

Chapter 16

Interest Rate Stochastic Models – Application to the Bond Pricing Problem

This chapter first presents some basic definitions on bond investments and interest rates. The second part is devoted to the two basic interest rate stochastic models: the *Ornstein-Uhlenbeck-Vasicek* (OUV) and the *Cox-Ingersoll-Ross* (CIR).

In the third part, we use these two models to describe the stochastic dynamics of zero-bonds applied to the pricing problem of bonds.

16.1. The bond investments

16.1.1. Introduction

A bond of *nominal value* P with *coupons of value* C and of *maturity date* $s+S$ gives the right for the investor buying this bond at time s , to receive the coupon value C at times $\{s+1, s+2, \dots, s+S\}$ and the nominal value P at time $s+S$.

In the following, a will represent the cost of this investment at time s , in general fixed by the bond market of the Stock exchange.

It follows that the successive cash flows of this investment are given by:

- at times $s, s+1, \dots, s+S-1$: the coupon value C ;
- at time $s+S$: the amount $P+C$.

If $A(t, T)$ ($s \leq t \leq s + S$) represents at time t the value of the bond issuing at time $t + T$, the main problem is to evaluate its fair value in view of comparing it to the proposed market value at time t .

A *zero-bond*, an investment made for example at time s and of maturity date $s + S$, is a very simple investment for which it is paid the sum $P(s, S)$ at time s in view of receiving €1 at the maturity dates $s + S$.

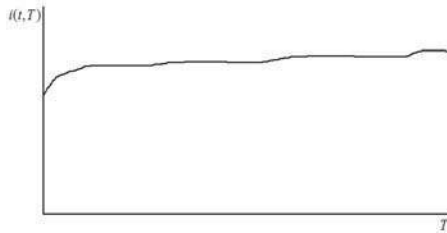
Thus, we can calculate the value of the above bond $A(t, T)$ with the following formula:

$$A(t, T) = P(t, 1)C + P(t, 2)C + \dots + P(t, T - 1)C + P(t, T)(C + P). \tag{16.1}$$

16.1.2. Yield curve

It is well known that the interest rate for a deposit at time t depends not only on this time t but also on its maturity T , so that this annual rate can be written as $i(t, T)$.

For a fixed time t , the graph of the function $T \mapsto i(t, T)$ represents the yield curve a time t and generally has the following form



Given this curve, we obtain the following value for a zero-bond:

$$P(t, T) = (1 + i(t, T))^{-T} \tag{16.2}$$

and using formula (16.1) for different bonds of different maturity times, we can calculate the values of the zero-bonds according to the market values of the observed bonds.

Example 16.1 Let us consider the case of $T = 2$ and let us suppose we have two bonds, the first with a coupon of 5.2%, with 100 as nominal value and with 1 as maturity, the second with a coupon of 5.6%, with 100 as nominal value and with 2 as maturity. The market values of these two bonds at time T are respectively of 100 and 102.

Using formula (16.1) twice, we obtain:

$$\begin{aligned} -100 &= P(t,1)(5.2 + 100); \\ -102 &= P(t,1)5.6 + P(t,2)(5.6 + 100). \end{aligned}$$

From the first equation, we obtain $P(t,1)$

$$P(t,1) = \frac{100}{105.2} (= 0.950570) \quad (16.3)$$

and then from the second, the value of $P(t,2)$:

$$P(t,2) = \frac{102 - 0.950570 \times 5.6}{105.6} (= 0.915228). \quad (16.4)$$

Consequently, the yield rates for one and two years are given by:

$$\begin{aligned} i(t,1) &= (0.950570)^{-1} - 1 = 5.2\%, \\ i(t,2) &= (0.915228)^{-1/2} - 1 = 4.53\%. \end{aligned} \quad (16.5)$$

Let us point out that, in this example, there is a phenomenon of *inversion of the yield curve* as the yield for a maturity of two years is smaller than the yield for a maturity of one year.

Of course, in practice, this method needs a bond market liquid enough to have all the data available for all maturities, and moreover a statistical treatment with the least squares method can be used to improve the method.

16.1.3. Yield to maturity for a financial investment and for a bond

Let us consider a financial investment of present value C generating the following financial flow:

$$F = \left\{ (F_j, t_j) \mid j = 1, \dots, n \right\}. \quad (16.6)$$

The *yield to maturity* is the constant discount rate or actuarial rate, the $i(F)$ solution of the polynomial equation:

$$C = \sum_{j=1}^n (1 + i(F))^{-t_j} F_j. \quad (16.7)$$

Using the traditional Newton interpolation with the nominal or coupon rate as initial value, this solution is easily given.

For the particular case of a bond of *subscription price* A at time t and maturity time $t+T$, and with *coupon value* C and *nominal value* P , the corresponding financial flow is given by:

- at times $t+1, \dots, t+T-1$: payment of the coupon C ;
- at time $t+T$: payment of the coupon C , and of the nominal value.

Equation (16.1) or (16.7) becomes:

$$\begin{aligned} A(t, T) = & (1+i(F))^{-1} C + (1+i(F))^{-2} C + \dots \\ & + (1+i(F))^{-(t+T-1)} C + (1+i(F))^{-(t+T)} (C + P). \end{aligned} \quad (16.8)$$

It is clear that the yield to maturity $i(F)$ is also a function of t and T .

16.2. Dynamic deterministic continuous time model for instantaneous interest rate

16.2.1. Instantaneous interest rate

In this section, we recall briefly some basic concepts fully described in section 3.7 using a discrete time model for the financial flows and the interest rates.

Now, we will use the traditional deterministic continuous time model (DCTM) for an investment on $[t, t+T]$ of amount $C(t)$ at time t producing a continuous yield of rate $r(s; t, T)$ at time s .

So, we see that this rate depends of t and T and on the “small” time interval $[s, s + \Delta s] \subset [t, t + T]$, one monetary unit at time t produces at the end of the interval a yield of value $r(s; t, T)\Delta s$. This rate is called the *continuous time instantaneous rate* or, in short, the *instantaneous rate* for an investment at time t and of maturity time $t+T$.

Let $C(s)$ be the capitalization value of $C(t)$ at time $s, s > t$.

From the definition of the instantaneous rate, it is clear that:

$$C(s + \Delta s) = C(s) + r(s; t, T)C(s)\Delta s. \quad (16.9)$$

With traditional limit reasoning, we obtain the following relation:

$$\frac{C'(s)}{C(s)} = r(s; t, T) \quad (16.10)$$

and by integration:

$$C(s) = C(t)e^{\int_t^{t+s} r(u;t,T)du} \quad (16.11)$$

In particular, at maturity, we obtain:

$$C(t+T) = C(t)e^{\int_t^{t+T} r(u;t,T)du} \quad (16.12)$$

16.2.2. Particular cases

As r is a function of three variables, it is useful to distinguish the four following cases:

- a) *stationarity in time*: r does not depend on t :
 $r(s; t, T) = r(s; T)$,
- b) *stationarity in maturity*: r does not depend on T :
 $r(s; t, T) = r(s; t)$,
- c) *stationarity in time and in maturity*: r does not depend both on t and T :
 $r(s; t, T) = r(s)$,
- d) *constant case*: r is independent of the three considered variables:
 $r(s; t, T) = \delta$.

For the last case, we get back the well known result of section 3.7:

$$C(t) = C(0)e^{\delta t}.$$

16.2.3. Yield curve associated with instantaneous interest rate

For the preceding section, we know that for an investment of €1 at time t and of maturity time $t+T$, the capitalization value at maturity is given by

$$e^{\int_t^{t+T} r(u;t,T)du} \quad (16.13)$$

Using the *yield curve* $T \mapsto i(t, T), T \geq 0$, for a fixed t , corresponding to this investment for which $i(t, T)$ represents the corresponding annual interest rate on $[t, t+T]$ given the same capitalization values as (16.13), we obtain:

$$(1 + i(t, T))^T = e^{\int_t^{t+T} r(u; t, T) du} \quad (16.14)$$

and so:

$$i(t, T) = e^{\frac{1}{T} \int_t^{t+T} r(u; t, T) du} - 1 \quad (16.15)$$

The constant instantaneous rate $\delta(t, T)$ on $[t, t+T]$ corresponding to this yield curve is defined as follows:

$$e^{\int_t^{t+T} \delta(t, T) du} = e^{\int_t^{t+T} r(u; t, T) du}$$

or

$$e^{\delta(t, T)T} = e^{\int_t^{t+T} r(u; t, T) du} \quad (16.16)$$

i.e.:

$$\delta(t, T) = \frac{1}{T} \int_t^{t+T} r(u; t, T) du$$

16.2.4. Examples of theoretical models

1) Constant case

For $r(s, t; T) = \delta$, relation (16.16) gives for the yield curve of the traditional case of deterministic traditional finance:

$$i(t, T) = e^{\frac{1}{T} \int_t^{t+T} \delta du} - 1$$

or

$$i(t, T) = e^{\delta} - 1 \quad (16.17)$$

From this last relation, we obtain:

$$\delta = \ln(1+i) \quad (16.18)$$

2) *Deterministic Ornstein-Uhlenbeck-Vasicek model (1973), (Janssen and Janssen (1996))*

Starting from the following relation:

$$r(t + \Delta t) - r(t) \approx a(b - r(t))\Delta t, \Delta t > 0, t \geq 0 \quad (16.19)$$

we obtain for $\Delta t \rightarrow 0$ the following differential equation:

$$dr(t) = a(b - r(t))dt \quad (16.20)$$

for which the general solution is given by:

$$r(t) = b - Ke^{-at} \quad (16.21)$$

With the initial condition:

$$r(0) = r_0$$

where r_0 is the observed instantaneous rate or *spot rate* observed at $t=0$, the constant K can be calculated to find the following unique solution:

$$r(t) = b + (r_0 - b)e^{-at} \quad (16.22)$$

or

$$r(t) = r_0e^{-at} + b(1 - e^{-at}) \quad (16.23)$$

So, the function r is a linear convex combination of r_0 and parameter b .

