Chapter 14

Option Theory

14.1. Introduction

During the last 30 years, financial innovation has generalized the systematic use of new financial instruments called *derivative instruments* such as *options* and *swaps*, mainly used for hedging but also, sometimes, used as speculative tools. This matter is now essential in mathematical finance and will be fully developed here following the presentation of Janssen and Manca (2007).

However, we will also develop some main results concerning *exotic options* and foreign currency options with the presentation of the *Garman-Kohlagen formula* and some important results on *American options.*

The first basic derivative instruments are now called *plain vanilla options*: the two types of such options are now defined.

Definition 14.1 *A* call option (*respectively* put option) *is a contract giving the right to buy* (*respectively to sell*) *a financial asset, called an underlying asset, for a fixed price, called an exercise price, at the end of the contract time, called maturity time, also laid down in the contract.*

If we can exercise the option at any time before maturity, this type of option is said to be of an *American type*; if we can exercise it only at maturity, the option is said to be of a *European type.*

We will use the following notation: K for the exercise price, T for the maturity time and S for the value of the underlying asset at maturity.

The “gain” of the holder of a European option at maturity time T is represented by the following graph.

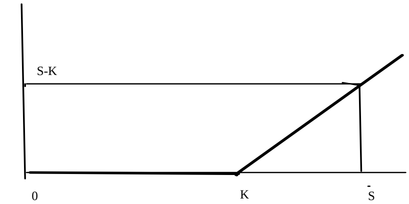

Figure 14.1. *Call option: holder’s gain at maturity*

For the holder of a put, this graph becomes the following.

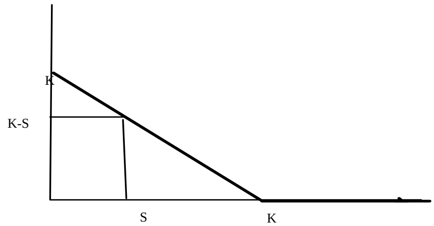

Figure 14.2. *Put option: holder’s gain at maturity*

Of course, to obtain the “net gain”, we must estimate the cost of the option, often called *option premium*, and furthermore *transaction costs* and *taxes*.

Let us represent respectively by C and P the premiums of call and put options.

So, we obtain, without taking into account transaction costs and taxes, the following two graphical representations.

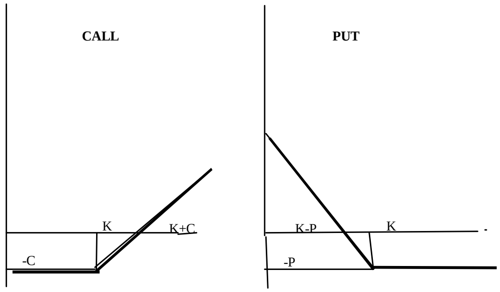

Figure 14.3. *Call and put options: net gains at maturity for the holder*

We will now cover the main problem for plain vanilla options, that is, the *pricing of such optional products*. We have to give within an economic-financial theory framework, the values of premiums C and P as a function of the maturity T and the value S of the asset at time 0.

More generally, as the holder of an option can sell his option on the option market at any time t, $0<t<T$, it is also necessary to give the "fair" value of the option at this time t knowing that the underlying asset has, at this time, the value $S = S(t)$, the fair market value represented by

$$C(S,\tau) \tag{14.1}$$

where

$$\tau = T - t \tag{14.2}$$

represents the maturity calculated at time t.

Sometimes, it is also useful to represent the call value as a function of the time $C(S, t)$.

To discuss this pricing problem, it is absolutely necessary to give assumptions about the stochastic process

$$S = \left(S(t), 0 \le t \le T\right). \tag{14.3}$$

Concerning the economic-financial theory framework, we adopt the assumption of *efficient market*, meaning that all the information available at time t is reflected in the values of the assets and so, transactions having an abnormal high profitability are not possible.

More precisely, an efficient market satisfies the following assumptions:

1) absence of transaction costs;

2) possibility of short sales;

3) availability of all information to all the economic agents;

4) perfect divisibility of assets;

5) continuous time financial market.

Furthermore, the market is *complete*, meaning that there exists zero-coupon bonds without risk for all possible maturities.

A zero-coupon bond is merely an asset giving the right to receive €1 and time $t+\tau$ for the payment of the sum B at time t.

Let us note that the word "information" used in point 3 can have different interpretations: weak, semi-strong or strong, depending on if it is based on past prices, on all public information or finally on all possible information that the agent can find.

According to Fama (1965), the efficient assumption justifies the "random walk" model in discrete time, saying that if $\Delta R_i(s)$ represents the increment of an asset i between s and $s+1$, we have:

$$\Delta R_i(s) = \mu_i + \varepsilon_i(s), \tag{14.4}$$

μ_i being a constant and $(\varepsilon_i(s))$ a sequence of uncorrelated r.v. of mean 0, sometimes called *errors*.

If we add the assumptions of equality of variances and of normality of the sequence $(\varepsilon_i(s))$, we obtain a special case of the traditional random walk.

Even if the efficiency assumption seems to be natural, some empirical studies show that this is not always the case, particularly, since some agents can have access to preferential information in principle forbidden by the legal authority control.

Nevertheless, should such agents use the pertinent information, it will be quickly noticed by those markets and balance between agents will be restored.

This possibility, also called the case of *asymmetric information*, was studied by Spencer, Akerlof and Stiglitz, who were awarded the Nobel Prize in Economics in 2001.

We feel that the efficiency assumption seems quite normal for the long term, i.e. with a large enough time unit, as it does not always seem to be true locally, i.e. with a short time unit. Indeed, deregulation of markets where investors want to secure very small benefits in a short time but in a lot of transactions plainly explains the intense activity of, for example, the currency markets receiving very small benefits.

