Chapter 15

Markov and Semi-Markov Option Models

15.1. The Janssen-Manca model

In this section, we present a new extension of the fundamental Black and Scholes (1973) formula in stochastic finance with the introduction of a random economic and financial environment using Markov processes, which we owe to Janssen and Manca (1999).

In preceding papers (Janssen, Manca and De Medici (1995), Janssen, Manca and Di Biase (1997), Janssen, Manca and Di Biase (1998), Janssen and Manca (2000)), these authors already show how it is useful to introduce Markov and semi-Markov theory to finance, with the assumption that the evolution of the asset follows a semi-Markov process, homogenous or non-homogenous, and how to price options in such new models. The main idea of this approach is to insert a strong dependence of the asset evolution as a function of the preceding value.

The construction of this new model starts from the traditional CRR model with one period to obtain a new continuous time model satisfying the absence of arbitrage assumption.

One of the main potential applications of our model concerns the possibility of obtaining a new way of using the Black and Scholes formula with information related to the economic and financial environment, particularly concerning the volatility of the underlying asset.

This new model also provides the possibility to take into account *anticipations* of investors in such a way as to incorporate them in their own option pricing.

In the same philosophy, the model can be used to construct scenarios, particularly in the case of stress in a VaR approach.

15.1.1. *The Markov extension of the one-period CRR model*

15.1.1.1. *The model*

Starting on a complete probability space $(\Omega, \Im, P)$, let us consider a one-period model for the evolution of one asset having the known value $S(0) = S_0$ at time 0 and random value $S(1)$ at time 1.

The economic and financial environment is defined with random variables J_0, J_1 representing the environment states respectively at time 0 and time 1. These random variables take their values in the state space $E = \{1,...,m\}$ and are defined on the probability space by:

$$\begin{aligned} &P\left(J_0 = i\right) = a_i, i = 1,...,m; \\ &P\left(J_1 \middle| J_0 = i\right) = p_{ij}, i, j = 1,...,m, \end{aligned} \tag{15.1}$$

where:

$$\begin{aligned} &a_i \geq 0, i = 1,...,m; \\ &\sum_{i=1}^{m} a_i = 1, \\ &p_{ij} \geq 0, i, j = 1,...,m, \\ &\sum_{j=1}^{m} p_{ij} = 1, \; i = 1,...,m. \end{aligned} \tag{15.2}$$

Furthermore, let us introduce the following function of J_0, J_1: $u_{J_0 J_1}, d_{J_0 J_1}$, $q_{J_0 J_1}$ such that, a.s.:

$$\begin{aligned} &0 < d_{J_0 J_1} < r_{J_0 J_1} < u_{J_0 J_1}, \\ &d_{J_0 J_1} < 1, \; 1 < r_{J_0 J_1}, \end{aligned} \tag{15.3}$$

$$0 < q_{J_0 J_1} < 1. \tag{15.4}$$

The *one-period model*, related to the process $\{S(0), S(1)\}$, is the following: given J_0, J_1 and that $S(0) = S_0$, the asset has the following evolution: it goes up from S_0 to $u_{J_0 J_1} S_0$ with the conditional probability $q_{J_0 J_1}$ or goes down from S_0 to

$d_{J_0 J_1} S_0$ with the conditional probability $1 - q_{J_0 J_1}$; moreover, the non-risky interest rate of this period has the value $\nu_{J_0 J_1}$ defined by:

$$\nu_{J_0 J_1} = r_{J_0 J_1} - 1. \tag{15.5}$$

Given J_0, J_1, we have:

$$\begin{aligned}
&P\left(S(1) = u_{J_0 J_1} S_0 \,|\, J_0, J_1, S_0\right) = q_{J_0 J_1},\\
&P\left(S(1) = d_{J_0 J_1} S_0 \,\|\, J_0, J_1, S_0\right) = 1 - q_{J_0 J_1},\\
&E\left(S(1) \| J_0, J_1, S_0\right) = q_{J_0 J_1} u_{J_0 J_1} S_0 + (1 - q_{J_0 J_1}) d_{J_0 J_1} S_0,\\
&E\left(S(1) \| J_0, S_0\right) = \sum_{j=1}^{m} p_{J_0 j} (q_{J_0 j} u_{J_0 j} S_0 + (1 - q_{J_0 j}) d_{J_0 j} S_0),\\
&E\left(S(1) | S_0\right) = \sum_{i=1}^{m} P\left(J_0 = i\right) \sum_{j=1}^{m} \left[p_{ij} (q_{ij} u_{ij} + (1 - q_{ij}) d_{ij}) \right] S_0.
\end{aligned} \tag{15.6}$$

One of the basic concepts of stochastic finance is the *absence of arbitrage possibility*. In fact, it is equivalent to state that the process $\{rS(0), S(1)\}$ is a martingale where $r = 1 + \rho$ and ρ is an adequate non-risky interest rate for calculating the present value of $S(1)$ at time 0.

Here, we must take into account the possible information of the investor concerning the environment; at time 0, in addition to the knowledge of S_0, different information sets may be available. Three cases are possible:

1) *Knowledge of* $\left(J_0, J_1\right)$

In this case, the martingale condition:

$$E\left(S(1) \| J_0, J_1, S_0\right) = r_{J_0 J_1} S_0 \tag{15.7}$$

becomes:

$$r_{J_0 J_1} S_0 = q_{J_0 J_1} u_{J_0 J_1} S_0 + (1 - q_{J_0 J_1}) d_{J_0 J_1} S_0 \tag{15.8}$$

or

$$r_{J_0 J_1} = q_{J_0 J_1} u_{J_0 J_1} + (1 - q_{J_0 J_1}) d_{J_0 J_1}. \tag{15.9}$$

This last condition is exactly the same as the CRR model; this means that the new conditional probability for which the martingale condition is satisfied is given by:

$$\tilde{q}_{J_0J_1} = \frac{r_{J_0J_1} - d_{J_0J_1}}{u_{J_0J_1} - d_{J_0J_1}}. \tag{15.10}$$

This value defines the *risk neutral conditional probability measure.*

As an example of its application in *option pricing*, let us consider that we want to study a European call option of maturity $T = 1$ and exercise price K bought at time 0.

It follows that at time 1 or at the end of the maturity, the value of the option will be given by the random variable:

$$C(S(1),0) = \max\{0, S(1) - K\}. \tag{15.11}$$

We calculate the price of the option at time 0 with a maturity period of value 1 as the conditional expectation under the risk neutral conditional probability measure, denoted $C_{J_0,J_1}(S_0,1)$, of the present value of the gain at time 1:

$$\begin{aligned} C_{J_0,J_1}(S_0,1) &= E\left(r_{J_0J_1}^{-1} \max\{0, S(1) - K\} \middle| J_0, J_1\right) \\ &= r_{J_0J_1}^{-1}\left[\tilde{q}_{J_0J_1} \max\left\{0, u_{J_0J_1}S_0 - K\right\} + (1 - \tilde{q}_{J_0J_1}) \max\left\{0, d_{J_0J_1}S_0 - K\right\}\right]. \end{aligned} \tag{15.12}$$

2) *Knowledge of* J_0

Let us begin to see what the martingale condition becomes.

We have:

$$E\left(S(1) \| J_0, S_0\right) = E\left(E\left(S(1) \| J_0, J_1, S_0\right) \middle| J_0, S_0\right). \tag{15.13}$$

As the assumption of AOA is now satisfied for the conditioning with J_0 and J_1, we can write that

$$E\left(S(1) \| J_0, S_0\right) = E\left(r_{J_0J_1} S_0 \middle| J_0, S_0\right), \tag{15.14}$$

and so:

$$E\left(S(1) \| J_{0,}, S_0\right) = S_0 E\left(r_{J_0J_1} \middle| J_0, S_0\right), \tag{15.15}$$

and finally:

$$E\left(S(1)\left\|J_{0,},S_{0}\right.\right)=\varsigma_{J_{0}}S_{0} \tag{15.16}$$

where:

$$\varsigma_{J_{0}}=\sum_{j=1}^{m}p_{J_{0}j}r_{J_{0}j}\ . \tag{15.17}$$

These last two formulae show that, given, at time 0, the initial environment state, the AOA is still valid with risk neutral interest

$$\rho_{J_{0}}=1-\varsigma_{J_{0}}, \tag{15.18}$$

or

$$\rho_{J_{0}}=\sum_{j=1}^{m}p_{J_{0}j}\nu_{J_{0}j}, \tag{15.19}$$

with $r_{J_0 j}$ given by relation (15.5) which is perfectly coherent as relation (15.19) represents the conditional mean of the non-risky interest rate given J_0.