To find the economic-financial significance of this last parameter, it suffices to let $t \rightarrow \infty$ to see that:

$$b = \lim_{t \rightarrow \infty} r(t) \quad (16.24)$$

which is the anticipated value of the long term spot rate.

To see what the other parameter represents, we obtain from relation (16.23):

$$r'(t) = -ae^{-at} (r_0 - b) \quad (16.25)$$

and so the sign of the derivative function of r is those of a if $r_0 < b$ or of $-a$ if $r_0 > b$, and moreover:

$$r'(0) = -a(r_0 - b) \quad (16.26)$$

In conclusion, if $r_0 < b$, function r is strictly increasing, starting from r_0 at $t=0$ and tending towards b for large t ; on the other hand, if $r_0 > b$, function r is strictly decreasing, starting from r_0 at $t=0$ and tending towards b for large t .

In the two cases the absolute value of the slope at $t=0$ is an increasing function of a ; this means that the convergence is faster for large values of a than for small values. This is why parameter a is often called the *convergence parameter*.

The frontier case $r_0 = b$ gives the very special case of a *flat yield curve*.

To obtain the yield curve corresponding to the instantaneous rate given by relation (16.23), it suffices to substitute the value of r in relation (16.15); this calculation (see Janssen and Janssen (1996)) gives the following result:

$$i(0, T) = e^{\frac{b-r_0}{bT}(e^{-aT} - 1)} - 1 \quad (16.27)$$

More generally, starting from t with r_0 as in initial rate, this last formula becomes

$$i(t, T) = e^{\frac{b-r_0}{bT}(e^{-aT} - 1)} - 1 \quad (16.28)$$

So, we see that we have a *stationary model*, as on $[t, t + T]$ there is no influence of time t .

16.3. Stochastic continuous time dynamic model for instantaneous interest rate

In finance, it is well known that the future values of the rates are uncertain as there is a large influence from many financial and economic parameters, also depending on political factors.

It follows that deterministic models are unsatisfactory and so the new discipline of mathematical finance called “stochastic finance” was born from the results of Samuelson (1965), Black, Merton and Scholes (1973).

In this section, we will present the three most important stochastic models used in practice: the *Ornstein-Uhlenbeck-Vasicek (OUV)* model, the *Cox, Ingersoll and Ross (CIR)* model and the *Heath, Jarrow and Morton (HJM)* model.

The first two models are related to the instantaneous rate or spot rate, and the last starts from the yield curve at time 0 to model this entire yield curve at time t .

Other models are possible; for example the *Brennan and Schwartz* model considers two rates: the spot and the long term rates both modeled with a system of two SDEs.

16.3.1. The OUV stochastic model

16.3.1.1. The model

As usual, we consider a complete filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), P)$ on which all the defined stochastic processes will be adapted, in particular, the following standard Brownian motion.

The considered OUV model starts with the following stochastic dynamic for the spot rate process $r = (r(t), t \geq 0)$

$$\begin{aligned} dr(t) &= a(b - r(t))dt + \sigma dB(t) \\ r(0) &= r_0. \end{aligned} \tag{16.29}$$

This means that r is a special diffusion process extending the deterministic OUV case depending on four parameters: a, b, r_0, σ , assumed to be constant and known.

Using Itô’s calculus, it is possible to show (see Appendix to this chapter) that the unique solution of this SDE is given by:

$$r(t) = b + (r_0 - b)e^{-at} + \sigma e^{-at} \int_0^t e^{as} dB(s) \tag{16.30}$$

Indeed, if we differentiate this function, we obtain:

$$dr(t) = -a(r_0 - b)e^{-at} - a\sigma e^{-at} \int_0^t e^{as} dB(s) + \sigma e^{-at} e^{at} dB(s)$$

and as

$$r(t) - b - (r_0 - b)e^{-at} = \sigma e^{-at} \int_0^t e^{as} dB(s)$$

$$dr(t) = -a(r_0 - b)e^{-at} - a(r(t) - b - (r_0 - b)e^{-at}) + \sigma dB(t)$$

and so:

$$dr(t) = a(b - r(t))dt + \sigma dB(t)$$

16.3.1.2. Model and parameters interpretation

From result (16.30) and the traditional rules of Itô differentiation seen in Chapter 4, we obtain the *mean* of $r(t)$:

$$m(t) = E[r(t)] = b + (r_0 - b)e^{-at} \quad (16.31)$$

So this mean is nothing other than the value of r in the deterministic OUV model and moreover $m(t)$ tends towards b for $t \rightarrow \infty$.

Consequently, the interpretation of the parameter b is the same as in the deterministic case that is the *anticipated spot rate for long term*.

Concerning the *variance* of $r(t)$, we still use Itô differentiation:

$$\text{var } r(t) = \text{var} \left(\sigma e^{-at} \int_0^t e^{as} dB(s) \right) \quad (16.32)$$

So:

$$\text{var } r(t) = \sigma^2 e^{-2at} \int_0^t e^{2as} dB(s) \quad (16.33)$$

and finally:

$$\text{var } r(t) = \frac{\sigma^2}{2a} (1 - e^{-2at}) \quad (16.34)$$

Results (16.31) and (16.34) show that the financial and economic interpretations of parameters a and b are identical as in the deterministic OUV model but also that the *key parameter* here is

$$\lambda = \frac{\sigma^2}{2a} \quad (16.35)$$

as indeed, it represents the value of the asymptotic variance of $r(t)$ as t tends to $+\infty$, and moreover this asymptotic variance is a linear function of λ .

So, this variance is smaller for a weakly volatile market and larger for a market with large volatility, in conformity with n empirical studies.

Parameter a has an opposite effect: large (small) values of a give smaller (larger) values of the variance of $r(t)$.

To conclude, we see that:

- (i) the variance of $r(t)$ is increasing with time, confirming the fact that the uncertainty on the rate values increases with time;
- (ii) the larger parameter σ , called *volatility*, is, the greater the impact of randomness;
- (iii) the larger parameter a , called *convergence parameter*, is, greater the convergence of the spot rate towards b .

16.3.1.3. Marginal distribution of $r(t)$, fixed t

To calculate the distribution of $r(t)$, for all fixed t , it suffices from relation via relation (16.30) to calculate one of the r.v. of $X(t)$ defined by:

$$X(t) = \int_0^t e^{as} dB(s). \quad (16.36)$$

Coming back to the definition of stochastic integral given in Chapter 13, let us consider a sequence of subdivisions of $[0, t]$:

$$\Pi_n = (t_0, \dots, t_n), t_0 = 0, t_n = t, n \in \mathbb{N}_0.$$

Then we know that

$$\int_0^t e^{as} dB(s) = \lim_{v_n} \sum_{i=0}^n e^{at_i} [B(t_{i+1}) - B(t_i)], \quad (16.37)$$

ν_n being the norm of subdivision Π_n and using the uniform convergence in probability.

However, from the properties of the standard Brownian motion (see Chapter 10), we know that for each such subdivision, the distribution of the sum

$$\sum_{i=0}^n e^{at_i} [B(t_{i+1}) - B(t_i)] \tag{16.38}$$

is normal with zero mean and with variance

$$\sum_{i=0}^{n-1} e^{2at_i} (t_{i+1} - t_i). \tag{16.39}$$

As $\nu \rightarrow 0$, this variance converges to

$$\text{var}(X(t)) = \int_0^t e^{2as} ds (= \frac{e^{2at} - 1}{2}). \tag{16.40}$$

As $X(t) \prec N(0, \frac{e^{2at} - 1}{2})$, we obtain from results (16.30), (16.31) and (16.33)

that

$$r(t) \prec N\left(b + (r_0 - b)e^{-at}, \frac{\sigma^2}{2a}(1 - e^{-2at})\right). \tag{16.41}$$

16.3.1.4. Confidence interval for $r(t)$, fixed t .

From result (16.41), we can easily give a confidence interval at level $1 - \alpha$, for example with $\alpha = 5\%$. Indeed, if λ_α is a quantile of a r.v. $X \prec N(0,1)$ such that:

$$P[|X| \leq \lambda_\alpha] = 1 - \alpha, \tag{16.42}$$

we obtain

$$P\left[\frac{r(t) - b - (r_0 - b)e^{-at}}{\sigma \sqrt{\frac{1 - e^{-2at}}{2a}}} \leq \lambda_\alpha\right] = 1 - \alpha. \tag{16.43}$$

Consequently, the confidence interval at level $1 - \alpha$ is given by

$$\begin{aligned} & \left(b + (r_0 - b)e^{-at} \right) - \lambda_\alpha \sigma \sqrt{\frac{1}{2a}(1 - e^{-2at})}, \\ & \left(b + (r_0 - b)e^{-at} \right) + \lambda_\alpha \sigma \sqrt{\frac{1}{2a}(1 - e^{-2at})} \end{aligned} \quad (16.44)$$

From relation (16.15), we also find a confidence interval at level $1 - \alpha$ for $i(t, T)$ given by:

$$e^{b + \frac{b-r_0}{aT}(e^{-\lambda T} - 1)} e^{-\frac{\lambda_\alpha \sigma}{T} \int_0^T \sqrt{\frac{1-e^{-2as}}{2a}} ds} - 1 \leq i(t, T) \leq e^{b + \frac{b-r_0}{aT}(e^{-aT} - 1)} e^{\frac{\lambda_\alpha \sigma}{T} \int_0^T \sqrt{\frac{1-e^{-2as}}{2a}} ds} - 1. \quad (16.45)$$

As the length of the half interval for $r(t)$ quickly tends $\frac{\lambda_\alpha \sigma}{\sqrt{2a}}$ for $T \rightarrow \infty$, we obtain approximatively:

$$e^{\frac{1}{T} \int_0^T \left(r_0 e^{-as} + (1 - e^{-as})b - \lambda_\alpha \frac{\sigma}{\sqrt{2a}} \right) ds} - 1 \leq i(t, T) \leq e^{\frac{1}{T} \int_0^T \left(r_0 e^{-as} + (1 - e^{-as})b + \lambda_\alpha \frac{\sigma}{\sqrt{2a}} \right) ds} - 1, \quad (16.46)$$

and finally:

$$e^{b + \frac{b-r_0}{aT}(e^{-aT} - 1)} e^{-\frac{\lambda_\alpha \sigma}{\sqrt{2a}}} - 1 \leq i(t, T) \leq e^{b + \frac{b-r_0}{aT}(e^{-aT} - 1)} e^{\frac{\lambda_\alpha \sigma}{\sqrt{2a}}} - 1. \quad (16.47)$$

In particular, if the basic coefficient $\frac{\lambda_\alpha \sigma}{\sqrt{2a}}$ is such that:

$$\frac{\lambda_\alpha \sigma}{\sqrt{2a}} \lll 1, \quad (16.48)$$

we have with a probability near to $1 - \alpha$

$$e^{b + \frac{b-r_0}{aT}(e^{-aT} - 1)} \approx 1 + i(t, T) \quad (16.49)$$

and so with condition (16.48), we see that the deterministic OUV model is a good approximation of the stochastic model.