That is why models for *asymmetric information* should always be short term models rejecting the *Absence Of Arbitrage* (AOA) assumption, that is, making money without any investment otherwise known as a "free lunch".

To be complete, let us note that it is now possible to construct models without the AOA assumption but with assumptions on the time asset evolution and a selection of different possible scenarios, so that the investor can predict what will happen if such scenarios occur (see Janssen, Manca and Di Biase (1997) and Jousseaume (1995)).

In this chapter, we will give the two most commonly used traditional models in option theory: the Cox, Ross, Rubinstein model in discrete time and the Black-Scholes model in continuous time.

14.2. The Cox, Ross, Rubinstein (CRR) or binomial model

The model we will present here has the advantage of being quite simple in a financial world not always open to the use of sophisticated mathematical tools such as those used by Black and Scholes in 1973 to obtain their famous formula. Thus, the CRR model, though coming later, was very good for the use of the Black-Scholes formula since, in the limit, the CRR model provides this formula again.

Moreover, the CRR model has still its own utility for financial institutions using discrete time models even with a short time period.

14.2.1. *One-period model*

To begin with, let us consider a model with only one time period, from time 0 to time 1; the time unit can be chosen as the user wishes: a quarter, a month, a week, a day, an hour, etc.

The basic assumption concerning the stochastic evolution of the underlying asset is that, starting from value $S(0)=S_0$ at time 0, it can only obtain two values at the end of the time period: $uS_0(u>1)$ if there is an up movement or $dS_0(0<d<1)$ in the case of a down movement, parameter u and d assumed to be known for the moment.

The probability measure is thus defined by the probability q of an up movement and to avoid trivialities, we will assume that:

$$0<q<1. \tag{14.5}$$

The next figure shows the two possible trajectories with of course

$$p=1-q. \tag{14.6}$$

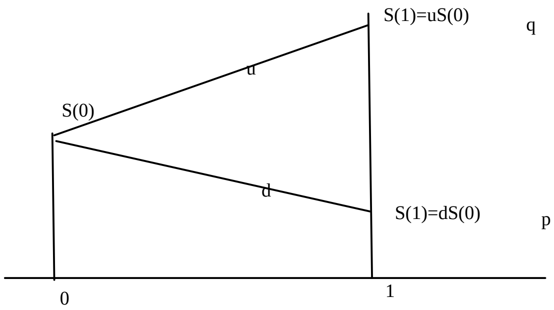

Figure 14.4. *One-period binomial model*

If we prefer to work with the percentages x and y respectively of gain and loss, we can express u and d as follows:

$$u = \left(1 + \frac{x}{100}\right), d = \left(1 - \frac{y}{100}\right). \tag{14.7}$$

We also suppose that there is no dividend repartition during the period.

Let us now consider an investor wishing to buy a European call at time 0 with maturity 1 and with K as exercise price.

The problem is thus to fix the *premium* of this call, which the investor has to pay at time 0 to buy this call, knowing the value S_0 of the underlying asset at time 0.

14.2.1.1. *The arbitrage model*

If the investor wants to buy a call, it is clear that he anticipates an up movement of the call so that exercising the call at the end of the period will be advantageous for him, and of course for the seller of the call the reverse will happen.

Nevertheless, the investor would take as little risk as possible knowing that he has always the possibility to invest on the non-risky market giving a fixed interest rate i per period.

In order to build a theory taking into account the apparently contradictory points of view, modern financial theory is based on the AOA principle meaning that there is no possibility to gain money without any investment, that is, there is no possibility of getting a *free lunch*.

This principle implies that the parameters *d*, *u* and *i* of the model must satisfy the following inequalities:

$$d < 1 + i < u\,. \tag{14.8}$$

Indeed, let us suppose for example that the first inequality is wrong. In this case the investment in the asset is always better than that on the non-risky market. If at time 0 we borrow the sum S_0 from the bank to by a share, at the end of the period obtaining the investment on assets, a free lunch of at least the amount $\left(d-(1+i)\right)S_0$ always exists.

Similarly, if the right inequality is false, we can sell the asset at time 0 to get it to the seller at time 1 and so, the minimum value of the free lunch is, in this case, $(1+i-u)S_0$, *so that in both cases* the AOA principle is not satisfied.

The seller of a call option, for example, has the obligation to sell the shares if the holder of the call exercises his right, he must be able to do it whatever the value of the considered share is; that is why we have to introduce the important concept of *hedging*.

To do so, let us consider a portfolio in which at time 0 we have Δ shares and an amount *B* of money invested at the non-risky rate *i* per period.

B may be negative in case of a loan given by the bank.

Under the AOA assumption, the investment in the call must follow the same random evolution as the considered portfolio so that we have the following relations for *t* = 1:

$$\begin{aligned} C_u(1) &= uS_0 + (1+i)B_0, \\ C_d(1) &= dS_0 + (1+i)B_0, \end{aligned} \tag{14.9}$$

where

$$\begin{aligned} C_u(1) &= \max\{0, uS_0 - K\}, \\ C_d(1) &= \max\{0, dS_0 - K\}. \end{aligned} \tag{14.10}$$

System (14.9) is a linear system with two unknown values Δ, B.

