frac{\mu - \mu_1}{\mu_2 - \mu_1} = \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}\tag{17.34}$$

or:

$$\mu = \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}(\mu_2 - \mu_1) + \mu_1.\tag{17.35}$$

The efficient frontier is given by the part of this straight line between the two points $(\mu_1, \sigma_1), (\mu_2, \sigma_2)$.

It follows that the portfolio of minimum risk consists of investing all in asset A and that of maximum risk and also with maximum mean return all in asset B.

Case 2: $\rho = -1$

Here, we have:

$$\begin{aligned} \sigma^2 &= (1-x)^2 \sigma_1^2 + x^2 \sigma_2^2 - 2x(1-x)\sigma_1\sigma_2, \\ &= (x\sigma_2 - (1-x)\sigma_1)^2, \end{aligned} \tag{17.36}$$

or

$$\sigma = |x(\sigma_2 + \sigma_1) - \sigma_1|. \tag{17.37}$$

In this case, it is possible to select a non-risky portfolio taking for x^* the value such that σ equals 0:

$$x^* = \frac{\sigma_1}{\sigma_1 + \sigma_2}. \tag{17.38}$$

The parametric equations of the efficient frontier are:

$$\left\{ \begin{array}{l} \mu = x(\mu_2 - \mu_1) + \mu_1, \\ \sigma = \begin{cases} x(\sigma_2 + \sigma_1) - \sigma_1, & x > \frac{\sigma_1}{\sigma_2 + \sigma_1}, \\ -x(\sigma_2 - \sigma_1) + \sigma_1, & x \leq \frac{\sigma_1}{\sigma_2 + \sigma_1}. \end{cases} \end{array} \right. \tag{17.39}$$

Thus, it is formed by two straight line segments, the first one between the representative point of asset A (σ_1, μ_1) to the point representative of the portfolio without any risk, and the second from this last point and the point representative of asset B (σ_2, μ_2) .

Figures 17.1 and 17.2 show the corresponding graphs.

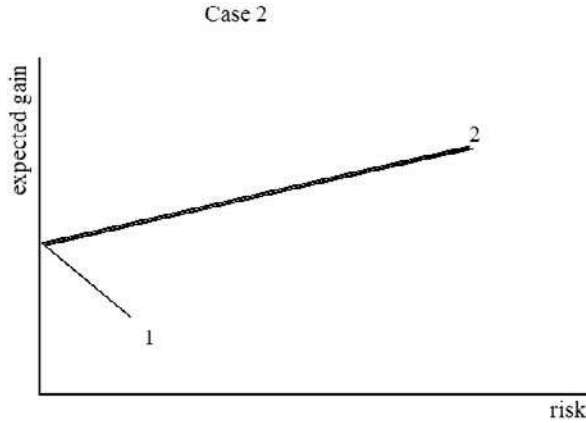


Figure 17.1. Case 2

Case 3: $-1 < \rho < 1$

In this case, the variance of the portfolio is given by:

$$\sigma^2 = (1-x)^2 \sigma_1^2 + x^2 \sigma_2^2 + 2x(1-x)\rho\sigma_1\sigma_2. \tag{17.40}$$

So, the parametric equations of the efficient frontier are:

$$\begin{cases} \mu = x\mu_1 + (1-x)\mu_2, \\ \sigma^2 = (1-x)^2 \sigma_1^2 + x^2 \sigma_2^2 + 2x(1-x)\rho\sigma_1\sigma_2. \end{cases} \tag{17.41}$$

To find the portfolio of minimum risk or minimum variance, we must find x^* such that (17.40) is minimum. After some basic calculations, we obtain:

$$\begin{aligned} x^* &= \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, \\ \sigma_{\min}^2 &= \frac{\sigma_1^2 \sigma_2^2 (1 - \rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \end{aligned} \tag{17.42}$$

It is easy to show that this variance is smaller than the minimum of the two variances of the assets, here σ_1^2 .

This portfolio gives for a minimum risk a return larger than the minimum of the two asset returns, but it cannot exceed the value μ_2 . To do that, we know that we must allow values of $x > 1$.

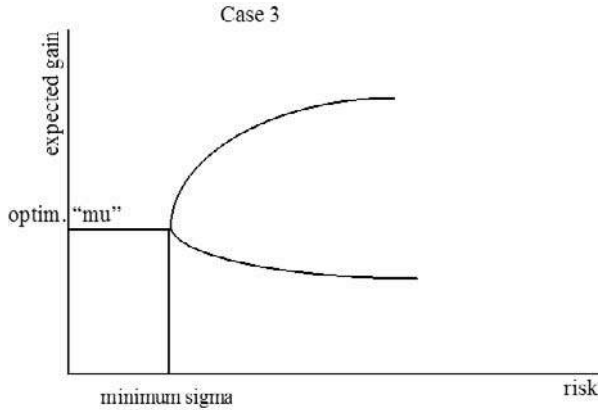


Figure 17.2. Case 3

17.5. Case of one risky asset and one non-risky asset

Let us consider a risky asset, which may be a portfolio having, on $[0, T]$, a mean return and standard deviation μ_a and σ_a respectively. The non-risky asset will have a return r on the time period.