More precisely, we have:

$$me^{b+\frac{b-r_0}{aT}(e^{-aT}-1)} \leq 1+i(t,T) \leq Me^{b+\frac{b-r_0}{aT}(e^{-aT}-1)},$$

$$m = e^{-\frac{\lambda_a \sigma}{\sqrt{2a}}}, M = e^{\frac{\lambda_a \sigma}{\sqrt{2a}}} (= 1/m).$$

The following table gives some numerical values for m and M .

| $1-\alpha$ | λ | σ | a | coefficient | λ coefficient | m | M |
|------------|-----------|----------|-----|-------------|-----------------------|-------|-------|
| 0.05 | 1.960 | 0.1 | 0.1 | 0.2236 | 0.01118 | 0.989 | 1.011 |
| 0.025 | 2.241 | 0.2 | 1.1 | 0.1348 | 0.00337 | 0.997 | 1.003 |
| 0.05 | 1.960 | 0.1 | 5 | 0.0316 | 0.00158 | 0.998 | 1.002 |
| 0.025 | 2.241 | 0.2 | 5 | 0.0632 | 0.00158 | 0.998 | 1.002 |

16.3.1.5. Monte Carlo simulation method

From the result (16.30) and the definition of the stochastic integral, we obtain the following approximation

$$r(t) \approx b + (r_0 - b)e^{-at} + \sigma e^{-at} \sum_{i=0}^n e^{at_i} [B(t_{i+1}) - B(t_i)], \tag{16.50}$$

corresponding to the following subdivision of the interval $[0,t]$

$$\begin{aligned} \Pi_n &= (t_0, \dots, t_n), \\ t_0 &= 0, t_n = t. \end{aligned} \tag{16.51}$$

To obtain a simulation of a sample path for the r -process on $[0,T]$, it suffices to simulate a sample path ω of the standard Brownian process $(B(t), t \geq 0)$ giving the observed values

$$\{B(t_i, \omega), t_i \in \{t_0, \dots, t_n\}, t_0 = 0, t_n = T\} \tag{16.52}$$

from which we deduce the observed values for the r -process given by:

$$\left\{ \begin{aligned} r(t_i, \omega) &= b + (r_0 - b)e^{-at_i} + \sigma e^{-at_i} \sum_{i=1}^n e^{at_i} [B(t_{i+1}, \omega) - B(t_i, \omega)], \\ t_i &\in \{t_0, \dots, t_n\}, t_0 = 0, t_n = T \end{aligned} \right\}. \tag{16.53}$$

16.3.2. The CIR model (1985)

16.3.2.1. The model

In 1985, Cox, Ingersoll and Ross presented a new model for the temporal structure of interest rates for which the inconvenience of having negative values, as is the case for the OUV model, were dropped.

To obtain the model, the authors introduced a factor $\sqrt{r(t)}$ in the coefficient of dB , which is why their model is also called the *square root model*.

The stochastic differential equation governing this model is the following one:

$$\begin{aligned} dr(t) &= a(b - r(t))dt + \sigma\sqrt{r(t)}dB(t), \\ r(0) &= r_0. \end{aligned} \tag{16.54}$$

with the same assumptions as in the OUV model.

16.3.2.2. Model and parameters interpretation

Now, we have a non-linear stochastic differential equation, but as the spirit of this model is the same for the OUV model as for $\sigma = 0$, we still obtain the deterministic OUV model.

16.3.2.3. Marginal distribution of $r(t)$, fixed t

This non-linearity implies that there does not exist a simple “explicit” form of the solution of problem (3.26); nevertheless, the authors obtained the following explicit form of the conditional density function of $r(t)$, giving $r(s)$ ($s < t$):

$$\begin{aligned} f(y, t | r(s) = x) &= Ke^{-K(u+v)} \left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}), \\ K &= \frac{2a}{\sigma^2(1 - e^{-a(t-s)})}, \\ u &= Kxe^{-a(t-s)}, v = Ky, \\ q &= \frac{2ab}{\sigma^2} - 1. \end{aligned} \tag{16.55}$$

I_q being the modified Bessel function of second kind of order q defined as the following convergent series:

$$I_n(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2k}}{k! \Gamma(n+k+1)},$$

$$I_{-n}(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{-n+2k}}{k! \Gamma(-n+k+1)}$$
(16.56)

n being a positive natural number and Γ the usual Eulerian function of the second kind.

From result (16.55), it is possible to obtain the following expressions for the conditional mean and variance of $r(t)$:

$$E[r(t)|r(s) = x] = xe^{-a(t-s)} + b(1 - e^{-a(t-s)}),$$

$$\text{var}[r(t)|r(s) = x] = x \frac{\sigma^2}{a} (e^{-a(t-s)} - e^{-2a(t-s)}) + b \frac{\sigma^2}{2a} (1 - e^{-a(t-s)})^2.$$
(16.57)

Remark 16.1 These last results immediately give the asymptotic forms of the mean and the variances as follows:

$$\lim_{t \rightarrow \infty} E[r(t)|r(s) = x] = b$$

$$\text{var}[r(t)|r(s) = x] = b \frac{\sigma^2}{2a}.$$
(16.58)

These results show that the interpretation of the two parameters a and b is the same as the OUV models, and we see that the conditional variance is inversely proportional to a .

Let us also point out that for $a \rightarrow 0$, we obtain for fixed t :

$$E[r(t)|r(s) = x] \rightarrow x,$$

$$\text{var}[r(t)|r(s) = x] \rightarrow x\sigma^2(t-s).$$
(16.59)

Remark 16.2 The authors show that:

- (i) if $2ab \geq \sigma^2$, then the solution of the SDE (16.54) never becomes zero starting with a strictly positive value at time 0;
- (ii) if $2ab < \sigma^2$, then it is possible that the spot rate takes on a 0 value but it will never take negative values as it could be for the OUV model.

16.3.2.4. Confidence interval for $r(t)$, fixed t

As the conditional distribution of $r(t)$ is given by result (16.55), it is possible to construct a confidence interval for this spot rate at time t .

Moreover, asymptotically, the authors proved that:

$$\lim_{t \rightarrow \infty} f(y, t | r(s) = x) = \frac{\theta^\nu}{\Gamma(\nu)} y^{\nu-1} e^{-\theta y}, \quad (16.60)$$

$$\theta = \frac{2a}{\sigma^2}, \nu = \frac{2ab}{\sigma^2}.$$

Of course, this last result gives results (16.58) for the asymptotic mean and variances.

From the practical point of view, it is seen that the bounds of the confidence interval for fixed t , quickly converge to the bounds of the asymptotic one.

16.3.2.5. Monte Carlo simulation method

As for the preceding model, the simulation of trajectories of the r -process is done with time discretization of the stochastic differential equation (16.54) with a fixed partition of $[0, t]$:

$$\begin{aligned} \Pi_n &= (t_0, \dots, t_n), \\ t_0 &= 0, t_n = t, \end{aligned} \quad (16.61)$$

often with equal subintervals of length t/n .

This leads to the following non-linear system:

$$\left\{ \begin{aligned} r(t_{i+1}, \omega) &= r(t_i, \omega) + a(b - r(t_i, \omega))(t_{i+1} - t_i) \\ &\quad + \sigma \sqrt{r(t_i, \omega)} [B(t_{i+1}, \omega) - B(t_i, \omega)], \\ r(0) &= r_0, t_i \in \{t_0, \dots, t_n\}, t_0 = 0, t_n = T. \end{aligned} \right\}. \quad (16.62)$$

This system being recursive is easily solved and so we obtain the following simulated trajectory:

$$(r_0, r(t_1, \omega), \dots, r(t_i, \omega), \dots, r(t_n, \omega)). \quad (16.63)$$

16.3.3. The HJM model (1992)

16.3.3.1. Motivation

This model starts from a quite different point of view than the two preceding models as the authors want to model the *entire yield curve* starting with the “actual” given yield curve, that is, at time 0.

Their general result is overall theoretical, and it provides two particular models known as the *Ho and Lee* and the *generalized Vasicek* model.

16.3.3.2. The forward rates

Let $f(t, s)$ ($t < s$) be the instantaneous forward rate at time t , which will be attributed at time s .

This means that on the future interval $(s, s + \Delta s)$, the attributed yield will be approximatively $f(t, s)\Delta s$.