The unique solution is given by:

$$\Delta = \frac{C_u(1) - C_d(1)}{(u-d)S_0},$$
$$B = \frac{uC_d(1) - dC_u(1)}{(u-d)(1+i)}. \tag{14.11}$$

Now, as stated above, from the AOA assumption, the value of the call at $t = 0$, denoted for the moment by $C\left(S_0, 0\right)$, is equal to the initial value of the portfolio so that:

$$C(S_0,1) = S_0\Delta + B_0,$$
$$C(S_0,1) = S_0\frac{C_u(1) - C_d(1)}{(u-d)S_0} + \frac{uC_d(1) - dC_u(1)}{(u-d)(1+i)}. \tag{14.12}$$

We can also write this value in the following form:

$$C(S_0,0) = \frac{1}{1+i}\left[\overline{q}C_u(1) + (1-\overline{q})C_d(1)\right],$$
$$\overline{q} = \frac{1+i-d}{u-d}. \tag{14.13}$$

This last expression shows that the value of the call at the beginning of the period can be seen as the *present value of the expected value of the "gain" at the end of the period.* However, this expectation is calculated under a new probability measure defined by $\overline{q}$, called *risk neutral measure* in opposition to the initial measure defined by *q*, and called the *historical* or *physical measure.*

From assumption (14.8), this risk neutral measure is *unique* and moreover independent of *q*, that is, on the historical measure.

This shows that whatever the investor has as anticipation about the price of the considered underlying asset, using this model, he will always get the same result as another investor.

However, it must be clear that this risk neutral measure only gives an easy way to calculate the "fair" value of the call, but if we want to calculate probabilities of events, such as for example the probability of exercising the call at the end of the period, then it is the historical measure that must be used.

14.2.1.2. *Numerical example*

Let us consider the following data:

$$S_0 = 80, K = 80, u = 1.5, d = 0.5, i = 3\%. \tag{14.14}$$

It follows from the model that:

$$\begin{aligned} C_u(1) &= \max\{0.80 \times 1.5 - 80\} = 40, \\ C_d(1) &= \max\{0.80 \times 0.5 - 80\} = 0. \end{aligned} \tag{14.15}$$

The value of $\overline{q}$ is obtained, i.e.

$$\overline{q} = \frac{1.03 - 0.5}{1.5 - 0.5} = 0.53 \tag{14.16}$$

and so we obtain the option value

$$C_{fin}(80,0) = \frac{1}{1.03}\left[\overline{q} \times 40 + (1-\overline{q}) \times 0\right] = 20.5825. \tag{14.17}$$

14.2.2. *Multi-period model*

14.2.2.1. *Case of two periods*

The two following figures show how the model with two periods works.

Here we have to evaluate not only the value of the call at the origin but also at the intermediary time $t = 1$.

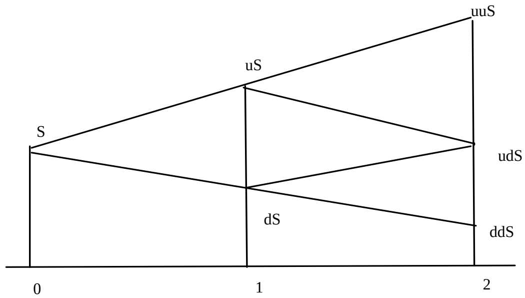

Figure 14.5. *Two-period model: scenarios for the underlying asset*

Using the notation $C(S,t)$, $t = 0,1,2$ in which the second variable represents the time, here 0, 1 or 2, the first variable is the value of the underlying asset at this considered time.

Here too, as in the case of only one period, the call values will be assessed with the risk neutral measure as the present values at time t of the “gains” at maturity t = 2, i.e.:

$$E_{\overline{q}}\left(C(S,2)\right). \tag{14.18}$$

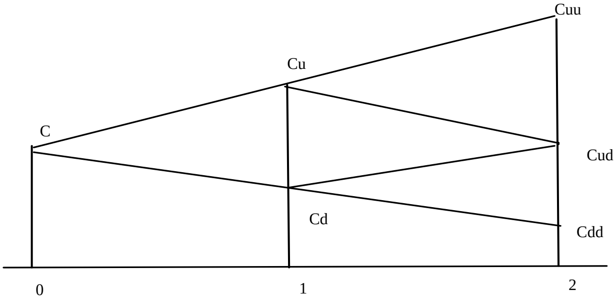

Figure.14.6. *Two-period model: values of the call*

For example, we obtain for t = 0:

$$C(S_0,0)=\frac{1}{(1+i)^2}\begin{bmatrix}\overline{q}^2\max\left\{0,u^2S_0-K\right\}+2\overline{q}(1-\overline{q})\\ \cdot\max\left\{0,udS_0-K\right\}+(1\text{-}\overline{q})^2\max\left\{0,d^2S_0-K\right\}\end{bmatrix}. \tag{14.19}$$

Remark 14.1 Using a backward reasoning from t = 2 to t = 1 and from t = 1 to t = 0, it is also possible to obtain this last result:

$$\begin{aligned}
C(uS_0,1)&=\frac{1}{1+i}\left[\overline{q}C\ (u^2S_0,2)+(1-\overline{q})C(udS_0,2)\right],\\
C(dS_0,1)&=\frac{1}{1+i}\left[\overline{q}C(udS_0,2)+(1-\overline{q})C(d^2S_0,2)\right],\\
C(S_0,0)&=\frac{1}{1+i}\left[\overline{q}C(uS_0,1)+(1-\overline{q})C(dS_0,1)\right].
\end{aligned} \tag{14.20}$$

Substituting the first two values in the last equality given above, we rediscover relation (14.19).