Here too, let x represent the proportion of the risky asset in the global portfolio; it follows that the mean return on $[0, T]$ is given by

$$R(x) = xX + (1 - x)r, \tag{17.43}$$

x being the random variable of the return of the risky asset on $[0, T]$.

The mean and variances of the return of the global portfolio are given by:

$$\begin{aligned} \mu &= x\mu_a + (1 - x)r, \\ \sigma &= x\sigma_a. \end{aligned} \tag{17.44}$$

By eliminating x , we obtain:

$$\mu = \frac{\mu_a - r}{\sigma_a} \sigma + \frac{\sigma_a}{\sigma} r. \quad (17.45)$$

This equation represents a straight line containing the point representative of the risky asset (σ_a, μ_a) with a slope of

$$\frac{\mu_a - r}{\sigma_a}. \quad (17.46)$$

This slope represents the *risk premium* for the investment in the risky asset and is in principle strictly positive except on very disturbed financial markets.

Remark 17.3

(i) If we introduce a second risky asset, we know the efficient frontier is a branch of a hyperbola and that we must consider the tangent with the maximum slope issuing from the point $(0, r)$.

The optimal portfolios are now on this half tangent and after on the part of the efficient frontier for two risky assets (see Figure 17.3 on case 4).

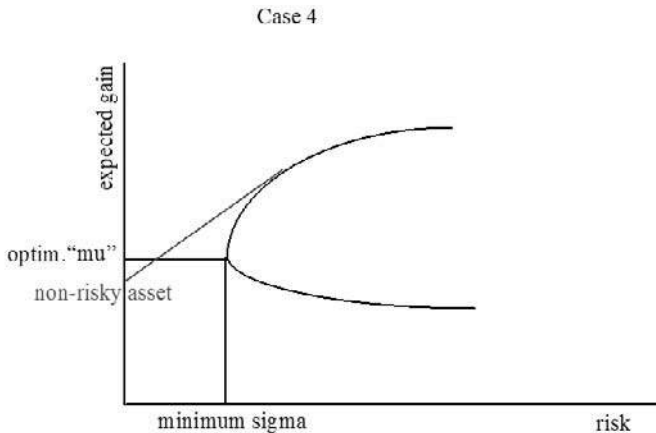


Figure 17.3. Case 4

Numerical example:

The following table gives data, exact results and simulation results for the efficient frontier in the case of two assets.

resolution data			simulation				
probability	active 1	active 2	proportion in the portfolio		average rend.	variance	standard deviation
			active 1	active 2			
			1	0	0.125	0.026875	0.16393596
0.25	0.1	0.8	0.9	0.1	0.1375	0.01741875	0.13198011
0.5	0	0.2	0.8	0.2	0.15	0.0123	0.11090537
0.25	0.4	-0.2	0.7	0.3	0.1625	0.01151875	0.10732544
			0.6	0.4	0.175	0.015075	0.12278029
			0.5	0.5	0.1875	0.02296875	0.15155445
mean	0.125	0.25	0.4	0.6	0.2	0.0352	0.18761663
			0.3	0.7	0.2125	0.05176875	0.22752747
	0.0425	0.19	0.2	0.8	0.225	0.072675	0.26958301
			0.1	0.9	0.2375	0.09791875	0.31291972
variance	0.026875	0.1275	0	1	0.25	0.1275	0.35707142
standard deviation	0.16393596	0.35707142					
covariance	-0.03125						
correlation	-0.53385178						

opt. approached

xopt(ac2)	0.26801153
1-xopt(ac1)	0.73198847
sig2min	0.01129683
sigmin	0.10628655
value opt.	0.15850144

(ii) A numerical example with simulation is given as an exercise.

Exercise

An investor wants to invest a sum of €100,000 in a portfolio formed from two risky assets *A* and *B*.

The two-dimensional discrete distribution of the “returns” is given by the following table.

Probability	Possible values for A	Possible values for B
.1	-.05	0
.2	0	.05
.4	.1125	.0875
.2	.15	.10
.1	.20	.15

a) Show that the mean and variance covariance are given by:

$$\boldsymbol{\mu} = \begin{bmatrix} .09 \\ .08 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} .00571 & .00267 \\ .00267 & .000141 \end{bmatrix}.$$

b) Give the graph of the efficient frontier.