Under our AOA assumption, the investment of one monetary unit on $[0, s]$ must produce the same yield as an investment of one monetary unit on $[0, t]$ followed by the investment of the capitalized value at time t on the time interval $[t, s]$; so we must have the following relation:

$$e^{\int_0^t r(u)du} \times e^{\int_t^s f(t, u)du} = e^{\int_0^s r(u)du}, \quad (16.64)$$

or

$$e^{\int_t^s f(t, u)du} = e^{\int_t^s r(u)du} (= 1/P(t, s)), \quad (16.65)$$

$P(t, s)$ being the value of a zero coupon at time t of time maturity s years and r the instantaneous continuous rate function.

This last relation is equivalent to:

$$\int_t^s f(t, u)du = -\ln P(t, s), \quad (16.66)$$

or still by derivation:

$$f(t, s) = -\frac{\partial \ln P}{\partial s}(t, s). \quad (16.67)$$

Relation (16.67) also gives the *forward value of a zero-coupon*, as usual without default risk, calculated at $t=0$

$$P(t, T) = e^{-\int_t^T f(t, u) du}. \quad (16.68)$$

The instantaneous continuous rate $r(t)$ is given by

$$r(t) = \lim_{T \rightarrow t} f(t, T) = f(t, t) \quad (16.69)$$

assuming that function f is continuous.

Let us also recall the following links with the yield curve i , $i(t, s)$ being the equivalent annual interest rate for an investment of one monetary unit decided at time t up to time s .

$$(1 + i(t, s))^{-(t-s)} = P(t, s). \quad (16.70)$$

From relation (16.67), we also obtain:

$$\begin{aligned} f(t, s) &= -\frac{\partial}{\partial s} \ln(1 + i(t, s))^{-(t-s)}, \\ f(t, s) &= (t-s) \ln(1 + i(t, s)). \end{aligned} \quad (16.71)$$

16.3.3.3. The HJM methodology

As already pointed out in section 3.3.1, the main idea of the authors is to build a stochastic model for the forward instantaneous rate curve at time t , $f(0, T), T \geq 0$ given at time 0, and the observable forward instantaneous rate curve $f(0, T), T \geq 0$.

From the preceding section, we now know that:

$$P(t, T) = e^{-\int_t^T f(t, u) du} \quad (16.72)$$

and

$$f(t, s) = -\frac{\partial \ln P}{\partial s}(t, s). \quad (16.73)$$

We assume that the stochastic dynamics of the zero-coupon is governed by the following SDE

$$dP(t, T) = \mu(t, T)P(t, T)dt + \sigma(t, T)P(t, T)dB(t), \tag{16.74}$$

B being a standard Brownian motion in the considered complete filtered probability space.

From Itô's formula, we obtain:

$$df(t, T) = \frac{\partial}{\partial T} \left(\frac{\sigma^2(t, T)}{2} - \mu(t, T) \right) dt - \frac{\partial}{\partial T} (\sigma(t, T) dB(t)), \tag{16.75}$$

a relation representing the SDE for the stochastic dynamics of the process $f(t, T), T > t$.

Remark 16.3 From relation (16.69), we know that

$$\begin{aligned} r(t) &= f(t, t), \\ \text{or} & \\ &= f(t^*, t) + \int_{t^*}^t df(s, t), (t^* < t). \end{aligned} \tag{16.76}$$

assuming that at time t^* , the values $f(t^*, t)$ are known for all $t \geq t^*$.

Using relation (16.75), we obtain:

$$r(t) = f(t^*, t) + \int_{t^*}^t \left[\sigma(s, t) \frac{\partial}{\partial t} \sigma(s, t) - \frac{\partial}{\partial t} \mu(t, s) \right] ds - \int_{t^*}^t \frac{\partial}{\partial T} (\sigma(t, s) dB(s)), \tag{16.77}$$

The calculation of the Itô differential of r (see Wilmott (2000)) gives:

$$dr = \left(\begin{aligned} &\left[\frac{\partial f(t^*, t)}{\partial t} - \frac{\partial \mu(t, s)}{\partial s} \right]_{s=t} \\ &+ \int_{t^*}^t \left[\sigma(s, t) \frac{\partial^2}{\partial t^2} \sigma(s, t) + \left(\frac{\partial \sigma(s, t)}{\partial t} \right)^2 \right. \\ &\left. - \frac{\partial^2 \mu(s, t)}{\partial t^2} \right] ds - \int_{t^*}^t \left[\frac{\partial^2 \sigma(s, t)}{\partial t^2} dB(s) \right] \end{aligned} \right) dt - \frac{\partial \sigma(t, s)}{\partial s} \Big|_{s=t} dB(t). \tag{16.78}$$

We give this result to observe that the last coefficient of dt depends on all the past processes from t^* up to time t , and so dynamics no longer defines a Markov process.

This is a serious complication for the HJM model as an infinite number of variables are needed to solve this equation. That is why we can only consider particular cases.

Under the risk-neutral measure Q , we know that the volatility still remains identical and the trend is the riskless instantaneous interest rate $m(t)$, so that:

$$\begin{aligned}df(t, T) &= m(t, T)dt + v(t, T)d\tilde{B}(t), \\v(t, T) &= -\frac{\partial}{\partial T}\sigma(t, T), \\f(0, T) &= f^*(0, T)\end{aligned}\tag{16.79}$$

where f^* is the forward rate curve at time 0 and from the Girsanov theorem \tilde{B} is a new standard Brownian motion.

To find the value of the trend under Q , let us start with the drift under the historical measure P given in relation (16.75):

$$\frac{\partial}{\partial T}\left(\frac{\sigma^2(t, T)}{2} - \mu(t, T)\right)\tag{16.80}$$

and as under the risk-neutral measure Q , the drift μ of the dynamics of the zero coupons is nothing other than $r(t)$, we obtain from the second equality of (16.79) and taking into account that $\sigma(t, t) = 0$, as $P(t, t) = 1$:

$$\begin{aligned}\frac{\partial}{\partial T}\left(\frac{\sigma^2(t, T)}{2} - \mu(t, T)\right) &= v(t, T)\int_t^T v(t, s)ds - \frac{\partial}{\partial T}r(t), \\ &= v(t, T)\int_t^T v(t, s)ds.\end{aligned}\tag{16.81}$$

So, under the measure Q , the stochastic dynamics of the HJM model become:

$$\begin{aligned}df(t, T) &= \tilde{\mu}(t, T)dt + v(t, T)d\tilde{B}(t), \\ \mu(t, T) &= v(t, T)\int_t^T v(s, T)ds, \\ f(0, T) &= f^*(0, T).\end{aligned}\tag{16.82}$$

where, from the second equality of (16.79):

$$v(t, T) = -\frac{\partial}{\partial T} \sigma(t, T). \tag{16.83}$$

Thus, we can now provide the way to obtain the pricing of the zero coupons without default risk:

- 1) observe the forward instantaneous yield curve of the market at time $t=0$: $f^*(0, T)$;
- 2) determine the volatility $v(t, T)$;
- 3) calculate drift $\mu(t, T)$ from (16.82);
- 4) evaluate the forward instantaneous yield curve at time t under the risk-neutral measure:

$$f(t, T) = f^*(0, T) + \int_0^t \mu(s, T) ds + \int_0^t v(s, T) d\tilde{B}(s) \tag{16.84}$$

- 5) evaluate the instantaneous short term or spot rate at time t under the risk-neutral measure:

$$\begin{aligned} r(t) &= f(t, t), \\ &= f^*(0, t) + \int_0^t \mu(s, t) ds + \int_0^t v(s, t) d\tilde{B}(s) \end{aligned} \tag{16.85}$$

- 6) evaluate zero coupons at time t of several maturities T under the risk-neutral measure:

$$\begin{aligned} P(t, T) &= e^{-\int_t^T f(t, s) ds} \\ &= e^{-\left[\int_t^T f^*(0, u) du + \int_t^T \int_0^u \mu(s, u) ds + \int_t^T \int_0^u v(s, u) d\tilde{B}(s) \right]} \end{aligned} \tag{16.86}$$

16.3.3.4. *Particular cases of the HJM model: the Ho and Lee and generalized Vasicek models*

16.3.3.4.1. The Ho and Lee model

This is the simplest and the most useful model with the very particular assumption that the volatility is constant: $v(t, T) = v$.

With this assumption, the methodology given above provides many simplifications and leads to the following one:

- 1) observe the forward instantaneous yield curve of the market at time $t=0$: $f^*(0, T)$;
- 2) determine the volatility $v(t, T) = v$;
- 3) calculate drift $\mu(t, T)$ from (3.54): $\mu(t, T) = v^2(T - t)$;
- 4) evaluate the forward instantaneous yield curve at time t under the risk-neutral measure:

$$\begin{aligned} f(t, T) &= f^*(0, T) + \int_0^t \mu(s, T) ds + \int_0^t v(s, T) d\tilde{B}(s) \\ &= f^*(0, T) + v^2 t \left(T - \frac{t}{2} \right) + v \tilde{B}(t) \end{aligned} \quad (16.87)$$

- 5) evaluate the instantaneous short term or spot rate at time t under the risk-neutral measure:

$$\begin{aligned} r(t) &= f(t, t), \\ &= f^*(0, t) + \int_0^t \mu(s, t) ds + \int_0^t v(s, t) d\tilde{B}(s) \\ &= f^*(0, t) + \int_0^t v^2(t - s) ds + \int_0^t v d\tilde{B}(s) \quad ; \\ &= f^*(0, t) + \frac{v^2 t^2}{2} + v \tilde{B}(t) \end{aligned} \quad (16.88)$$

- 6) evaluate zero coupons at time t of several maturities T under the risk-neutral measure:

$$\begin{aligned} P(t, T) &= e^{-\int_t^T f(t, s) ds} \\ &= e^{-\left[\int_t^T f^*(0, s) ds + \int_t^T v^2 t \left(s - \frac{t}{2} \right) ds + \int_t^T v \tilde{B}(t) ds \right]} \\ &= e^{-\int_t^T f^*(0, s) ds - \frac{v^2 T t (T - t)}{2} - v \tilde{B}(t) (T - t)} \end{aligned} \quad (16.89)$$