3) *No environment knowledge*

In this last case, the investor merely observes the initial value of the stock S_0 as in the CRR or the Black and Scholes models. As above, we can calculate the expectation of S(1) as follows:

$$E\left(S(1)\left|S_{0}\right.\right)=E\left(E\left(S(1)\left|J_{0}\right.\right)\left|S_{0}\right.\right), \tag{15.20}$$

and from relation (15.16):

$$E\left(S(1)\left|S_{0}\right.\right)=S_{0}E\left(\varsigma_{J_{0}}\left|S_{0}\right.\right). \tag{15.21}$$

As, from relation (15.17), we obtain:

$$E\left(\varsigma_{J_{0}}\left|S_{0}\right.\right).=\sum_{i=1}^{m}a_{i}\sum_{j=1}^{m}p_{ij}r_{ij}\ , \tag{15.22}$$

it follows that the AOA is still true in this case with a non-risky interest rate ρ defined by:

$$\rho = 1 - \sum_{i=1}^{m} a_i \sum_{j=1}^{m} p_{ij} r_{ij}. \tag{15.23}$$

From this last relation and relation (15.19), we obtain

$$\begin{aligned}\rho &= \sum_{i=1}^{m} a_{\mathrm{i}} - \sum_{i=1}^{m} a_i \sum_{j=1}^{m} p_{ij}(1-\nu_{ij}) \\ &= \sum_{i=1}^{m} a_i \sum_{j=1}^{m} p_{ij}\nu_{ij} \\ &= \sum_{i=1}^{m} a_i \nu_i.\end{aligned} \tag{15.24}$$

Once more, these last two relations show the perfect coherence concerning the non-risky interest rates to be used with regard to the three environment information sets we can have.

15.1.1.2. *Calculational option pricing formula for the one-period model*

In the preceding section, relation (15.12) gives the value of a call option at time 0 given the initial and final environment states J_0 and J_1. We now calculate the price of the option, firstly with only the knowledge at time 0 of the initial environment state J_0, then with only the knowledge of the final state J_1 and finally with no knowledge of the initial and final states:

1) *with the knowledge of* J_0

This value, denoted by $C_{J_0}(S_0,1)$, is nothing other than the conditional expectation of $C_{J_0J_1}(S_0,1)$ given J_0:

$$C_{J_0}(S_0,1) = E\left(C_{J_0J_1}(S_0,1) \middle| J_0, S_0\right), \tag{15.25}$$

or

$$C_{J_0}(S_0,1) = \sum_{j=1}^{m} p_{J_0 j} C_{J_0 j}(S_0,1). \tag{15.26}$$

2) *with the knowledge of* J_1

Let $C^j(S_0,1)$ represent the value of the call, in this case when $J_1 = j$; we have:

$$C^j(S_0,1) = \sum_{i=1}^{m} P\left(J_0 = i \middle| J_1 = j\right) C_{ij}(S_0,1) . \tag{15.27}$$

From the Bayes formula, we obtain:

$$\begin{aligned} P\left(J_0 = i \middle| J_1 = j\right) &= \frac{P\left(J_0 = i, J_1 = j\right)}{P\left(J_1 = j\right)} \\ &= \frac{a_i p_{ij}}{\sum_{k=1}^{m} a_k p_{kj}} \end{aligned} \tag{15.28}$$

and so, from relation (15.27):

$$C^j(S_0,1) = \sum_{i=1}^{m} \frac{a_i p_{ij}}{\sum_{k=1}^{m} a_k p_{kj}} C_{ij}(S_0,1). \tag{15.29}$$

Let us note that this case is useful if the investor wants to anticipate the final value of the environment state at time 0.

3) *with no knowledge of* J_0 *and* J_1

In this case, with the help of relation (15.26), we can write that the call value represented by $C(S_0,1)$ is given by:

$$C(S_0,1) = \sum_{i=1}^{m} a_i C_i(S_0,1), \tag{15.30}$$

or with the help of relation (15.29) by:

$$C(S_0,1) = \sum_{j=1}^{m} \sum_{k=1}^{m} a_k p_{kj} C^j(S_0,0). \tag{15.31}$$

15.1.1.3. *Examples*

The application of our one-period model is already useful with only two or three states. Indeed, it is quite natural to consider one state, for example, state 0 to model the *normal* economic and financial environment; then we can add a supplementary state –1 to represent an *abnormal* situation like a crash or a doped situation.

With three states, we can separate the crash possibility represented by state –1 from the doped situation represented by state 1, state 0 always being the normal case.

Example 15.1 *A two-state model*

As stated just above, let the state set be:

$$I = \{0,1\} \tag{15.32}$$

with state 0 as the *normal* economic and financial situation environment and state 1 as the *exceptional* in the sense of, for example, a crash or doped situation.

Numerical data are the following:

$$\begin{aligned}
&\mathbf{a} = (0.95, 0.05),\\
&\mathbf{P} = \begin{bmatrix} 0.98 & 0.02 \\ 0.60 & 0.4 \end{bmatrix}, \ \underline{\mathbf{v}} = \begin{bmatrix} 1.03 & 1.05 \\ 1.05 & 1.03 \end{bmatrix},\\
&\underline{\mathbf{U}} = \begin{bmatrix} 1.3 & 1.1. \\ 1.06 & 1.2 \end{bmatrix}, \ \underline{\mathbf{D}} = \begin{bmatrix} 0.7 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}.
\end{aligned} \tag{15.33}$$

Example 15.2 *A three-state model*

Here, let the state set be:

$$I = \{-1,0,1\}. \tag{15.34}$$

State 0 represents the *normal* economic and financial situation environment, state –1 the *exceptionally bad* situation in the sense of, for example, a crash situation and state 1 as *exceptionally good* as a doped effect of the Stock Exchange, for example.

Numerical data are the following:

$$\begin{aligned}
&\mathbf{a} = (0.05, 0.90, 0.05),\\
&\mathbf{P} = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.02 & 0.96 & 0.02 \\ 0.6 & 0.35 & 0.05 \end{bmatrix}, \ \underline{\mathbf{v}} = \begin{bmatrix} 1.05 & 1.03 & 10.2 \\ 1.05 & 1.03 & 10.2 \\ 1.06 & 1.04 & 10.3 \end{bmatrix},\\
&\underline{\mathbf{U}} = \begin{bmatrix} 1.07 & 1.10 & 1.20 \\ 1.07 & 1.10 & 1.20 \\ 1.07 & 1.09 & 1.15 \end{bmatrix}, \ \underline{\mathbf{D}} = \begin{bmatrix} 0.5 & 0.7 & 0.8 \\ 0.6 & 0.7 & 0.8 \\ 0.65 & 0.7 & 0.8 \end{bmatrix}.
\end{aligned} \tag{15.35}$$

For both examples, we will consider a European call option with $S_0 = 100$ and $K = 80$ and 95.

Results are given in Table 15.1.