14.2.2.2. *Case of n periods*

$C_{u^jd^{n-j}}(S_0,n)$ represents the call value at t = n if the underlying asset has had j up movements and n-j down movements and with an initial value of the underlying asset of $S(0)$, that is:

$$C_{u^j d^{n-j}}(S_0, n) = \max\left\{0, u^j d^{n-j} S_0 - K\right\}, j = 0,1,...,n. \tag{14.21}$$

A straightforward extension of the case of two periods gives the following result:

$$C(S_0, 0) = \frac{1}{(1+i)^n} \sum_{j=0}^{n} \binom{n}{j} \overline{q}^j (1-\overline{q})^{n-j} C_{u^j d^{n-j}}(n) \tag{14.22}$$

and similar results for intermediary time values.

From the calculational point of view, Cox and Rubinstein introduce the minimum number of up movements a so that the call will be "in the money", which will mean that the holder has the advantage to exercise his option; clearly, this integer is given by:

$$a = \min\left\{ j \in N : u^j d^{n-j} S_0 > K \right\}. \tag{14.23}$$

Of course, if a is strictly larger than n, the call will always finish "out of the money" so that the call value at $t = n$ is zero.

From relation (14.23), we obtain:

$$u^j d^{n-j} S_0 = K \Leftrightarrow a = \left\lfloor \frac{\log KS_0^{-1} d^{-n}}{\log ud^{-1}} \right\rfloor + 1, \tag{14.24}$$

$\lfloor x \rfloor$ representing the larger integer smaller than or equal to the real x.

From section 10.1, we know that if X is an r.v. having a binomial distribution with parameters (n, q), we have:

$$P\left(X > a-1\right) = \sum_{j=a}^{n} \binom{n}{j} q^j (1-q)^{n-j} (= \overline{B}(n, q; a)). \tag{14.25}$$

As we have (see Cox, Rubinstein (1985)):

$$\overline{q} < \frac{1+i}{u} < 1, \tag{14.26}$$

it follows that the quantity $\overline{q}\,'$ defined here below is such that $0 < \overline{q}\,' < 1$ and so the call value can be written in the form:

$$C_{fin}(S_0,0) = S_0\overline{B}(n,\overline{q}';a) - \frac{K}{(1+i)^n}\overline{B}(n,\overline{q};a),$$
$$\overline{q} = \frac{1+i-d}{u-d}, \overline{q}' = \frac{u}{1+i}q. \tag{14.27}$$

In conclusion, the binomial distribution is sufficient to calculate the call values.

14.2.2.3. *Numerical example*

Coming back to the preceding example for which

$$S_0 = 80, K = 80, u = 1.5, d = 0.5, i = 3\%, \tag{14.28}$$

and $\overline{q} = 0.53$ but now for *n=2,* we obtain:

$$\overline{q}' = \frac{1.5}{1.03} \times 0.6 = 0.7718 \tag{14.29}$$

and consequently

$$C(80,0) = 26.4775. \tag{14.30}$$

14.3. The Black-Scholes formula as the limit of the binomial model

14.3.1. *The log-normality of the underlying asset*

Since nowadays financial markets operate in continuous time, we will study the asymptotical behavior of CRR formula (14.27) to obtain the value of a call at time 0 and of maturity *T*.

To begin with, we will work with a discrete time scale on [*0,T*] with a unit time period *h* defined by $n = T/h$ or more precisely $n = \lfloor T/h \rfloor$.

Moreover, if *i* represents the annual interest rate, the rate for a period of length *h* called $\hat{\imath}$ is defined by the relation:

$$(1+\hat{\imath})^n = (1+i)^T, \tag{14.31}$$

so that

$$\hat{\imath} = (1+i)^{T/n} - 1. \tag{14.32}$$

If J_n represents the r.v. giving the number of ascending movements of the underlying asset, we know that:

$$J_n \prec B(n,q) \tag{14.33}$$

and so, starting from S_0, the value of the underlying asset at the end of period *n* is given by

$$S(n) = u^{J_{,n}} d^{n-J_{,n}} S_0. \tag{14.34}$$

It follows that

$$\log \frac{S(n)}{S_0} = J_n \log \frac{u}{d} + n \log d. \tag{14.35}$$

The results of the binomial distribution (see section 10.5.1) imply that

$$\begin{aligned} &E\left(\log \frac{S(n)}{S_0}\right) = \hat{\mu} n, \\ &\operatorname{var}\left(\log \frac{S(n)}{S_0}\right) = \hat{\sigma}^2 n, \\ &\hat{\mu} = q\hat{\sigma}^2 + \log d, \\ &\hat{\sigma}^2 = q(1-q)\left(\log \frac{u}{d}\right)^2. \end{aligned} \tag{14.36}$$

To obtain a limit behavior, for every fixed *n*, we must introduce a dependence of *u*, *d* and *q* with respect to $n = \lfloor T/h \rfloor$ so that

$$\begin{aligned} &\lim_{n\to\infty} \hat{\mu}(n) n = \alpha T, \\ &\lim_{n\to\infty} \hat{\sigma}^2(n) n = \sigma^2 T, \end{aligned} \tag{14.37}$$

α, σ being constant values as parameters of the limit model. As an example, Cox and Rubinstein (1985) select the values

$$u = e^{\sigma\sqrt{T/n}}, d = \frac{1}{u}(= e^{-\sigma\sqrt{T/n}}),$$
$$q = \frac{1}{2} + \frac{1}{2}\frac{\alpha}{\sigma}\sqrt{T/n}. \tag{14.38}$$

This choice leads to the values:

$$\hat{\mu}(n)n = \alpha T,$$
$$\hat{\sigma}^2(n)n = \left(\sigma^2 - \alpha^2 \frac{T}{n}\right)T. \tag{14.39}$$

Using a version of the central limit theorem for independent but non-identically distributed r.vs., the authors show that $S(n)/S_0$ converges in law to a lognormal distribution for $n \to \infty$. More precisely, we have:

$$P\left(\frac{\log\frac{S(n)}{S_0} - \hat{\mu}(n)n}{\hat{\sigma}\sqrt{n}} \leq x\right) \to \Phi(x), \tag{14.40}$$

Φ, being as defined in section 10.3, is the distribution function of the reduced normal distribution provided that the following condition is satisfied:

$$\frac{q\left|\log u - \hat{\mu}\right|^3 + (1-q)\left|\log u - \hat{\mu}\right|^3}{\hat{\sigma}^3\sqrt{n}} \xrightarrow[n\to\infty]{} 0. \tag{14.41}$$

This condition is equivalent to

$$\frac{(1-q)^2 + q^2}{\sqrt{nq(1-q)}} \to 0 \tag{14.42}$$

which is true from assumption (14.38).