From (16.87), we immediately obtain:

$$(i) f(t, T) \prec N \left(f(0, T) + \frac{v^2}{2} t \left(T - \frac{t}{2} \right), v^2 t \right) \tag{16.90}$$

showing that, in the Vasicek model, negative values for f could be observed;

$$(ii) r \prec N \left(f(0, t) + \frac{v^2 t^2}{2}, v^2 t \right) \tag{16.91}$$

showing that, in the Vasicek model, negative values for r could be observed;

(iii) (distribution lognormality of the zero coupons)

Relation (16.89) leads to:

$$P(t, T) = e^{-\int_t^T f^*(0, s) ds - \frac{v^2 T t (T-t)}{2} - v \tilde{B}(t)(T-t)}, \tag{16.92}$$

and so

$$\ln P(t, T) = -\int_t^T f^*(0, s) ds - \frac{v^2 T t (T-t)}{2} - v \tilde{B}(t)(T-t). \tag{16.93}$$

As

$$\begin{aligned} \ln P(t, T) &= -\int_t^T f^*(0, s) ds - \frac{v^2 T t (T-t)}{2} - v \tilde{B}(t)(T-t) \\ E \left[-\int_t^T f^*(0, s) ds - \frac{v^2 T t (T-t)}{2} - v \tilde{B}(t)(T-t) \right] &= -\int_t^T f^*(0, s) ds - \frac{v^2 T t (T-t)}{2} \\ \text{var} \left[-\int_t^T f^*(0, s) ds - \frac{v^2 T t (T-t)}{2} - v \tilde{B}(t)(T-t) \right] &= v^2 (T-t)^2 t, \end{aligned} \tag{16.94}$$

it follows that:

$$P(t, T) \prec LN \left(-\int_t^T f^*(0, s) ds - \frac{v^2 T t (T-t)}{2}, v^2 (T-t)^2 t \right); \tag{16.95}$$

(iv) as $P(0, u) = \int_0^u f^*(0, s) ds, \forall u \geq 0$, we still have that:

$$e^{-\int_0^T f^*(0, s) ds} = \frac{P(0, T)}{P(0, t)} \quad (16.96)$$

and thus from relation (3.61), we can write:

$$P(t, T) = \frac{P(0, T)}{P(0, t)} e^{-\frac{v^2 T t (T-t)}{2} - v \tilde{B}(t)(T-t)}. \quad (16.97)$$

This last result gives the possibility to calculate the forward values of zero coupons without the forward instantaneous yield curve of the market at time $t=0$: $f^*(0, T)$, and to easily simulate these values.

16.3.3.4.2. The generalized Vasicek model

For this model, the volatility is given by:

$$v(t, T) = v e^{-k(T-t)} \quad (16.98)$$

the volatility tending to 0 as $t \rightarrow T$.

The general HJM methodology now becomes:

$$(i) \mu(t, T) = \frac{v^2}{k} (e^{-k(T-t)} - e^{-2k(T-t)})$$

$$(ii) df(t, T) = \left[\frac{v^2}{k} (e^{-k(T-t)} - e^{-2k(T-t)}) \right] dt + v e^{-k(T-t)} d\tilde{B}(t)$$

$$(iii) f(t, T) = f^*(0, T) - \frac{v^2}{2k^2} (1 - e^{-k(T-t)})^2 + \frac{v^2}{2k^2} (1 - e^{-kT})^2 + v \int_0^t e^{-k(T-s)} d\tilde{B}(s)$$

$$(iv) r(t) = f^*(0, T) + \frac{v^2}{2k^2} (1 - e^{-kt})^2 + v \int_0^t e^{-k(t-s)} d\tilde{B}(s)$$

$$(v) P(t, T) = e^{-\int_t^T f(t, s) ds} = \frac{P(0, T)}{P(0, t)} e^{\left[-\frac{K^2(t, T)}{2} L(t) + K(t, T)(f(0, t) - r(t)) \right]}$$

$$K(t, T) = \frac{1 - e^{-k(T-t)}}{t}, L(t) = v^2 \int_0^t e^{-2k(T-s)} ds = v^2 e^{-2kT} \frac{e^{2kt} - 1}{2k}$$

$$(vi) f(t, T) \prec N(f^*(0, T) - \frac{v^2}{2k^2} (1 - e^{-k(T-t)})^2 + \frac{v^2}{2k^2} (1 - e^{-kT}), v^2 e^{-2kT} \frac{e^{2kt} - 1}{2k})$$

$$(vii) r(t) \prec N(f^*(0, T) + \frac{v^2}{2k^2} (1 - e^{-kt})^2, v^2 \frac{1 - e^{-2kt}}{2k})$$

(viii) $P(t, T)$ has a lognormal distribution.

Exercise 16.2 For this model, calculate the parameters of the distribution of $P(t, T)$.

Remark 16.4 There are many other rate models. For example, the discrete time *Ho and Lee model* (Ho and Lee (1986) uses a binomial tree and the *Hull and White model* (Hull-White (1996)), uses the following stochastic dynamics for the spot rate:

$$\begin{aligned} \Delta r &= (\theta(t) - ar)\Delta t + \sigma \Delta B, \\ \Delta r &= r(t + \Delta t) - r(t), \\ \theta(t) &> 0, \forall t, \\ a &> 0, \\ \sigma &> 0 \text{ (volatility)}, \\ \Delta B &= B(t + \Delta t) - B(t), B \text{ MBS.} \end{aligned} \tag{16.99}$$

Let us also mention the *Black, Derman and Toy model* (Black, Derman and Toy (1990)), starting with the following discrete time model for the spot rate:

$$\Delta \ln r(t) = \mu(r, t)\Delta t + \sigma(t)\Delta B(t). \tag{16.100}$$

16.4. Zero-coupon pricing under the assumption of no arbitrage

In the HJM model, we have introduced the dynamics of the zero coupons; in this section, we will do the same in the general case and finally for our two basic models, the OUV and CIR models.

16.4.1. Stochastic dynamics of zero-coupons

As before, let $P(t,s)$ ($t < s$) represent at time t the value of a zero-coupon of time maturity s , thus, of maturity $T=s-t$ at time t . Let $T=s-t$ so that with our preceding notation:

$$P(t,s)=P(t,t+T) \quad (16.101)$$

and so

$$P(s,s)=1. \quad (16.102)$$

The general problem of the evaluation of zero coupons consists of studying the stochastic process P defined on the filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), P)$ as follows:

$$P = \{P(t, s, \omega), t \in [0, s]\}. \quad (16.103)$$

To study this process is equivalent to the study of process R of the equivalent instantaneous yields:

$$R = \{R(t, s, \omega), t \in [0, s]\}, \quad (16.104)$$

where:

$$\begin{aligned} P(t, s, \omega) &= e^{-(s-t)R(t, s-t, \omega)}, \\ &= e^{-TR(t, T, \omega)}. \end{aligned} \quad (16.105)$$

Let us repeat that $R(t, T, \omega)$ is the equivalent instantaneous rate constant on the time interval $[t, t+T]$ given by present value, the value of the zero coupon at time t , $P(t, T, \omega)$.

From this last relation, we can provide the value of R :

$$R(t, T, \omega) = -\frac{1}{T} \log P(t, t+T, \omega). \quad (16.106)$$

For the *spot* rate r , we have:

$$\lim_{T \rightarrow 0} R(t, T, \omega) = r(t, \omega). \quad (16.107)$$

The equivalent annual rate is given by:

$$e^{R(t,T,\omega)} = 1 + i(t, T, \omega). \quad (16.108)$$

Let us now assume a *general stochastic dynamics* for the spot rate process r defined by the following SDE:

$$dr(t) = f(r, t)dt + \rho(r, t)dB(t). \quad (16.109)$$

Observing P also as a function of r , Itô's formula provides the value of the stochastic differential $dP(t, s, r)$:

$$dP(t, s, r) = \left[\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r} f(r, t) + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \rho^2(r, t) \right] dt + \frac{\partial P}{\partial r} \rho(r, t) dB(t). \quad (16.110)$$

With

$$\begin{aligned} \mu(t, s, r) &= \frac{1}{P(t, s, r)} \left(\frac{\partial}{\partial t} + f \frac{\partial}{\partial r} + \frac{1}{2} \rho^2 \frac{\partial^2}{\partial r^2} \right) P(t, s, r), \\ \sigma(t, s, r) &= -\frac{1}{P(t, s, r)} \rho \frac{\partial}{\partial r} P(t, s, r), \end{aligned} \quad (16.111)$$

we obtain

$$dP(t, s, r) = P(t, s, r) \mu(t, s, r) dt - P(t, s, r) \sigma(t, s, r) dB(t). \quad (16.112)$$

16.4.2. Application of the no arbitrage principle and risk premium

Let us study an investor issuing at time t , x zero coupon bonds expiring at time s_1 and investing y in zero coupon bonds expiring at time s_2 .

The value $W(t)$ of this portfolio at time t is given by

$$W(t) = -xP(t, s_1, r) + yP(t, s_2, r). \quad (16.113)$$

From the linearity property of Itô's formula and from relation (16.112), we obtain

$$\begin{aligned} dW(t) &= [yP(t, s_2, r)\mu(t, s_2, r) - xP(t, s_1, r)\mu(t, s_1, r)] dt \\ &\quad - [yP(t, s_2, r)\sigma(t, s_2, r) - xP(t, s_1, r)\sigma(t, s_1, r)] dB(t). \end{aligned} \quad (16.114)$$

Now, the assumption of AOA has two consequences:

(i) firstly, we have to cancel the risk component and so the coefficient of dB must have a 0 value;

(ii) then, the instantaneous yield of this portfolio must be, in every time interval $(t, t+dt)$, the same as a riskless investment at the spot rate $r(t)$.