S	100										
K	95										
Example 1											
transition	A1	a2	a3	p(ij)	r(ij)	u(ij)	d(ij)	q(ij)	Cij(100,1)	Ci(100,1)	C(100,1)
0 to 0	0.95	0.05		0.98	1.03	1.3	0.7	0.55	2.6699	2.7038	
0 to 1				0.02	1.05	1.1	0.5	0.9167	4.3651		
1 to 0				0.6	1.05	1.06	0.4	0.9848	4.6898	4.2054	
1 to 1				0.4	1.03	1.2	0.6	0.7167	3.4790		
											2.7789
Example 2											
	0.05	0.9	0.05								
bad to bad				0.6	1.05	1.07	0.5	0.9649	4.5948	4.2280	
bad to normal				0.3	1.03	1.1	0.7	0.825	4.0049		
bad to good				0.1	1.02	1.2	0.8	0.55	2.6961		
normal to bad				0.02	1.05	1.07	0.6	0.9574	4.5594	4.3275	
normal to normal				0.96	1.03	1.07	0.7	0.8919	4.3296		
normal to good				0.02	1.02	1.07	0.8	0.8148	3.9942		
good to bad				0.6	1.02	1.2	0.65	0.6727	3.2977	3.2361	
good to normal				0.35	1.02	1.2	0.7	0.64	3.1373		
good to good				0.05	1.03	1.15	0.8	0.6571	3.1900		
											4.2679

Table 15.1. *European call option examples*

15.1.2. *The multi-period discrete Markov chain model*

Let us now consider a multi-period model over the time interval $[0,n]$, n being an integer larger than 1, always under the assumption of absence of arbitrage.

To obtain useful results, we will still follow the fundamental presentation of the CRR model (Cox, Rubinstein (1985)) but adapted for our Markov environment in such a way that tractable results may be found:

1) *result with knowledge of* $J_0,...,J_n$

Let us begin with a discrete time model with n periods and suppose that given $J_0,...,J_n,S(0)=S_0$ with $J_0=i,J_n=j,$ the up and down parameters, the non-risky interest rate and the probabilities of an up movement for each period are the same for all periods and given respectively by u_{ij},d_{ij},r_{ij} and q_{ij}.

Then, the asset value $S(n)$ at time n is given by:

$$S(n)=V_{j_0 j_1}\cdots\cdots V_{j_{n-1}j_n}S_0 \tag{15.36}$$

where the conditional distributions of the random variables V are defined as:

$$V_{J_{n-1}J_n}=\begin{cases} u_{ij} \text{ with probability } q_{ij}, \\ d_{ij} \text{ with probability } 1\text{-}q_{ij}, \end{cases} i,j\in I. \tag{15.37}$$

Moreover, we suppose that, for each n, the random variables $V_{J_0J_1},...,V_{J_{n-1}J_n}$ are conditionally independent given $J_0,...,J_n$.

If the random variable M_n represents the total number of up movements on $[0,n]$, the asset value at time n is given by:

$$S(n)=(u_{ij})^{M_n}(d_{ij})^{n-M_n}S_0 \tag{15.38}$$

and consequently:

$$\ln\frac{S(n)}{S_0}=M_n\ln u_{ij}+(n-M_n)\ln d_{ij}. \tag{15.39}$$

Given $J_0 = j_0,...,J_n = j_n, S(0) = S_0$, the conditional distribution of M_n is a binomial distribution with parameters (n, q_{ij}). It follows that:

$$E\left(\ln\frac{S(n)}{S_0}\Big|J_0 = j_0,...,J_n = j_n, S(0) = S_0\right) = n(q_{ij}\ln u_{ij} + (1-q_{ij})\ln d_{ij}). \quad (15.40)$$

Concerning the conditional variance, we obtain:

$$\text{var}\left(\ln\frac{S(n)}{S_0}\Big|J_0 = j_0,...,J_n = j_n, S(0) = S_0\right) = n\left[q_{ij}(1-q_{ij})\left(\ln\frac{u_{ij}}{d_{ij}}\right)^2\right]. \quad (15.41)$$

Choosing now for the up probability on the *n* periods, the risk neutral probability given by relation (15.10):

$$\tilde{q}_{ij_1} = \frac{r_{ij_1} - d_{ij_1}}{u_{ij_1} - d_{ij_1}}, \quad (15.42)$$

it is clear that, under our assumptions, for each *n*, given $J_0,...,J_n, S(0) = S_0$ with $J_0 = i,\ J_n = j,$ we have a CRR model, so that their results recalled in the beginning of this chapter concerning the European call are valid. Consequently, we obtain the value of the European call with exercise price and maturity *n* as the present value of the expectation of the "gain" at time *n* under the risk neutral measure, that is:

$$\begin{aligned} &C(S_0, 0 | J_0 = i, J_1,..., J_n = j) \\ &= \frac{1}{v_{ij}^n}\sum_{k=0}^{n}\binom{n}{k}\tilde{q}_{ij}^k(1-\tilde{q}_{ij})^{n-k}\max\left\{u_{ij}^k d_{ij}^{n-k} S_0 - K\right\}. \end{aligned} \quad (15.43)$$

After some calculation, we can obtain the following expression (see Cox and Rubinstein (1985)):

$$\begin{aligned} &C(S_0, n | J_0 = i, J_1,..., J_n = j) \\ &= \begin{cases} S_0 B(a_{ij}; n, \tilde{q}'_{ij}) - \dfrac{K}{v_{ij}^n} B(a_{ij}; n, \tilde{q}_{ij}), & \text{if } a_{ij} < n, \\ 0 & \text{if } a_{ij} > n, \end{cases} \end{aligned} \quad (15.44)$$

where $B(x;m,\alpha)$ is the value of the complementary binomial distribution function complementary with parameters m,α at point x and

$$a_{ij}=\left[\frac{\ln(K/d_{ij}^{n}S_0)}{\ln(u_{ij}/d_{ij})}+1\right],$$
$$\tilde{q}'_{ij}=\frac{u_{ij}}{r_{ij}}q_{ij}. \tag{15.45}$$

Result (15.44) can be seen as the *discrete time extension of the Black and Scholes formula* given the environment:

$$J_0=i,...,J_n=j,S(0)=S_0\,. \tag{15.46}$$

2) *result with knowledge of* $J_0=i$

If we only know the initial state of the environment $J_0=i$, it is clear that the value of the call is given by

$$C_i(S_0,n)=\sum_{j=1}^{m}p_{ij}^{(n)}C_{ij}(S_0,n) \tag{15.47}$$

where, of course:

$$\left[p_{ij}^{(n)}\right]=\mathbf{P}^n. \tag{15.48}$$

3) *result with knowledge of* $J_n=j$

Proceeding as in the previous section, the use of the Bayes formula provides the following result, now on *n* periods instead of one:

$$P\left(J_0=i\middle|J_n=j\right)=\frac{P\left(J_0=i,J_n=j\right)}{P\left(J_n=j\right)}$$
$$=\frac{a_i\,p_{ij}^{(n)}}{\sum_{k=0}^{m}a_k\,p_{kj}^{(n)}} \tag{15.49}$$

and so the value of the call given J_n=j, represented by $C^j(S_0,n)$, is given by:

$$C^j(S_0,n)=\sum_{i=1}^{m}\frac{a_i p_{ij}^{(n)}}{\sum_{k=0}^{m} a_k p_{kj}^{(n)}}C_{ij}(S_0,n). \tag{15.50}$$

4) *result with no environment knowledge*

Finally, if we have no knowledge on the initial environment state but know its probability distribution given by (15.1), the value of the call denoted $C(S_0,n)$ is given by

$$C(S_0,n)=\sum_{i=1}^{m} a_i C_i(S_0,n) \tag{15.51}$$

or by

$$C(S_0,n)=\sum_{j=1}^{m}\sum_{k=1}^{m} a_k p_{kj}^{(n)} C^j(S_0,n). \tag{15.52}$$

15.1.3. *The multi-period discrete Markov chain limit model*

To construct our continuous time model on the time interval [0,t], let us begin to consider a multi-period discrete Markov chain model with n periods, where each period has length h so that we have equidistant observations at time 0,h,2h,...,nh with $n=\lfloor t/h \rfloor$.