This result and the definition given in section 10.4, gives the next proposition.

Proposition 14.1 (*Cox and Rubinstein* (*1985*)) *Under assumption (14.38), the limit law of the underlying asset is a lognormal law with parameters* $(\alpha T, \sigma^2 T)$ *or*

$$P\left(\frac{\log\frac{S(T)}{S_0}-\alpha T}{\sigma\sqrt{T}}\le x\right)=\Phi(x). \tag{14.43}$$

In particular, it follows that:

$$\begin{aligned} E\left(\frac{S(T)}{S_0}\right) &= e^{\alpha T+\frac{\sigma^2}{2}T}, \\ \operatorname{var}\left(\frac{S(T)}{S_0}\right) &= e^{2\alpha T+\sigma^2 T}(e^{\sigma^2 T}-1). \end{aligned} \tag{14.44}$$

14.3.2. *The Black-Scholes formula*

Starting from result (14.25) and using Proposition 14.1 under the risk neutral measure, Cox and Rubinstein (1985) proved that the asymptotical value of the call is given by the famous Black and Scholes (1973) formula:

$$\begin{aligned} C(S,T) &= S\Phi(x)-K(1+i)^{-T}\Phi(x-\sigma\sqrt{T}), \\ x &= \frac{\ln\left(S/K(1+i)^{-T}\right)}{\sigma\sqrt{T}}+\frac{1}{2}\sigma\sqrt{T}. \end{aligned} \tag{14.45}$$

Here, we note the call using the maturity as a second variable and S representing the value of the underlying asset at time 0.

The interpretation of the Black and Scholes formula can be given in the concept of a hedging portfolio.

Indeed, we already know that in the CRR model, the value of the call takes the form:

$$C = S\Delta + B, \tag{14.46}$$

Δ representing the proportion of assets in the portfolio and B the quantity invested on the non-risky rate at t = 0.

From result (14.46), at the limit, we obtain:

$$\begin{aligned} &\Delta = \Phi(x), \\ &B = -K(1+i)^{-T}\,\Phi(x - \sigma\sqrt{T}). \end{aligned} \tag{14.47}$$

So, under the assumption of an efficient market, the hedging portfolio is also known in continuous time.

Remark 14.2 This hedging portfolio must of course, at least theoretically, be rebalanced at every time s on $[0, T]$. Rewriting the Black and Scholes formula in order to calculate the call at time s, the underlying asset having the value S, we obtain:

$$\begin{aligned} &\Delta = \Phi(x), B = -K(1+i)^{-(T-s)}\,\Phi\left(x - \sigma\sqrt{T-s}\right), \\ &x = \frac{\ln\left(S / K(1+i)^{-(T-s)}\right)}{\sigma\sqrt{T-s}} + \frac{1}{2}\sigma\sqrt{T-s}. \end{aligned} \tag{14.48}$$

Of course, a continuous rebalancing and even a portfolio with frequent time changes are not possible due to the costs of transaction.

14.4. The Black-Scholes continuous time model

14.4.1. *The model*

In fact, Black and Scholes used a continuous time model for the underlying asset introduced by Samuelson (1965).

On a complete filtered probability space $\left(\Omega, \Im, \left(\Im_t, t \geq 0\right), P\right)$ (see Definition 10.13) the stochastic process

$$S = \left(S(t), t \geq 0\right) \tag{14.49}$$

will now represent the time evolution of the underlying asset.

The basic assumption is that the stochastic dynamic of the S-process is given by

$$\begin{aligned} &dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \\ &S(0) = S_0, \end{aligned} \tag{14.50}$$

where the process $B = (B(t), t \in [0,T])$ is a standard Brownian process (see section 10.9 which is adapted to the considered filtration).

14.4.2. *The solution of the Black-Scholes-Samuelson model*

Let us go back to model (14.50). Using the Itô formula of Chapter 13 for *lnS*(*t*), we obtain:

$$d\ln S(t) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dB(t) \tag{14.51}$$

and so by integration:

$$\ln S(t) - \ln S_0 = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t). \tag{14.52}$$

As, for every fixed *t*, *B*(*t*) has a normal distribution with parameters (0, *t*) – *t* for the variance – (see Chapter 13), this last result shows that the r.v. $S(t)/S_0$ has a lognormal distribution with parameters $\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$ and so:

$$\begin{aligned} &E\left(\log\frac{S(t)}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right)t, \\ &\operatorname{var}\left(\log\frac{S(t)}{S_0}\right) = \sigma^2 t. \end{aligned} \tag{14.53}$$

Of course, from result (14.52), we obtain the explicit form of the trajectories of the *S*-process:

$$S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t} e^{\sigma B(t)}. \tag{14.54}$$

This process is called a *geometric Brownian motion.*

The fact of having the lognormality confirms the CRR process at the limit as, indeed, a lot of empirical studies show that, for an efficient market, stock prices are well adjusted with such a distribution.