Thus, from relation (16.114) we obtain:

$$\begin{aligned} yP(t, s_2, r)\sigma(t, s_2, r) - xP(t, s_1, r)\sigma(t, s_1, r) &= 0, \\ yP(t, s_2, r)\mu(t, s_2, r) - xP(t, s_1, r)\mu(t, s_1, r) &= rW(t). \end{aligned} \quad (16.115)$$

Moreover, as the value of $W(t)$ is given by relation (16.113), we obtain the following linear system for the two unknown values x and y :

$$\begin{aligned} yP(t, s_2, r)\sigma(t, s_2, r) - xP(t, s_1, r)\sigma(t, s_1, r) &= 0, \\ yP(t, s_2, r)[\mu(t, s_2, r) - r] - xP(t, s_1, r)[\mu(t, s_1, r) - r] &= 0. \end{aligned} \quad (16.116)$$

As this system is homogenous, from Rouchè's theorem, we know that there exists a non-trivial solution, i.e. a solution with at least one value different from 0, if and only if the determinant of the system is different from 0.

Thus, the condition to have a financial market is

$$\frac{\mu(t, s_1, r) - r}{\sigma(t, s_1, r)} = \frac{\mu(t, s_2, r) - r}{\sigma(t, s_2, r)}, \quad (16.117)$$

for all t, s_1 and s_2 .

This condition means that the function $\frac{\mu(t, s, r) - r}{\sigma(t, s, r)}$ is independent of s or that the function λ defined by

$$\lambda(t, r) = \frac{\mu(t, s, r) - r}{\sigma(t, s, r)} \quad (16.118)$$

is independent of s . Function λ represents the *risk premium of the market* as the difference between the instantaneous yield of the bond and the riskless rate r , normed by the volatility value σ .

16.4.3. Partial differential equation for the structure of zero coupons

Substituting the values of μ, σ from relations (16.111) into relation (16.118), we obtain the following *structural partial differential equation* (PDE) of zero-coupon bonds:

$$\begin{aligned}
 rP(t, s, r) &= \frac{\partial}{\partial t} P(t, s, r) + (f(r, t) + \rho(r, t)\lambda(r, t)) \frac{\partial}{\partial r} P(t, s, r) \\
 &+ \frac{1}{2} \rho^2 \frac{\partial^2}{\partial r^2} P(t, s, r), \\
 P(s, s, r) &= 1.
 \end{aligned}
 \tag{16.119}$$

The next proposition gives its solution.

Proposition 16.1 Under the traditional regularity conditions on the coefficients, the solution of the structural PDE (16.119) is given, for all $t \leq s$ by:

$$P(t, s, r) = E_{\mathfrak{F}_t} \left[e^{-\int_t^s r(\tau) d\tau - \frac{1}{2} \int_t^s \lambda^2(\tau, r(\tau)) d\tau + \int_t^s \lambda(\tau, r(\tau)) dB(\tau)} \right],
 \tag{16.120}$$

where $\mathfrak{F}_t = \sigma(B(u), u \leq t)$.

Proof If we introduce process V defined by

$$V(u) = e^{-\int_t^u r(\tau) d\tau - \frac{1}{2} \int_t^u \lambda^2(\tau, r(\tau)) d\tau + \int_t^u \lambda(\tau, r(\tau)) dB(\tau)},
 \tag{16.121}$$

we can also write:

$$V(u) = e^{g(u)},
 \tag{16.122}$$

where

$$g(u) = -\int_t^u r(\tau) d\tau - \frac{1}{2} \int_t^u \lambda^2(\tau, r(\tau)) d\tau + \int_t^u \lambda(\tau, r(\tau)) dB(\tau),
 \tag{16.123}$$

and so

$$dg(u) = -\left[r(u) + \frac{1}{2}\lambda^2(u, r(u))\right]du + \lambda(u, r(u))dB(u). \quad (16.124)$$

Using Itô's calculus rules, we have

$$dV(u) = V(u)\left[-\left(r(u) + \frac{1}{2}\lambda^2(u, r(u))\right) + \frac{1}{2}\lambda^2(u, r(u))\right]du \\ + V(u)\lambda(u, r(u))dB(u). \quad (16.125)$$

As

$$dP(t, s, r) = \left[\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r}f(r, t) + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\rho^2(r, t)\right]dt + \frac{\partial P}{\partial r}\rho(r, t)dB(t), \quad (16.126)$$

we obtain:

$$d(P(u)V(u)) = P(u)dV(u) + V(u)dP(u) + V(u)\lambda(u, r(u))\frac{\partial P}{\partial r}\rho(r, u)du. \quad (16.127)$$

Substituting into this last equality $dP(u)$ and $dV(u)$, given by relations (16.125) and (16.126), we obtain, after some calculations that:

$$d(PV) = V\left(\frac{\partial P}{\partial t} + (f + \rho\lambda)\frac{\partial P}{\partial r} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2} - rP\right)du \\ + V\frac{\partial P}{\partial r}\rho dB + PV\lambda dB. \quad (16.128)$$

From PDE (16.119), this last equality becomes:

$$d(PV) = V\frac{\partial P}{\partial r}\rho dB + PV\lambda dB. \quad (16.129)$$

By integration from t to s ($t < s$), we obtain:

$$PV(s) - PV(t) = \int_t^s \left[\rho V\frac{\partial P}{\partial r} + PV\lambda\right]dB. \quad (16.130)$$

By conditional expectation with respect to \mathfrak{F}_t , we find:

$$\begin{aligned}
 E_{\mathfrak{F}_t} [P(s, s, r)V(s) - P(t, s, r)V(t)] &= E_{\mathfrak{F}_t} \left[\int_t^s \left[\rho V \frac{\partial P}{\partial r} + PV\lambda \right] dB \right], \\
 &= \int_t^s E_{\mathfrak{F}_t} \left[\rho V \frac{\partial P}{\partial r} + PV\lambda \right] dE_{\mathfrak{F}_t} B, \\
 &= 0,
 \end{aligned}
 \tag{16.131}$$

as B is a SBM.

As we know that $P(s, s, r) = 1$, relation (16.131) gives:

$$E_{\mathfrak{F}_t} [V(s)] = P(t, s, r)V(t) \tag{16.132}$$

and finally, as $V(t) = 1$:

$$E_{\mathfrak{F}_t} [V(s)] = P(t, s, r), \tag{16.133}$$

that is, relation (16.120) is proved. □

16.4.4. Values of zero coupons without arbitrage opportunity for particular cases

We will now make use of relation (16.120) to evaluate the risk-neutral value of a zero coupon.

16.4.4.1. *The risk premium is 0*

From (16.118), we obtain:

$$\mu(t, s) = r(t), t \geq s, \tag{16.134}$$

and from the fundamental result (16.120), we obtain the traditional form of a zero-coupon value:

$$P(t, s, r(t)) = E_{\mathfrak{F}_t} \left[e^{-\int_t^s r(u)du} \right]. \tag{16.135}$$

If, moreover, the spot rate is deterministic, we obtain the traditional formula:

$$P(t, s, r(t)) = e^{-\int_t^s r(u)du} \tag{16.136}$$

16.4.4.2. *Constant premium rate*

Here result (16.120) becomes:

$$P(t, s, r) = E_{\mathfrak{F}_t} \left[e^{-\int_t^s r(\tau) d\tau - \frac{1}{2} \lambda^2 (s-t) + \lambda [B(s) - B(t)]} \right]. \quad (16.137)$$

To provide an interpretation of this result with the introduction of the *risk-neutral measure*, let us now introduce *Girsanov's theorem*.

16.4.4.3. Girsanov's theorem (Gikhman and Skorohod (1980))

On $(\Omega, \mathfrak{F}, (\mathfrak{F}_t, t \geq 0), P)$, let us consider the adapted stochastic process $f = \{f(t), t \in [0, T]\}$ such that

$$\int_0^T f^2(s) ds < \infty \text{ a.s.} \quad (16.138)$$

Then, Girsanov's theorem introduces to the new stochastic process $\xi = \{\xi(t), t \in [0, T]\}$ defined by:

$$\xi(t) = \exp \left(\int_0^t f(s) dB(s) - \frac{1}{2} \int_0^t f^2(s) ds \right). \quad (16.139)$$

Moreover, process B is a SBM on $(\Omega, \mathfrak{F}, (\mathfrak{F}_t, t \geq 0), P)$.

Girsanov introduces a new probability measure dependent on f and noted $Q = Q(f)$ on $(\Omega, \mathfrak{F}, (\mathfrak{F}_t, t \geq 0))$ of density $\xi(T)$ with respect to the initial measure P , such that:

$$\frac{dQ(f)}{dP} = \xi(T). \quad (16.140)$$

This means that the expectation with respect to the new measure is related to the old one with the following relation:

$$\int X(\omega) dQ(f) = \int X(\omega) \xi(T, \omega) dP$$

or

$$E_Q[X(\omega)] = E_P[X(\omega) \xi(T, \omega)]. \quad (16.141)$$

Of course, we also have:

$$E_P[X(\omega)] = E_Q \left[X(\omega) (\xi(T, \omega))^{-1} \right]. \quad (16.142)$$

The result of Girsanov's theorem is if the following condition is fulfilled:

$$E[\xi(T, \omega)] = 1, \quad (16.143)$$

then the new process

$$\hat{B} = \{\hat{B}(t), t \in [0, T]\} \quad (16.144)$$

defined by

$$\hat{B}(t) = B(t) - \int_0^t f(s) ds \quad (16.145)$$

is still a SBM but now on $(\Omega, \mathfrak{F}, (\mathfrak{F}_t, t \geq 0), Q)$.

Under Novikov's condition:

$$E\left[\exp\left(\frac{1}{2}\int_0^T f^2(s) ds\right)\right] < \infty \quad (16.146)$$

process ξ defined by (16.135) satisfies condition (16.143) and so Girsanov's theorem applies.