We also assume that in the approximated discrete time model, the environment process is a homogenous ergodic Markov chain defined by relations (15.1) and (15.2) and that (see Cox and Rubinstein (1985)), for each n, given $J_0,\ldots,J_n,S(0)=S_0$ with $J_0=i, J_n=j$, we select, in each subinterval $\left[kh,(k+1)h\right]$, the following up and down parameters:

$$\begin{aligned} &u_{j_k j_{k+1}}=e^{\sigma_{ij}\sqrt{\frac{t}{n}}}, d_{j_k j_{k+1}}=e^{-\sigma_{ij}\sqrt{\frac{t}{n}}},\\ &q_{j_k j_{k+1}}=\frac{1}{2}+\frac{1}{2}\frac{\mu_{ij}}{\sigma_{ij}}\sqrt{\frac{t}{n}}, \end{aligned} \tag{15.53}$$

thus depending on the two $m \times m$ non-negative matrices:

$$\left[\mu_{ij}\right], \left[\sigma_{ij}\right]. \tag{15.54}$$

From relations (15.40) and (15.41), it follows that, for all *n*:

$$E\left(\ln \frac{S(n)}{S_0} \middle| J_0 = j_0, ..., J_n = j_n, S(0) = S_0 \right) = \mu_{ij} t, \tag{15.55}$$

$$\operatorname{var}\left(\ln \frac{S(n)}{S_0} \middle| J_0 = j_0, ..., J_n = j_n, S(0) = S_0 \right) = \sigma_{ij}^2 t. \tag{15.56}$$

As our conditioning implies that we can follow the reasoning of Cox and Rubinstein (1985), we know that, for $n \to +\infty$:

$$\ln \frac{S(t)}{S_0} \prec N(\mu_{ij} t, \sigma_{ij}^2 t), \tag{15.57}$$

where $j_0 = i$ as the initial environment state observed at $t = 0$ and *j* the environment state at time *t*.

Concerning the non-risky interest rates, we also suppose that, for all *i* and *j*, there exists $\nu_{ij} > 1$ such that the new return rate for all the periods $\left(kh, (k+1)h\right)$, denoted $\hat{r}_{ij}$, for $n \to +\infty$, satisfies the following condition:

$$(1 + r_{ij})^n \to (1 + \hat{r}_{ij})^t . \tag{15.58}$$

Now let $C_{ij}(S_0, n)$ represent the value at time 0 of a European call option with maturity *n* and exercise price *K*.

Using the proof of the Black and Scholes formula given by Cox and Rubinstein (1985)) but with our parameters depending on all on the environment states *i* and *j*, we obtain under conditions (15.53) and (15.58), for fixed *t*:

$$C_{ij}(S_0, n) \to C_{ij}(S_0, t) \tag{15.59}$$

where:

$$\begin{aligned}
&C_{ij}\ (S_0,t)=S_0\Phi(d_{ij_t1})-Kr_{ij}^{-t}\Phi(d_{ij_t,2}),\\
&d_{ij_t,1}=\frac{\ln\dfrac{S_0}{Kr_{ij}^{-1}}}{\sigma_{ij}\sqrt{t}}+\frac{1}{2}\sigma_{ij}\sqrt{t},\\
&d_{ij_t,2}=d_{ij,1}-\sigma_{ij}\sqrt{t}.
\end{aligned}\tag{15.60}$$

This result gives the value of the call at time 0 with *i* as the initial environment state and *j* as the environment state observed at time *t*, represented from now by J_t.

If we want to use the traditional notation in the Black and Scholes (1973) framework, we can define the instantaneous interest rate intensity ρ_{ij} such that:

$$r_{ij}=e^{\rho_{ij}}\tag{15.61}$$

so that the preceding formula (15.60) now becomes:

$$\begin{aligned}
&C_{ij_t}(S_0,t)=S_0\Phi(d_{ij_t1})-Ke^{-\rho_{ij}t}\Phi(d_{ij_t,2}),\\
&d_{ij_t,1}=\frac{1}{\sigma_{ij}\sqrt{t}}\left(\ln\frac{S}{K}+\left(\rho_{ij}-\frac{\sigma_{ij}^2}{2}\right)t\right),\\
&d_{ij_t,2}=d_{ij,1}-\sigma_{ij}\sqrt{t}.
\end{aligned}\tag{15.62}$$

15.1.4. *The extension of the Black-Scholes pricing formula with Markov environment: the Janssen-Manca formula*

The last result (15.62) gives a first extension of the Black and Scholes formula in continuous time from the knowledge of the initial and final environment states, respectively J_0 and J_t where J_t represents, as stated above, the state of the environment at time *t*.

Now, always with the assumption that the Markov chain with matrix **P** is ergodic, we can extend results (15.44), (15.50) and (15.52) valid for our discrete multi-period model to our continuous time model, thus giving the following main result.

Proposition 15.1 (Janssen and Manca (1999))

Under the assumption that the Markov chain of matrix **P** *of the environment process is ergodic and given that the initial environment state* $i \in I$ *and the environment state at time t is* $j \in I$, *the non-risky rate is given by* ρ_{ij} *and the annual volatility by* σ_{ij}, *then we have the following results concerning the European call price at time* 0 *with exercise price K and maturity t:*

(1) *with knowledge of state* $J_0 = i, J_t = j$, *the call value is given by result* (15.62),

(2) *with knowledge of state* $J_0 = i$, *the call value represented by* $C_i(S_0,t)$ *is given by:*

$$C_i(S_0,t) = \sum_{j=1}^{m} \pi_j C_{ij}(S_0,t), \tag{15.63}$$

(3) *with knowledge of state* $J_t = j$, *the call value represented by* $C^j(S_0,t)$ *is given by:*

$$C^j(S_0,t) = \sum_{i=1}^{m} a_i C_{ij}(S_0,t), \tag{15.64}$$

(4) *without any environment knowledge, the call value represented by* $C(S_0,t)$ *is given by:*

$$C(S_0,t) = \sum_{i=1}^{m} a_i C_i(S_0,t) \tag{15.65}$$

or

$$C(S_0,t) = \sum_{j=1}^{m} \pi_j C^j(S_0,t)\,. \tag{15.66}$$

Proof Result (1) is proved in the previous section.

Result (2) follows from relation (15.47), letting n go to $+\infty$ and then using result (1) and the assumption of ergodicity on the environment matrix chain **P**.

Result (3) can easily be deduced from result (2) and relation (15.50).

Finally, result (4) follows immediately from relations (15.51) or (15.52) and results (2) and (3). □

Example

Examples 15.1 and 15.2 of the preceding section are covered in Table 15.2 where “?” means “unknown”.

Example 1				
K	80		K	80
S	100		S	100
0 to 0			0 to 0	
	t	Cij(100,t)	t	Cij(100,t)
	0.25	22.18	0.25	11.84
	0.5	24.87	0.5	15.69
	0.75	27.24	0.75	18.7
	1	29.35	1	21.26
1 to 0	0.25	22.01	0.25	11.18
	0.5	24.54	0.5	14.86
	0.75	26.83	0.75	17.8
	1	28.91	1	20.32
? to 1	0.25	21.57	0.25	10.17
	0.5	23.64	0.5	13.42
	0.75	25.61	0.75	16.03
	1	27.43	1	18.29
? to ?	0.25	22.11	0.25	11.31
	0.5	24.35	0.5	14.54
	0.75	26.58	0.75	17.43
	1	28.62	1	19.93

Table 15.2. *Janssen Manca option model results*

In conclusion, the Janssen-Manca approach gives for the first time a new family of Black and Scholes formulae taking into account the economic and social environment showing that:

– a “good” extension of the traditional Cox Rubinstein model is possible;

– the model also extends the Black and Scholes model;

– numerical results are possible.

Moreover, as the Janssen-Manca formulae are linear combinations of the traditional Black-Scholes results, the Greek parameters can also be calculated and will be linear combinations of the Greek parameters given in section 14.6 and similarly for hedging coefficients.