From properties of the lognormal distribution, we obtain:

$$E\left(\frac{S(t)}{S_0}\right)=e^{\mu t},$$
$$\operatorname{var}\left(\frac{S(t)}{S_0}\right)=e^{2\mu t}(e^{\sigma^2 t}-1). \tag{14.55}$$

So, we see that the mean value of the asset at time t is given as if the initial amount S_0 was invested at the non-risky instantaneous interest rate μ and that its value is above or below S_0 following the "hazard" variations modeled with the Brownian motion.

From the second result of (14.55), it is also clear that the expectations of large gains – and losses! – are better for large values of σ; that is why σ is called the *volatility* of the considered asset.

It follows that a market with high volatility will attract *risk lover* investors and not *risk adverse* investors.

From the explicit form, it is not difficult to simulate trajectories of the *S*-process. The next figure shows a typical form.

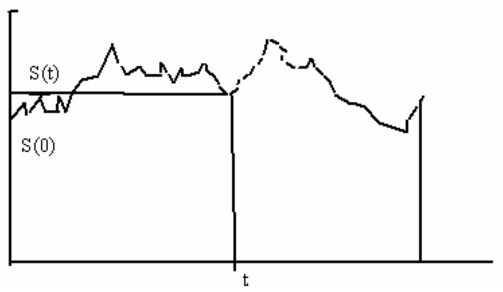

Figure 14.7. *A typical trajectory*

14.4.3. *Pricing the call with the Black-Scholes-Samuelson model*

14.4.3.1. *The hedging portfolio*

The problem consists of pricing the value of a European call of maturity T and exercise price K at every time t belonging to $[0, T]$ as a function of t or the maturity at time t, $\tau = T - t$, and of the value of the asset at time t, $S = S(t)$, knowing that the non-risky instantaneous interest rate is r, so that if i is the non-risky annual rate, we have:

$$e^r = 1 + i\,. \tag{14.56}$$

We will use the notations $C(S, t)$ or, more frequently, $C(S,\tau)$.

As in the CRR model, we introduce a portfolio P containing, at every time t of a call and a proportion α, which may be negative, of shares of the underlying asset.

The stochastic differential of $P(t)$ is given by:

$$dP(t) = dC(S,t) + \alpha dS(t) \tag{14.57}$$

or, from relation (14.50):

$$dP(t) = dC(S,t) + \alpha\mu S(t)dt + \alpha\sigma S(t)dB(t). \tag{14.58}$$

Using Itô's formula, in a correct form as proved by Bartels (1995) of the first initial form given by Black and Scholes (1973), we obtain:

$$\begin{aligned} dP(t) &= \left[\frac{\partial C}{\partial S}(S,t)\mu S + \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2 + \alpha\mu S(t)\right]dt \\ &+ \left[\alpha\sigma S(t) + \frac{\partial C}{\partial S}(S,t)\sigma S\right]dB(t). \end{aligned} \tag{14.59}$$

Now, using the principle of AOA, this variation must be identical to that of the same amount invested at the non-risky interest, that is:

$$rP(t)dt = r\left[C(S,t) + \alpha S\right]dt. \tag{14.60}$$

So, we obtain the following relation:

$$rP(t)dt = dP(t), \tag{14.61}$$

$$\begin{aligned} & r\left[C(S,t) + \alpha S\right]dt = \\ & \left[\frac{\partial C}{\partial S}(S,t)\mu S + \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2 + \alpha\mu S(t)\right]dt \\ & + \left[\alpha\sigma S(t) + \frac{\partial C}{\partial S}(S,t)\sigma S\right]dB(t). \end{aligned} \tag{14.62}$$

By identification, we obtain:

$$
\begin{aligned}
& r\left[C(S,t)+\alpha S\right]dt - \\
& \left[\frac{\partial C}{\partial S}(S,t)\mu S + \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2 + \alpha\mu S(t)\right]dt = 0, \\
& \left[\alpha\sigma S(t) + \frac{\partial C}{\partial S}(S,t)\sigma S\right] = 0.
\end{aligned}
\tag{14.63}
$$

From the last equality, we obtain:

$$
\alpha = -\frac{\partial C}{\partial S}(S,t). \tag{14.64}
$$

Substituting this value in the first equality of (14.63), we obtain after simplification:

$$
r\left[C(S,t) - \frac{\partial C}{\partial S}(S,t)S\right] - \left[\frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2\right] = 0, \tag{14.65}
$$

or finally

$$
-rC(S,t) + r\frac{\partial C}{\partial S}(S,t)S + \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2 = 0, \tag{14.66}
$$

a *linear partial differential equation of order* 2 for the unknown function $C(S, t)$ with as initial condition

$$
C(S,t) = \begin{cases} 0, t \in [0,T), \\ \max\{0, S-K\}, t = T \end{cases}. \tag{14.67}
$$

Using results from the heat equation in physics, for which an explicit solution is given in terms of a Green function, Black and Scholes (1973) obtained the following explicit form for the call value:

$$C(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$
$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S}{K} + (r+\frac{\sigma^2}{2})(T-t)\right], \tag{14.68}$$
$$d_2 = d_1 - \sigma\sqrt{T-t},$$
$$S = S(t).$$

Remark 14.3

(i) Using relation (14.61), we obtain relation (14.45) for t = 0 or $\tau = T$. The interpretation is, of course, already given in section 14.3.2.

(ii) The differentiation in relation (14.57) is correct only if we assume that the supplementary terms produced by Itô's calculus (see relation (13.108)) are zero. In fact, this assumption is equivalent to assuming that the used portfolio strategy is *self financing*; this means that each rebalancing of the portfolio has no cost.