16.4.4.4. Neutral risk measure

Let us recall the stochastic dynamics of the zero coupons is defined by the SDE:

$$dP(t, s, r) = P(t, s, r)\mu(t, s, r)dt - P(t, s, r)\sigma(t, s, r)dB(t) \quad (16.147)$$

where

$$\mu(t, s, r) = r(t) + \lambda(t)\sigma(t, s, r). \quad (16.148)$$

Let us now introduce the new probability measure Q on $(\Omega, \mathfrak{F}, (\mathfrak{F}_t, t \geq 0))$ defined by:

$$\frac{dQ}{dP} = e^{\int_0^s \lambda(u)dB(u) - \frac{1}{2}\int_0^s \lambda^2(u)du} \quad (= \xi(T)). \quad (16.149)$$

Through Girsanov's theorem, we know that process $\hat{B} = \{\hat{B}(u), u \in [0, T]\}$ defined by

$$\widehat{B}(u) = B(u) - \int_0^u \lambda(v) dv \quad (16.150)$$

is also a SBM but now on $(\Omega, \mathfrak{F}, (\mathfrak{F}_t, t \geq 0), Q)$ such that:

$$d\widehat{B}(u) = dB(u) - \lambda(u) du. \quad (16.151)$$

Returning to relation (16.146), we now have:

$$dP(t, s, r) = P(t, s, r) \mu(t, s, r) dt - P(t, s, r) \sigma(t, s, r) [d\widehat{B}(t) + \lambda(t) dt] \quad (16.152)$$

or

$$\frac{dP(t, s, r)}{P(t, s, r)} = [\mu(t, s, r) - \sigma(t, s, r) \lambda(t)] dt - \sigma(t, s, r) d\widehat{B}(t). \quad (16.153)$$

From relation (16.148), we also have:

$$\frac{dP(t, s, r)}{P(t, s, r)} = r(t) dt - \sigma(t, s, r) d\widehat{B}(t). \quad (16.154)$$

From result (16.120) stating that:

$$P(0, s, r) = E_{\mathfrak{F}_t} \left[e^{-\int_0^s r(\tau) d\tau - \frac{1}{2} \int_0^s \lambda^2(\tau, r(\tau)) d\tau + \int_0^s \lambda(\tau, r(\tau)) dB(\tau)} \right], \quad (16.155)$$

we obtain from relation (16.149):

$$\begin{aligned} P(0, s, r) &= E_Q \left[e^{-\int_0^s r(\tau) d\tau - \frac{1}{2} \int_0^s \lambda^2(\tau, r(\tau)) d\tau + \int_0^s \lambda(\tau, r(\tau)) dB(\tau)} \cdot e^{\frac{1}{2} \int_0^s \lambda^2(\tau, r(\tau)) d\tau - \int_0^s \lambda(\tau, r(\tau)) dB(\tau)} \right], \\ &= E_Q \left[e^{-\int_0^s r(\tau) d\tau} \right]. \end{aligned} \quad (16.156)$$

Thus, under Q , the value of a zero coupon is formally given, as in the particular case of $\lambda = 0$, which is why the new measure Q is called the *risk-neutral measure*.

From the Markov property, this last result gives, starting at time t with $r(t)=r$, as the spot rate:

$$\begin{aligned}
 &P(t, s, r | \mathfrak{F}_t) \\
 &= E_Q \left[e^{\int_t^s r(\tau) d\tau - \frac{1}{2} \int_t^s \lambda^2(\tau, r(\tau)) d\tau + \int_t^s \lambda(\tau, r(\tau)) dB(\tau)} \cdot e^{\frac{1}{2} \int_t^s \lambda^2(\tau, r(\tau)) d\tau - \int_t^s \lambda(\tau, r(\tau)) dB(\tau)} \mid \mathfrak{F}_t \right], \quad (16.157) \\
 &= E_Q \left[e^{-\int_t^s r(\tau) d\tau} \mid \mathfrak{F}_t \right].
 \end{aligned}$$

16.4.4.5. Examples

Example 16.1 The OUV process as rate dynamics

We know that the OUV model is governed by the following SDE:

$$\begin{aligned}
 dr(t) &= a(b - r(t))dt + \sigma dB(t), \mid \mathfrak{F}_t \\
 r(0) &= r_0.
 \end{aligned} \quad (16.158)$$

Assuming that the risk premium is constant with value λ on $[0, t]$, the risk-neutral measure given by relation (16.149):

$$\frac{dQ}{dP} = e^{\int_0^s \lambda(u) dB(u) - \frac{1}{2} \int_0^s \lambda^2(u) du} \quad (= \xi(T)), \quad (16.159)$$

becomes:

$$dQ = e^{\lambda B(t) - \frac{1}{2} \lambda^2 t} dP \quad (16.160)$$

and by relation (16.150)

$$\hat{B}(u) = B(u) - \lambda(u). \quad (16.161)$$

Thus, under measure Q , the stochastic dynamics of process r are defined by the following SDE

$$\begin{aligned}
 dr(t) &= a(b - r(t))dt + \sigma [d\hat{B}(t) + \lambda dt], \\
 r(0) &= r_0,
 \end{aligned} \quad (16.162)$$

or even:

$$\begin{aligned} dr(t) &= a(\theta - r(t))dt + \sigma [d\widehat{B}(t) + \lambda dt], \\ r(0) &= r_0, \\ \theta &= b + \frac{\lambda\sigma}{a}. \end{aligned} \tag{16.163}$$

On the time interval $[t, s]$, under Q , the basic results of section 16.3.1.1 take the form:

$$\begin{aligned} r(s) &= \theta + (r_t - \theta)e^{-at} + \sigma e^{-at} \int_t^s e^{au} d\widehat{B}(u), \\ E_Q[r(s)] &= \theta + (r_t - \theta)e^{-as}, \\ \text{var}_Q r(s) &= \frac{\sigma^2}{2a} (1 - e^{-2as}). \end{aligned} \tag{16.164}$$

For the value of the zero coupon, we obtain from result (16.157):

$$P(t, s, r) = E_Q \left[e^{-\int_t^s r(\tau) d\tau} \middle| \mathfrak{F}_t \right]. \tag{16.165}$$

Let us now calculate the value of $P(0, s, r_0)$.

With

$$\beta(u) = \frac{1 - e^{-au}}{a}, \tag{16.166}$$

the first relation of relation (16.164) becomes:

$$r(u) = r_0 e^{-au} \theta + a\theta\beta(u) + \sigma \int_0^u e^{-a(u-v)} d\widehat{B}(v). \tag{16.167}$$

For

$$U(0, s) = \int_0^s r(u) du, \tag{16.168}$$

the Fubini theorem and result (16.167) lead to the following result:

$$U(0, s) = (r_0 - \theta)\beta(s) + \theta s + \sigma \int_0^s \beta(s-u) d\widehat{B}(u). \quad (16.169)$$

It follows that the distribution of $U(0, s)$ is normal with parameters given by:

$$E[U(0, s)] = (r_0 - \theta)\beta(s) + \theta s, \quad (16.170)$$

$$\begin{aligned} \text{var}[U(0, s)] &= E \left[\left(\sigma \int_0^s \beta(s-u) d\widehat{B}(u) \right)^2 \right], \\ &= \sigma^2 \int_0^s \beta^2(s-u) du. \end{aligned} \quad (16.171)$$

As β is given by relation (16.166), we obtain:

$$\text{var}[U(0, s)] = \frac{\sigma^2}{a^2} \int_0^s \left(1 - e^{-a(s-u)} \right)^2 du. \quad (16.172)$$

The calculation of this traditional integral leads to:

$$\text{var}[U(0, s)] = \frac{\sigma^2}{2a^3} \left[2as - e^{-2as} + 4e^{-as} - 3 \right]. \quad (16.173)$$

Returning to result (16.165) with $t=0$, we find that:

$$\begin{aligned} P(0, s, r_0) &= E_Q \left[e^{-\int_0^s r(\tau) d\tau} \mid \mathfrak{F}_0 \right], \\ &= E_Q \left[e^{-\int_0^s r(\tau) d\tau} \right], \\ &= E_Q \left[e^{-U(0, s)} \right]. \end{aligned} \quad (16.174)$$

Consequently, the value of this zero coupon is given by the value of the generating function of $U(0,s)$ at $s=-1$, that is,

$$P(0, s, r_0) = e^{-E_Q[U(0,s)] + \frac{1}{2} \text{var}_Q[U(0,s)]}. \quad (16.175)$$

Starting from a t different from 0, we obtain:

$$P(t, s, r_0) = e^{-E_Q[U(t,s)] + \frac{1}{2} \text{var}_Q[U(t,s)]}. \quad (16.176)$$

Using results (16.163), (16.170) and (16.171), we obtain:

$$P(t, s, r_t) = \exp \left[-(s-t)R_\infty - \beta(s-t)(r_t - R_\infty) - \frac{\sigma^2}{4a} \beta^2(s-t) \right],$$

where

$$R_\infty = b + \frac{\lambda\sigma}{a} - \frac{\sigma^2}{2a^2}, \quad (16.177)$$

$$r_t = r(t).$$

Using result (16.105), we obtain the instantaneous term structure:

$$R(s, s-t) = R_\infty + \frac{\beta(s-t)}{s-t}(r_t - R_\infty) + \frac{\sigma^2}{4a} \frac{\beta^2(s-t)}{s-t},$$

$$\beta(u) = \frac{1 - e^{-au}}{a}, \quad (16.178)$$

$$R(t, \infty) = R_\infty,$$

$$E = R_\infty - \frac{\sigma^2}{4a^2}, \quad F = R_\infty + \frac{\sigma^2}{2a^2}.$$

Example 16.2 The CIR process as rate dynamic

We can also use the CIR model defined and studied in section 16.3.2 by:

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dB(t),$$

$$r(0) = r_0. \quad (16.179)$$

We will express the premium risk in the form

$$\lambda(t, r) = -\frac{\pi}{\sigma} \sqrt{r(t)}. \quad (16.180)$$

It is clear that this premium risk is no longer constant as in the preceding example.