We also add that, from our point of view, one of the main potential applications of our new model concerns the possibility of obtaining a new way of using the Black and Scholes formula with information related to the economic, financial and even political environment, provided it can be modeled by an ergodic homogenous Markov chain.

This model also provides the possibility of taking into account *anticipations* made by the investors in such a way as to incorporate them in their own option pricing and can also be used for models with financial crashes as well as to construct scenarios, and particularly in the case of stress in a VaR type approach.

15.2. The extension of the Black-Scholes pricing formula with a semi-Markov environment: the Janssen-Manca-Volpe formula (Janssen and Manca (2007))

15.2.1. *Introduction*

In this section, we present the semi-Markov (SM) extension of the Black and Scholes formula to the Janssen-Manca-Volpe model to eliminate one of the restrictions of the Black and Scholes model, that is, the assumption of constant volatility over time.

There have been many attempts to slacken this condition, as for example in the model of Hull and White (1985) where the concept of stochastic volatility is introduced, but to our knowledge, in practice, no generalized model really supplants the traditional Black and Scholes model.

Whilst comparing the Markovian Janssen-Manca model of the preceding section, we developed another type of model. More precisely, we present new semi-Markov models for the evolution of the volatility of the underlying asset.

In fact, the SM model presented here assumes a type of SM evolution for the volatility of an initial Black-Scholes model presented at the ETH Zurich (1995) by Janssen, and in a different approach by E. Çinlar at the First Euro-Japanese meeting on Insurance, Finance and Reliability held in Brussels in 1998 which led to a generalization of the traditional Black and Scholes formula for the pricing of European calls with easy numerical applications.

15.2.2. *The Janssen-Manca-Çinlar model*

The semi-Markov extension of the Black and Scholes model assumes a type of SM evolution for the volatility of an initial Black and Scholes model presented by Janssen (1995) and, more recently, in a different approach by Çinlar (1998).

Hereby, we present Janssen's initial model which is similar to the presentation of Çinlar, however he provides the formula for the pricing of a call option using the Markov renewal theory.

15.2.2.1. *The JMC (Janssen-Manca-Çinlar) semi-Markov model (1995, 1998)*

Let us consider a *two-dimensional* positive (*J*-*X*) process of kernel **Q** with state space:

$$I = \{1, ..., m\}. \tag{15.67}$$

This means that on the probability space $(\Omega, \Im, \mathrm{P})$, we define the *three-dimensional* process

$$\left(\left(J_n, (X_n, \sigma_n)\right), n \geq 0\right) \tag{15.68}$$

with:

$$J_n \in I, (X_n, \sigma_n) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{15.69}$$

such that:

$$\begin{aligned} & P\left(X_n \leq x, \sigma_n \leq \sigma, J_n = j \middle| \left(J_k, (X_k, \sigma_k)\right), k = 0.1.....n-1\right) \\ & = Q_{J_{n-1}j}(x, \sigma), p.s. \end{aligned} \tag{15.70}$$

We know that the $Q_{ij}, i, j \in I$ can be written in the following form:

$$Q_{ij}(x, \sigma) = p_{ij} F_{ij}(x, \sigma) \tag{15.71}$$

where:

$$p_{ij} = P\left(J_n = j \middle| J_k, k \leq n-1, J_{n-1} = i\right), \tag{15.72}$$

$$F_{ij}(x, \sigma) = P\left(X_n \leq x, \sigma_n \leq \sigma \middle| (J_k, (X_k, \sigma_k)), k \leq n-1, J_{n-1} = i\right). \tag{15.73}$$

We also introduce the following r.v.:

$$\begin{aligned} &T_n = X_1 + \cdots + X_n, n \geq 0, \\ &N(t) = \sup\{n : T_n \leq t\}, t \geq 0, \\ &Z(t) = J_{N(t)}, t \geq 0. \end{aligned} \tag{15.74}$$

As usual, the transition probability for the process $Z = \left(Z(t), T \geq 0\right)$ is designed by:

$$\phi_{ij}(t) = P\left(Z(t) = j \middle| Z(t) = i\right) \tag{15.75}$$

and the stochastic processes $(N(t), t \in \mathbb{R}^+), (Z(t), t \in \mathbb{R}^+)$ are respectively the Markov renewal counting and the semi-Markov processes.

To give the financial interpretation of our model, let us define on the probability space $\left(\Omega, \mathfrak{I}, P\right)$, the following filtration $\mathfrak{I} = (\mathfrak{I}_t, t \in \mathbb{R}^+)$,

$$\mathfrak{I}_t = \sigma((J_n, (X_n, \sigma_n)), n \leq N(t)). \tag{15.76}$$

Given $\mathfrak{I}_{\mathrm{t}}$, let us consider the random time interval $\left[T_{N(t)}, T_{N(t)+1}\right]$ on which we define the new stochastic process $(S(t), t \in \mathbb{R}^+)$, representing the value of the considered financial asset, as the solution of the stochastic differential equation:

$$\begin{aligned} &\frac{dS}{S(t')} = \mu_{J_{N(t)}J_{N(t)+1}} dt' + \sigma_{J_{N(t)}J_{N(t)+1}} dW_{J_{N(t)}J_{N(t)+1}}(t' - T_{N(t)}), t' \in \left[T_{N(t)}, T_{N(t)+1}\right], \\ &S(T_{N(t)}) = S(T_{N(t)}-), \end{aligned} \tag{15.77}$$

where process $(W_{J_{N(t)}J_{N(t)+1}}(t'), t' \geq 0)$ is a standard Brownian motion on $\left[T_{N(t)}, T_{N(t)+1}\right]$ defined on the basic probability space stochastically independent on $\left(J_{N(t)}, X_{N(t)}\right)$.

This model has the following financial interpretation: at $t = 0$, the asset starts from the known initial value S_0, with the known initial *j*-state J_0 representing the state of the initial economic and financial environment. On the time interval X_1, the asset has the random volatility σ_1 and has as stochastic dynamics the SDE (15.77) with $t = 0$; at time X_1, the *J* process has a transition to state J_1 and on the time interval $\left[T_1, T_2\right)$, the asset has the random volatility σ_2 and has as stochastic dynamics the SDE (15.77) with $N(t) = 1$, etc.

We always define $X_0 = 0$, a.s.

So, it is now clear that we have in fact a disrupted Black and Scholes model due to this random change of volatility; note that this model is quite general as, in fact, we have a random volatility on each time interval $\left[T_{N(t)}, T_{N(t)+1}\right]$.

Of course, for m = 1, we recover the traditional Black-Scholes-Samuelson model for the description of an asset.

15.2.2.2. *The explicit expression of S(t)*

Given $J_{N(t)}, J_{N(t)+1}$, the Itô calculus gives the solution of the SDE (15.77):

$$S(t') = S_{N(t)} e^{\left(\mu_{J_{N(t)}J_{N(t)+1}} - \frac{\sigma^2_{J_{N(t)}J_{N(t)+1}}}{2}\right)t'} e^{\sigma_{J_{N(t)}J_{N(t)+1}} W(t'-T_{N(t)})}, \tag{15.78}$$
$$t' \in \left[T_{N(t)}, T_{N(t)+1}\right].$$

Starting from state S_0 at time 0 and given a scenario for the economic and financial environment $(J_0, J_1, ..., J_n, ...)$, this expression gives the explicit form of the trajectories of the process $(S(t), t \geq 0)$.