14.4.3.2. *The risk neutral measure and the martingale property*

As for the CRR model, it is possible to construct another probability measure Q on $\left(\Omega, \Im, (\Im_t)\right)$, called the *risk neutral measure*, such that the value of the call given by formula (14.68) is simply the expectation value of the present value of the "gain" at maturity time T.

Using a change of probability measure for going from P to Q, with the famous Girsanov theorem (see for example Gihman and Skorohod (1975) and Chapter 15) it can be shown that the new measure Q, which moreover is unique, can be defined by replacing in the stochastic differential equation (14.50) the trend μ by r.

Doing so, the explicit form of $S(t)$ given by relation (14.54) becomes:

$$S(t) = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t} e^{\sigma B'(t)} \tag{14.69}$$

where process B' is an adapted standard Brownian motion and the value of C can be calculated as the present value of the expectation of the final "gain" of the call at time T:

$$C(S,t) = e^{-r(T-t)} E_Q\left(\sup\left\{S(T) - K, 0\right\}\right). \tag{14.70}$$

The risk neutral measure gives another important property for the process of present values of the asset values on [0, T]:

$$\left\{e^{-rt}S(t), t \in [0,T]\right\} \tag{14.71}$$

Indeed, under Q, this process is a martingale, so that (see section 10.8) for all s and t such that $s < t$:

$$E\left(e^{-rt}S(t)\middle|\Im_s\right)\ (s<t) = S(s). \tag{14.72}$$

This means that at every time s, the best statistical estimation of $S(t)$ is given by the observed value at time s, a result consistent with the assumption of an efficient market.

From relation (14.72), we obtain in particular:

$$E\left(e^{-rt}S(t)\right) = S_0. \tag{14.73}$$

So, on average, the present value of the asset at any time t equals its value at time 0.

To conclude, we see that the knowledge of the risk neutral measure avoids the resolution of the partial differential equation and replaces it by the calculation of an expectation, which is in general easier, as it only uses the marginal distribution of $S(T)$.

However, we must add that, for more complicated derivative products, it may be more interesting, from the numerical point of view, to solve this partial differential equation with the finite difference method, and particularly in the case of American options.

14.4.3.3. *The call put parity relation*

From section 14.1, we know that the value of a put at maturity time T and exercise price K is given by:

$$P\left(S(T),K\right) = \max\left\{0, K - S(T)\right\}. \tag{14.74}$$

As for the call, we have:

$$C\left(S(T),K\right) = \max\left\{0, S(T) - K\right\}, \tag{14.75}$$

and so, we obtain:

$$C\left(S(T),K\right) - P\left(S(T),K\right) = S(T) - K. \tag{14.76}$$

And so, for the expectations:

$$E\left(C\left(S(T),K\right)\right)-E\left(P\left(S(T),K\right)\right)=E\left(S(T)\right)-K. \tag{14.77}$$

Using the principle of mathematical expectation for pricing the call and put, we obtain:

$$e^{rT}C(S_0,0)-e^{rT}P(S_0,0)=E\left(S(T)\right)-K. \tag{14.78}$$

We call this relation the *general call put parity relation* as it gives the value of the put knowing the value of the call and vice versa.

Now, under the assumption of an efficient market, we can use property (14.73) to get

$$e^{rT}C(S_0,0)-e^{rT}P(S_0,0)=S_0e^{rT}-K \tag{14.79}$$

and so the put value is given by:

$$P(S_0,0)=C(S_0,0)-S_0+e^{-rT}K. \tag{14.80}$$

Remark 14.4 We can interpret this relation as follows: assume a portfolio having at time 0 a share of value S_0, a put on the same asset with maturity T and an exercise price K, and a sold call with the same maturity and exercise price; the value of the portfolio at time T is always K, whatever the value of $S(T)$ is.

From the call put parity relation, we easily obtain the value of a put having the same maturity time T and exercise price K as for the call:

$$P(S,t)=C(S,t)-S+e^{-r(T-t)}K, \tag{14.81}$$

and using the Black and Scholes result, we obtain:

$$\begin{aligned}
&P(S,t)=Ke^{-r(T-t)}\Phi(-d_2)-S\Phi(-d_1),\\
&d_1=\frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)\right],\\
&d_2=d_1-\sigma\sqrt{T-t},\\
&S=S(t).
\end{aligned} \tag{14.82}$$

14.5. Exercises on option pricing

Exercise 14.1 Let us consider a portfolio with Δ shares of unit price €1,000 and an amount b invested at the non-risky interest rate of 4% per period.

1) What is the price C of a European call having €1,050 as the exercise price, of maturity two periods if, per period, the share increases by a quarter of its value with probability 0.75 and decreases by a third of its value with probability 0.25? What are the intermediate values of the call?

2) What is the composition of the hedging portfolio at time 0?

3) If the maturity has a value of 2 weeks and the period is the day, give an estimation of the volatility and the trend of the considered asset.