Here the PDE (16.119) for the zero coupon value takes the form:

$$\frac{\partial P}{\partial t} + [a(b-r-\pi r)] \frac{\partial P}{\partial r} + \frac{1}{2} \sigma^2 r \frac{\partial^2 P}{\partial r^2} - rP = 0, \tag{16.181}$$

$$P(r, s, s) = 1.$$

Cox, Ingersoll and Ross obtained the solution under the form:

$$P(r, t, s) = A(t, s)e^{-rD(t, s)},$$

where

$$A(t, s) = \left[\frac{2\gamma e^{\frac{(a+\pi+\gamma)(s-t)}{2}}}{(a+\pi+\gamma)(e^{\gamma(s-t)} - 1) + 2\gamma} \right]^{\frac{2ab}{\sigma^2}}, \tag{16.182}$$

$$D(t, s) = \frac{2(e^{\gamma(s-t)} - 1)}{(a+\pi+\gamma)(e^{\gamma(s-t)} - 1) + 2\gamma},$$

$$\gamma = \sqrt{(a+\pi)^2 + 2\sigma^2}.$$

So, as

$$P(r, t, s) = e^{-(s-t)R(r, t, s)}, \tag{16.183}$$

we obtain from (16.182):

$$R(r, t, s) = \frac{rD(t, s) - \ln A(t, s)}{(s-t)}. \tag{16.184}$$

The two limit cases give as results:

$$s \rightarrow t \Rightarrow R(r, t, s) \rightarrow r (= r(t)),$$

$$s \rightarrow \infty \Rightarrow R(r, t, s) \rightarrow \frac{2ab}{a+\pi+\gamma}. \tag{16.185}$$

Remark 16.5 If we assume that the spot rate satisfies the very simple model:

$$\begin{aligned} dr &= \mu dt + \sigma dB(t), \\ r(0) &= r_0, \end{aligned} \quad (16.186)$$

which is not incredibly adequate since it has a linear trend, it can be shown that the value of the zero-coupon is given by:

$$P(t, s) = \exp \left[-r(s-t) - \frac{\mu(s-t)^2}{2} + \frac{\sigma^2(s-t)^3}{6} \right], \quad (16.187)$$

which is a very unsatisfactory solution since

$$\lim_{s \rightarrow \infty} P(t, s) = \infty! \quad (16.188)$$

This is why such a simple model must be rejected in the case of interest rate modeling.

16.4.5. Value of a call on zero-coupon

16.4.5.1. General results

Let us give at time 0, a zero coupon expiring at time s . On this asset, we consider a call of maturity T where $s > T$, with K as the exercise price and the value $C(0, T)$ at $t = 0$.

From relation (16.112) canceling the dependence with respect to r , the dynamics for the zero coupon is given by:

$$\frac{dP}{P} = \mu(t, T)dt - \sigma(t, T)dB(t). \quad (16.189)$$

It is possible to show (Musielà and Rutkowski (1997)) that the value under AOA is given by

$$\begin{aligned} C(0, T) &= P(0, s)\Phi(d_1) - KP(0, T)\Phi(d_1 - H), \\ H &= \int_0^T [\sigma(u, s) - \sigma(u, T)]^2 du, \\ d_1 &= \frac{1}{H} \ln \frac{P(0, s)}{KP(0, T)} + \frac{H}{2}. \end{aligned} \quad (16.190)$$

More generally, for the evaluation at time t instead of 0, the preceding result becomes:

$$\begin{aligned}
 C(t, T) &= P(t, s)\Phi(h_+) - KP(t, T)\Phi(h_-), \\
 h_+ &= \frac{1}{V(s, T)} \ln \frac{P(t, s)}{KP(t, T)} + \frac{V(t, T)}{2}, \\
 h_- &= \frac{1}{V(s, T)} \ln \frac{P(t, s)}{KP(t, T)} - \frac{V(t, T)}{2},
 \end{aligned}
 \tag{16.191}$$

with

$$V(t, T)^2 = \int_t^T [\sigma(u, s) - \sigma(u, T)]^2 du.$$

For the puts, we use the call parity formula:

$$Call(t, T) - Put(t, T) = P(t, s) - KP(t, T),
 \tag{16.192}$$

and after calculation, the final result is given by

$$Put(t, T) = KP(t, T)N(-h_-) - P(t, s)N(-h_+).
 \tag{16.193}$$

16.4.5.2. Particular case of the OUV model

Here, we have

$$H^2 = \frac{\sigma^2}{a^2} \left(\frac{1 - e^{2aT}}{2a} \right) (1 - e^{-a(s-T)})^2.
 \tag{16.194}$$

16.4.6. Option on bond with coupons (Jamshidian (1989))

The exact value of an option on a bond with coupons was first given by Jamshidian as a linear combination of options on zero coupons.

Before giving his result, it is necessary to introduce some notions.

Let r^*b be the yield rate at time T , where T is the maturity of the option such that the price of the considered bond is equal to the exercise price of the call option.

Let s_j represent the j th date of coupon maturity with $j=1, \dots, n$, should be after time T : ($s_j > T, j=1, \dots, n$) and let $c_j, j=1, \dots, n$, be the value of the j th coupon.

Jamshidian also introduced the values $K_j, j = 1, \dots, n$ defined as:

$$K_j = P(r^*, T, s_j), j = 1, \dots, n \quad (16.195)$$

which is the value of a zero coupon at time T with $r(T)=r^*$ and of maturity S_j .

As the price of a bond is a decreasing function of the spot rate r , the investor will exercise the call if and only if $r < r^*$ and if a zero coupon of maturity S_j will be larger than $c_j K_j$, Jamshidian proved that the value $C(t, T)$ of the European call on the bond is given by the following linear combination:

$$C(t, T) = \sum_{j=1}^n C(t, T, s_j, K_j), \quad (16.196)$$

$C(t, T, s_j, K_j)$ being the value of an European call of maturity time T with K_j as exercise price on a zero coupon expiring at time s_j .

It can also be proved that (El Karoui and Rochet (1989)):

$$\begin{aligned} C(0, T) &= \sum_{j=1}^n [c_j P(0, s_j) \Phi(d_j) - KP(0, T) \Phi(d_0)], \\ d_j &= d_0 + H_j, j = 1, \dots, n, \\ H_j^2 &= \int_0^T [\sigma(u, s_j) - \sigma(u, T)]^2 du, \\ d_0 &: \sum_{i=1}^n c_i P(0, s_i) e^{-\frac{1}{2}H_i^2 + d_0 H_i} = KP(0, T). \end{aligned} \quad (16.197)$$

For the put, the relation of parity leads to the following result:

$$Put(0, T) = KP(0, T) \Phi(-d_0) - \sum_{j=1}^n c_j P(0, s_j) \Phi(-d_j). \quad (16.198)$$

16.4.7. A numerical example

The next table provides the result of zero coupon values with CIR models with four scenarios given by the five parameters, selected as given by lines 2 and 3.

| scen. | I | II | III | IV |
|-------------|-------------|-------------|-------------|------------|
| par. | 1-3.5%-3.5% | 1-3.5%-3.5% | 5-3.5%-3.5% | 1-6%-3.5% |
| | 3%-1% | 25%-1% | 3%-1% | 3%-1% |
| mat. | CIR | CIR | CIR | CIR |
| 0.25 | 0.99128664 | 0.99127952 | 0.99128701 | 0.98863114 |
| 0.5 | 0.98264681 | 0.98263459 | 0.98264889 | 0.97490507 |
| 1 | 0.96558803 | 0.96561683 | 0.96559715 | 0.94644076 |
| 3 | 0.90024357 | 0.90102057 | 0.90029773 | 0.8394155 |
| 5 | 0.83931463 | 0.84105891 | 0.83941429 | 0.74446771 |
| 7 | 0.78325086 | 0.78512914 | 0.78264807 | 0.66025964 |
| 10 | 0.70442635 | 0.7081375 | 0.70461487 | 0.55146515 |
| 20 | 0.49620034 | 0.5020042 | 0.49648109 | 0.30266638 |

Table 16.1. Zero coupons values with CIR

16.5. Appendix (solution of the OUV equation)

To solve the OUV equation:

$$\begin{aligned} dr(t) &= a(b - r(t))dt + \sigma dB(t), \\ r(0) &= r_0, \end{aligned} \tag{16.199}$$

let us start from the deterministic version

$$\begin{aligned} dr(t) &= a(b - r(t))dt \\ r(0) &= r_0, \end{aligned} \tag{16.200}$$

for which the general solution is given by

$$\begin{aligned} r(t) &= b + ce^{-at}, \\ c &\text{ constant.} \end{aligned} \tag{16.201}$$

Now let us suppose that c is also a function of t such that the function

$$r(t) = b + c(t)e^{-at}$$

is the solution of the SDE (16.199).

From Proposition 7.1 in Chapter 4, we know that:

$$dr = e^{-at} dc - ae^{-at} c(t) dt \quad (16.202)$$

and from relation (16.201), we obtain:

$$dr = e^{-at} dc + a(b - r(t)) dt. \quad (16.203)$$

and comparing with relation (16.199), we obtain:

$$e^{-at} dc + a(b - r(t)) dt = a(b - r(t)) dt + \sigma dB(t), \quad (16.204)$$

and so

$$e^{-at} dc = \sigma dB(t).$$

It follows that:

$$dc(t) = \sigma e^{at} dB(t)$$

and by relation (A.3)

$$r(t) = b + e^{-at} (c_0 + \sigma \int_0^t e^{as} dB(s)), \quad (16.205)$$

with

$$r_0 = b + c_0$$

or

$$c_0 = r_0 - b.$$

Substituting this last value in the first equality of relation (16.205), we obtain the announced solution in section 16.3.1.1:

$$r(t) = b + (r_0 - b)e^{-at} + \sigma e^{-at} \int_0^t e^{as} dB(s).$$