Now, given $(J_0, X_0, J_1, X_1, ..., J_{N(t)}, X_{N(t)}, J_{N(t)+1}, X_{N(t)+1})$, from relation (15.78), we obtain:

$$\ln \frac{S(t')}{S_{N(t)}} = \left(\mu_{J_{N(t)}J_{N(t)+1}} - \frac{\sigma^2_{J_{N(t)}J_{N(t)+1}}}{2}) t' + \sigma_{J_{N(t)}J_{N(t)+1}} W\left(t' - T_{N(t)}\right)\right), \tag{15.79}$$
$$t' \in \left[T_{N(t)}, T_{N(t)+1}\right],$$

so that for $t' \in \left[T_{N(t)}, T_{N(t)+1}\right]$:

$$\ln \frac{S(t')}{S_{N(t)}} \prec N\left(\mu_{J_{N(t)}J_{N(t)+1}} - \frac{\sigma^2_{J_{N(t)}J_{N(t)+1}}}{2}\right)(t' - T_{N(t)}), \tag{15.80}$$
$$\sigma^2_{J_{N(t)}J_{N(t)+1}}(t' - T_{N(t)}).$$

$$E\left(\frac{S(t)}{S_{N(t)}} \middle| \Im_t, J_{N(t)+1}\right) = e^{\mu_{J_{N(t)},J_{N(t)+1}}(t' - T_{N(t)})}, \tag{15.81}$$

$$\operatorname{var}\left(\frac{S(t)}{S_{N(t)}}\middle|\Im_t, J_{N(t)+1}\right) = e^{2\mu_{J_{N(t)},J_{N(t)+1}}(t'-T_{N(t)})}\left(e^{\sigma^2_{J_{N(t)}J_{N(t)+1}}(t'-T_{N(t)})} - 1\right). \quad (15.82)$$

Let us suppose that the random variables

$$S_0, J_0, X_1, J_1, ..., J_{N(t)}, X_{N(t)+1}, J_{N(t)+1}$$

are given; it follows that the conditional distribution function of $\frac{S(t)}{S_0}$ is a lognormal distribution, i.e.:

$$\ln\frac{S(t)}{S_0} \prec N\left(\mu_{J_0J_1}X_1 + \cdots + \mu_{J_{N(t)}J_{N(t)+1}}(t - T_{N(t)}), \sigma^2_{J_0J_1}X_1 + \cdots + \sigma^2_{J_{N(t)}J_{N(t)+1}}(t - T_{N(t)})\right). \quad (15.83)$$

15.2.3. ***Call option pricing***

Now to obtain a useful model, let us proceed as in Janssen and Manca (1999); for a fixed *t*, we assume that all the parameters μ, σ only depend on $J_0, J_{N(t)}, J_{N(t)+1}$, and *t* is represented by

$$\mu_{J_0J_{N(t)}J_{N(t)+1}}, \sigma_{J_0J_{N(t)}J_{N(t)+1}} \quad (15.84)$$

so that from relation (15.83):

$$\ln\frac{S(t)}{S_0} \prec N\left(\left(\mu_{J_0J_{N(t)}J_{N(t)+1}} - \frac{1}{2}\sigma^2_{J_0J_{N(t)}J_{N(t)+1}}\right)t, \sigma^2_{J_0J_{N(t)}J_{N(t)+1}}t\right). \quad (15.85)$$

Of course, we can always simplify our basic assumption by suppressing the dependence with respect to $J_{N(t)+1}$ and even to $J_{N(t)}$.

Nevertheless, we think that the dependence from the future environment state $J_{N(t)+1}$ is quite important as it gives for the first time the possibility of modeling the stochastic asset evolution taking into account this anticipation of the next future state.

Let us now consider a European call option with *t* as the maturity time, and *K* as the exercise price that we must price at time 0.

If we want to assume that there is no arbitrage possibility, we must impose that

$$\mu_{J_0 J_{N(t)} J_{N(t)+1}} = \delta_{J_0 J_{N(t)} J_{N(t)+1}} \tag{15.86}$$

where $\delta_{J_0 J_{N(t)} J_{N(t)+1}}$ represents the equivalent instantaneous non-risky return on $[0,t]$ given $J_0, J_{N(t)}, J_{N(t)+1}$. Doing so, we will use the risk-neutral measure under which the forward value of the asset is a martingale, otherwise we work with the initial "physical" measure more appropriate for insurance than for finance.

Knowing $J_0, J_{N(t)}, J_{N(t)+1}$ and working with the risk neutral measure, we can calculate the value of the call at time 0 using the traditional Black and Scholes formula:

$$\begin{aligned}
&C_{J_0 J_{N(t)} J_{N(t)+1}}(S_0,t) = S_0 \Phi(d_{J_0 J_{N(t)} J_{N(t)+1},1}) - K r_{J_0 J_{N(t)} J_{N(t)+1}}^{-t} \Phi(d_{J_0 J_{N(t)} J_{N(t)+1},2}),\\
&d_{J_0 J_{N(t)} J_{N(t)+1},1} = \frac{\ln \dfrac{S_0}{K r_{J_0 J_{N(t)} J_{N(t)+1}}^{-1}}}{\sigma_{J_0 J_{N(t)} J_{N(t)+1}} \sqrt{t}} + \frac{1}{2} \sigma_{J_0 J_{N(t)} J_{N(t)+1}} \sqrt{t},\\
&d_{J_0 J_{N(t)} J_{N(t)+1},2} = d_{J_0 J_{N(t)} J_{N(t)+1},1} - \sigma_{J_0 J_{N(t)} J_{N(t)+1}} \sqrt{t},\\
&\nu_{J_0 J_{N(t)} J_{N(t)+1}} = e^{\delta_{J_0 J_{N(t)} J_{N(t)+1}}}.
\end{aligned} \tag{15.87}$$

To obtain the formula of the call only knowing S_0, J_0, we must use the following formula:

$$C_{J_0}(t) = E\left(C_{J_0 J_{N(t)} J_{N(t)+1}}(S_0,t) \middle| J_0, S_0\right). \tag{15.88}$$

From the theory of semi-Markov processes, we obtain:

$$\begin{aligned}
&C_{J_0}(t) = E\left(C_{J_0 J_{N(t)} J_{N(t)+1}}(S_0,t) \middle| J_0, S_0\right),\\
&C_{J_0}(t) = \sum_{j \in I} \sum_{k \in I} P_{J_0 j}(t) p_{jk} C_{J_0 jk}(S_0,t).
\end{aligned} \tag{15.89}$$

If we have no information about the initial state J_0, we of course obtain the following formula:

$$C(t) = E\left(C_{J_0}(t)\right) = E\left(E\left(C_{J_0 J_{N(t)} J_{N(t)+1}}(S_0, t) \middle| J_0, S_0\right)\right),$$
$$C(t) = \sum_{i \in I} a_i C_i(t). \tag{15.90}$$

Remark 15.1 Numerical treatments are possible.

15.2.4. *Stationary option pricing formula*

In option pricing, it is nonsense to let t tend towards $+\infty$; nevertheless, we can use the limit reasoning proposed by Janssen by supposing that on the time horizon [0,t], the semi-Markov environment has more and more transitions in this finite time period.

We can model this situation under the assumption that the conditional sojourn time means that $b_{ij}, i, j \in I$ satisfy the conditions

$$b_{ij} = \varepsilon \varsigma_{ij}, \ \varepsilon > 0,$$
$$b_{ij} = E\left(X_n \middle| J_{n-1} = i, J_n = j\right) \tag{15.91}$$

so that:

$$\eta_i = \sum_{j \in I} p_{ij} b_{ij} = \varepsilon \sum_{j \in I} p_{ij} \varsigma_{ij} = \varepsilon \theta_i, i \in I,$$
$$\theta_i = \sum_{j \in I} p_{ij} \varsigma_{ij}. \tag{15.92}$$

From the asymptotic theory of semi-Markov processes, we know that:

$$\lim_{\varepsilon \to 0} P\left(J_{N(t)} = j, J_{N(t)+1} = k\right) = \frac{\pi_i p_{jk} \varsigma_{jk}}{\sum_{l=1}^{m} \pi_l \theta_l}, i, j \in I, \tag{15.93}$$

where the vector $\left(\pi_1, ..., \pi_m\right)$ is the unique stationary distribution of the embedded Markov chain of matrix **P** assumed to be ergodic.

The new parameters ς_{jk}, $i, j, k \in I$ represent factors expressing the proportionality of the sojourn in each environment state.

Now result (15.89) becomes:

$$C_{J_0}(t) = \sum_{j \in I} \sum_{k \in I} \frac{\pi_j p_{jk} \varsigma_{jk}}{\sum_{l=1}^{m} \pi_l \theta_l} C_{J_0 jk}(S_0, t). \tag{15.94}$$

From (15.90), we obtain

$$C(t) = \sum_{i \in I} a_i \sum_{j \in I} \sum_{k \in I} \frac{\pi_j p_{jk} \varsigma_{jk}}{\sum_{l=1}^{m} \pi_l \theta_l} C_{ijk}(S_0, t). \tag{15.95}$$

This last formula replaces the Black and Scholes formula without any *a priori* information at time 0 except of course the initial value of the asset S_0.

In conclusion, the new model proposed here extends the traditional Black and Scholes formula in the case of the existence of an economic and financial environment modeled with a homogenous semi-Markov process taking into account this environment not only at the time of pricing but also before and after the maturity date.

This new family of Black and Scholes formulae seems to be more adapted to the reality, particularly when taking into account the anticipations of the investor or the consideration of stress scenario in the philosophy of the VaR approach.

15.3. Markov and semi-Markov option pricing models with arbitrage possibility

The aim of this last part is the presentation of new models for option pricing, discrete in time and within the framework of Markov and semi-Markov processes as an alternative to the traditional Cox-Rubinstein model and giving arbitrage possibilities. Both cases of European and American options are considered and possible extensions are given.

15.3.1. *Introduction*

Let us consider an asset observed on a discrete time scale

$$\{0,1,...,t,...T\}, T < \infty \tag{15.96}$$

having $S(t)$ as market value at time t. To model the basic stochastic process

$$(S(t), t = 0,1,...,T), \tag{15.97}$$

we suppose that the asset has known minimal and maximal values so that the set of all possible values is the closed interval $[S_{\min}, S_{\max}]$ partitioned in a subset of m subclasses.

For example, if S_0 is the value of the asset at time 0, we can put:

$$\begin{aligned} &S_0 = \frac{S_{\max} - S_{\min}}{2}, \\ &S_k = S_0 + k\Delta, k = 1,...,\upsilon, \\ &S_{-k} = S_0 - k\Delta, k = 1,...,\upsilon, \\ &\Delta = \frac{S_{\max} - S_{\min}}{2\upsilon}, \end{aligned} \tag{15.98}$$

υ being arbitrarily chosen.

This implies that the total number of states is $2\upsilon + 1$. In the following, we will order these states in the natural increasing order and use the following notation for the state space:

$$I = \{-\upsilon, -(\upsilon - 1), ..., 0, 1, ..., \upsilon\}. \tag{15.99}$$

We can also introduce different step lengths following up or down movements and so consider respectively Δ, Δ'.

It is also possible to let

$$S_{\max} \to +\infty \tag{15.100}$$

and

$$T \to +\infty \tag{15.101}$$

particularly to obtain good approximation results.

Let us suppose we want to study a call option of maturity T and exercise price $K = k_0\Delta$ in both European and American cases bought at time 0.

So, in the European case, the intrinsic value of the option is given by:

$$C(T) = \max\{0, S(T) - K\}. \tag{15.102}$$

For the American case, the optimal time for exercising is given by the random time τ such that:

$$\max_{t=1,\dots,T} \max\{0, S_t - K\} = \max\{0, S_\tau - K\}. \tag{15.103}$$

To obtain results, we must now introduce in the following section a stochastic model for the *S*-process.

15.3.2. *The homogenous Markov model for the underlying asset*

Let us suppose that we are working on the filtered probability space $(\Omega, \Im, (\Im_t)P)$.

In our first model, we will suppose that the underlying asset *S* is a homogenous Markov chain with matrix:

$$\mathbf{P} = \left[p_{ij} \right] \tag{15.104}$$

on the state space *I* given by relation (15.99).

It follows that, at time *t*, given the knowledge of the asset value $S(t) = S_t$, the market value of the option at time *t*, *C*(*t*), thus with a remaining maturity *T*-*t* and exercise price *K* given by $K = k_0\Delta,$ has as the probability distribution:

$$\begin{aligned} &P(C(T) = (j - k_0)\Delta) = p_{S,j}^{(T-t)},\ j > k_0, \\ &P(C(T) = 0) = \sum_{l \le k_0} p_{S,j}^{(T-t)}. \end{aligned} \tag{15.105}$$

This result gives the possibility to calculate all interesting parameters concerning *C*. For example, the mean of *C*(*t*) has the value:

$$E(C(T) \big| S(t) = S_t) = \sum_{l > k_0} p_{S,j}^{(T-t)} (l - k_0)\Delta. \tag{15.106}$$

Of course, we have to calculate the present value at time *t* with the non-risky unit period interest rate *r* so that the value of the call at time *t* is given by:

$$\begin{aligned} &C(t) = v^{T-t} E(C(T) \big| S(t) = S_t) = v^{T-t} \sum_{l > k_0} p_{S,j}^{(T-t)} (l - k_0)\Delta, \\ &v = \frac{1}{1+r}. \end{aligned} \tag{15.107}$$

If matrix **P** is ergodic, then if *T-t* is large enough, results (15.105) and (15.106) can be well approximated by:

$$\begin{aligned}
&P(C(T)=(j-k_0)\Delta)=\pi_j,\ j>k_0,\\
&P(C(T)=0)=\sum_{l\le k_0}\pi_l,\ j\le k_0,\\
&E(C(T)\,|\,S(t)=S_t)=\sum_{l>k_0}\pi_j(l-k_0)\Delta,\\
&C(t)=v^{T-t}\sum_{l>k_0}\pi_j(l-k_0)\Delta.
\end{aligned} \tag{15.108}$$

Of course, the vector

$$\boldsymbol{\pi}=(\pi_{-\nu},\ldots,\pi_0,\ldots,\pi_\nu) \tag{15.109}$$

is the steady-state vector related to the matrix **P**.

15.3.3. *Particular cases*

As we stated in our introduction, our homogenous Markov model contains as a very special case the famous CRR binomial model but with fixed minimal and maximal values. It suffices to select a Markov matrix **P** with the structure

$$\begin{bmatrix}
* & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
* & 0 & * & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & * & 0 & * & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & * & 0 & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 0 & * & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & * & 0 & * & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & * & 0 & *\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & *
\end{bmatrix} \tag{15.110}$$

and as the Cox-Rubinstein model has a multiplicative form, we can consider that:

$$\Delta=\begin{cases}(u-1)S_0, u>1, S>S_0,\\(1-d)S_0, d<1, S<S_0.\end{cases} \tag{15.111}$$

Remark 15.2 Under (15.100), matrix **P** has an infinite number of rows and columns.

We can also obtain the *trinomial model* if we put in (15.110) a non-zero main diagonal, etc.

15.3.4. *Numerical example for the Markov model*

To numerically illustrate our first model, let us suppose that we are interested in an asset whose possible values are restricted to the following ones:

– maximum value: state 3 = 1,650;

– intermediary values: state 2 = 1,600, state 1 = 1,550, state 0 = 1,500;

– state –1 = 1,450, state –2 = 1,400;

– minimum value: state –3 = 1,350.

With the used notation, this means that $S_0 = 1{,}500,\ \Delta = 50.$ Moreover, we also suppose that the transition matrix **P**, with the week as unit step, is given by

$$\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{7} & \frac{2}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{7} & \frac{3}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} & \frac{2}{7} & \frac{1}{7} & \frac{1}{7} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} \tag{15.112}$$

It is easily seen that matrix **P** is ergodic with as unique stationary distribution:

(0.10002, 0.13336, 0.27228, 0.23737, 0.16927, 0.07539, 0.01231).

Then, starting at time 0 in state 1,500 with a maturity time of 16 weeks, the asymptotic value of the European call option expectation with 1,500 as exercise price is 41.95 and the call value at time 0 is 41.328.

Table 15.3 gives option expectations and option values with different exercise prices.

Exercise price	Option expectation	Option value
1,350	174.106	171.512
1,400	124.721	122.826
1,450	79.1059	77.927
1,500	41.9538	41.328
1,550	16.6704	16.422
1,600	5.00113	4.927
1,650	0	0

Table 15.3. *Markov option calculation*

Let us now consider the transient behavior, meaning that we will consider the maturity as a parameter expressed in *n* weeks. Table 15.4, gives option expectations, Table 15.5 option values with as exercise price 1,500 and for different maturity times from 1 to 16 weeks.

	STATE						
n	-3	-2	-1	0	1	2	3
1	75.00	75.00	57.14	25.00	14.29	7.14	0.00
2	60.71	53.57	46.93	38.39	30.10	20.41	16.96
3	50.02	48.40	43.39	40.60	37.08	31.61	31.39
4	45.70	44.92	42.79	41.11	39.61	37.39	37.44
5	43.70	43.30	42.35	41.57	40.84	39.87	39.81
6	42.76	42.58	42.13	41.78	41.45	40.98	40.96
7	42.33	42.24	42.04	41.87	41.72	41.50	41.50
8	42.13	42.09	41.99	41.92	41.84	41.75	41.74
9	42.03	42.02	41.97	41.94	41.90	41.86	41.86
10	41.99	41.98	41.96	41.95	41.93	41.91	41.91
11	41.97	41.97	41.96	41.95	41.94	41.93	41.93
12	41.96	41.96	41.96	41.95	41.95	41.94	41.94
13	41.96	41.96	41.95	41.95	41.95	41.95	41.95
14	41.96	41.96	41.95	41.95	41.95	41.95	41.95
15	41.95	41.95	41.95	41.95	41.95	41.95	41.95
16	41.95	41.95	41.95	41.95	41.95	41.95	41.95

Table 15.4. *Option expectation*

	STATE						
n	–3	–2	–1	0	1	2	3
1	70.93	74.93	57.09	24.98	14.27	7.14	0.00
2	60.60	53.47	46.85	38.32	30.05	20.37	16.93
3	49.88	48.27	43.26	40.48	36.98	31.53	31.31
4	45.53	44.75	42.63	40.26	39.45	37.25	37.30
5	43.50	43.10	42.15	41.38	40.65	39.68	39.63
6	42.22	42.34	41.90	41.54	41.21	40.75	40.73
7	42.05	41.97	41.76	41.60	41.45	41.23	41.22
8	41.81	41.77	41.68	41.60	41.53	41.43	41.43
9	41.68	41.66	41.62	41.58	41.55	41.51	41.50
10	41.60	41.59	41.57	41.55	41.54	41.52	41.52
11	41.54	41.54	41.53	41.52	41.51	41.50	41.50
12	41.49	41.49	41.49	41.48	41.48	41.47	41.47
13	41.45	41.45	41.45	41.44	41.44	41.44	41.44
14	41.41	41.41	41.41	41.41	41.41	41.41	41.40
15	41.37	41.37	41.37	41.37	41.37	41.37	41.37
16	41.33	41.33	41.33	41.33	41.33	41.33	41.33

Table 15.5. *Option value*

15.3.5. *The continuous time homogenous semi-Markov model for the underlying asset*

With the generalization of electronic trading systems, it seems more adaptive to construct a time continuous model for which the changes in the values of the underlying process may depend on the time it remained unchanged before a transition.

Also, let

$$((S_n, T_n)\ n = 0,1,...) \tag{15.113}$$

be the successive states and time changes of the considered asset.

The Janssen-Manca semi-Markov continuous model without AOA starts from the basic assumption that process (15.113) is a semi-Markov process of kernel **Q**.

It follows that, at time t in state $S(t) = S_t$, the market value of the considered European option with maturity $T - t$ has as probability distribution at maturity time

$$P(C(T) = (j - k_0)) = \phi_{S_t j}(T - t),\ j > k_0,$$
$$P(C(T) = 0) = \sum_{l \le K_0} \phi_{S_t j}(T - t),\ j \le k_0. \tag{15.114}$$

Of course, matrix $\mathbf{\Phi}(t)$ represents the transition probabilities for the considered semi-Markov process (see relation (12.101)).

This result gives the possibility to calculate all interesting parameters concerning *C*. For example, the mean of $C(T)$ has the value:

$$E(C(T) = \left| S(t) = S_t \right) = \sum_{j > k_0} \phi_{S_t j}(T - t)(j - k_0)\Delta. \tag{15.115}$$

The pricing of the option at time *t* is here given by the conditional market value $C(t)$:

$$C(S_t, t) = v^{T-t} \sum_{j > k_0} \phi_{S_t j}(T - t)(j - k_0)\Delta \tag{15.116}$$

which is the Janssen-Manca-Di Biase formula for the considered semi-Markov model.

If the semi-Markov process is ergodic, then, if $(T - t)$ is large enough, results (15.114) can be well approximated by:

$$P(C(T) = (j - k_0)) = \tilde{\pi}_j,\ j > k_0,$$
$$P(C(T) = 0) = \sum_{l \le K_0} \tilde{\pi}_l,\ j \le k_0. \tag{15.117}$$

The *stationary* version of the Janssen-Manca-Di Biase formula is thus given by

$$C(S_t, t) = v^{T-t} \sum_{j > k_0} \tilde{\pi}_j \phi_{S_t j}(j - k_0)\Delta. \tag{15.118}$$

Of course the vector $(\tilde{\pi}_1, ..., \tilde{\pi}_m)$ is the asymptotic distribution of the embedded semi-Markov process given by relation (12.15) .

Formally the evaluation of assets is continuous, but substantially is given in the discrete case; furthermore, the numerical solution of a continuous time semi-Markov process causes problems of numerical and stochastic convergence. For these reasons, it may be useful to deal with our problem with the discrete time homogenous semi-Markov process as introduced in Janssen and Manca (2007).

15.3.6. *Numerical example for the semi-Markov model*

We will only provide a numerical example for the semi-Markov model in the asymptotic case, i.e. values of the option expectation and of the options for large maturities.

We merely need as supplementary information, the conditional mean sojourn times given by relations (12.25). The used values are given by the following matrix $\mathbf{\Sigma}$:

$$\mathbf{\Sigma} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 & 2 & 1 & 1 & 1 \\ 1 & \frac{1}{4} & \frac{1}{4} & 1 & 2 & 1 & 1 \\ 2 & 1 & \frac{1}{2} & 1 & 2 & 2 & 1 \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 \\ 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 2 & 1 \\ 1 & 1 & 2 & 1 & \frac{1}{2} & \frac{1}{2} & 2 \\ 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}. \quad (15.119)$$

In this case, the asymptotic distribution for the semi-Markov process is:

(0.09487, 0.12650, 0.38238, 0.15352, 0.15013, 0.08358, 0.00902).

Then, starting at time 0 in state 1,500, the asymptotic value of the call option expectation with 1,500 as the exercise price is 46 and the call value is 45.315.

The following table gives option expectations and option values with different exercise prices.

Exercise price	Option expectation	Option value
1,350	178.78	176.119
1,400	129.234	127.308
1,450	83.8638	82.614
1,500	46.0002	45.315
1,550	15.8126	15.577
1,600	4.74378	4.673
1,650	0	0

Table 15.6. *Semi-Markov option calculation*

15.3.7. *Conclusion*

The JMD models presented here provide a semi-Markov approach for the pricing of option financial products working in discrete time and with a finite number of possible values for the imbedded asset, which is always the case from the numerical point of view.

The main interest of these models is that they work even when there are possibilities of arbitrage, that is to say, for the most common cases. Of course, one of the main difficulties in applying this model is the fitting of the needed data and this is only of interest in the case of asymmetric information so that the economic agent can believe in his own information, knowing that he will always be in a risky situation to expect gain but still worried about the possibility of losing as in the case of a real life situation!

It is also important to point out that the numerical examples are coherent; nevertheless, there are significant differences according to the model used, Markov or semi-Markov, so that it is very important to select the most concrete one.