Solution:

1)

$$C_{uu} = 512.5, C_{ud} = C_{dd} = 0,$$
$$C_u = 315.38, C_d = 0,$$
$$C = 194.08.$$

2)

$C = \Delta S + B$ where:

$$\Delta = \frac{C_u - C_d}{S(u-d)} = 54.07\% \text{ (part of the asset)},$$

$$B = \frac{uC_d - dC_u}{(u-d)} = -346.57F \text{ (loan at the non-risky rate from the bank)}.$$

3) We know that:

$$1{,}000\times\frac{5}{4}=1{,}000\times e^{\mu\frac{t}{n}+\sigma\sqrt{\frac{t}{n}}},$$

$$1{,}000\times\frac{2}{3}=1{,}000\times e^{\mu\frac{t}{n}-\sigma\sqrt{\frac{t}{n}}},$$

or:

$t=14\ days,$

$n=1\ day,$

so:

$$\frac{5}{4}=e^{\mu 14+\sigma\sqrt{14}}\Rightarrow 14\mu+\sqrt{14}\sigma=\ln\frac{5}{4},$$

$$\frac{2}{3}=e^{\mu 14-\sigma\sqrt{14}}\Rightarrow 14\mu-\sqrt{14}\sigma=\ln\frac{5}{4}.$$

Finally,we obtain:

$$\mu=\frac{1}{28}0.2231436=0.0079694,$$

$$\mu_{year}=360\times 0.0079694=2.868994,$$

$$\sigma=\frac{1}{2\sqrt{14}}0.2231436=0.0298188,$$

$$\sigma_{year}=\sqrt{360}\times 0.0298188=0.565772.$$

14.6. The Greek parameters

14.6.1. *Introduction*

The technical management of the trader of options, particularly by the brokers, uses the *Greek parameters* to measure the impacts of small variations of parameters involved in formulas (4.20) and (4.34) for the pricing of options:

$$S,\sigma,\tau,r,K.$$

The delta coefficient

This is an indicator concerning the influence of small variations ΔS of the asset price defined as follows:

$$\begin{aligned} &C(S+\Delta S,t) \approx C(S,t) + \Delta(\Delta S), \\ &\Delta = \frac{\partial C}{\partial S}(S,t). \end{aligned} \tag{14.83}$$

This parameter is often used to cancel the variations of the asset value in the hedging portfolio.

The gamma coefficient

This is defined as:

$$\gamma = \frac{\partial^2 C}{\partial S^2}(S,t) \tag{14.84}$$

and so it may be seen as the *delta of the delta.*

It gives a measure of the acceleration of the variation of the call and a refinement of the measure of the variation of the call using the Taylor formula of order 2:

$$C(S+\Delta S,t) \approx C(S,t) + \Delta \Delta t + \frac{1}{2}\gamma \Delta t^2. \tag{14.85}$$

The theta coefficient

It gives the dependence of C with respect to the maturity $\tau(=T-t)$, and so also from time t:

$$\theta = -\frac{\partial C}{\partial t}\left(=\frac{\partial C}{\partial \tau}\right). \tag{14.86}$$

It follows the first order approximation:

$$C(S,t+\Delta t) \approx C(S,t) - \theta \Delta t. \tag{14.87}$$

For the maturity variations $\tau = T - t$, we obtain:

$$C(S,\tau+\Delta\tau) \approx C(S,\tau) + \theta \Delta \tau. \tag{14.88}$$

The elasticity coefficient

Recall the economic definition of this coefficient which gives:

$$e(S,t) = \frac{\partial C}{\partial S}(S,t) \times \frac{S}{C(S,t)} \tag{14.89}$$

and so:

$$\frac{\Delta C}{C}\left(= \frac{C(S+\Delta S,t) - C(S,t)}{C(S,t)}\right) \approx e(S,t)\frac{\Delta S}{S}. \tag{14.90}$$

The vega coefficient

This is the indicator concerning the measure of small variations of the volatility σ and so:

$$\upsilon = \frac{\partial C}{\partial \sigma}(S,t)\,. \tag{14.91}$$

Thus, we have approximately for small variations $\Delta\sigma$,

$$C(S+\Delta S,t) \approx C(S,t) + \upsilon \Delta o. \tag{14.92}$$

The rho coefficient

This concerns the non-risky instantaneous rate *r* and so:

$$\rho = \frac{\partial C}{\partial r}(S,t). \tag{14.93}$$

14.6.2. *Values of the Greek parameters*

The following table gives the values of the Greek parameters first for the call and then for the put.

I. For the calls:

$$1)\ \text{delta}\left(=\frac{\partial \mathrm{C}}{\partial \mathrm{S}}\right)=\Phi(d_1)>0$$

$$2)\ \text{gamma}\left(=\frac{\partial \Delta}{\partial \mathrm{S}}\right)=\frac{\Phi'(d_1)}{S\sigma\sqrt{\tau}}>0$$

$$3)\ \text{vega}\left(=\frac{\partial \mathrm{C}}{\partial \sigma}\right)=S\sqrt{\tau}\Phi'(d_1)>0$$

$$4)\ \text{rho}\left(=\frac{\partial \mathrm{C}}{\partial \mathrm{r}}\right)=K\tau \mathrm{e}^{-\mathrm{r}\tau}\Phi(d_2)>0$$

$$5)\ \text{theta}(=\frac{\partial \mathrm{C}}{\partial \tau})=\mathrm{rKe}^{-\mathrm{r}\tau}\Phi(d_2)+\frac{\sigma \mathrm{S}}{2\sqrt{\tau}}\Phi'(d_1)>0$$

$$6)\ \frac{\partial C}{\partial K}=-e^{-r\tau}\Phi(d_2)<0$$

II. For the puts:

$$1)\ \text{delta}\left(=\frac{\partial \mathrm{P}}{\partial \mathrm{S}}\right)=(\Phi(d_1)-1)=-\Phi(-d_1)(=\Delta_C-1)<0$$

$$2)\ \text{gamma}\left(=\frac{\partial \Delta}{\partial \mathrm{S}}\right)=\frac{\Phi'(d_1)}{S\sigma\sqrt{\tau}}(=gamma_C)>0$$

$$3)\ \text{vega}\left(=\frac{\partial \mathrm{P}}{\partial \sigma}\right)=S\sqrt{\tau}\Phi'(d_1)(=véga_C)>0$$

$$4)\ \text{rho}\left(=\frac{\partial \mathrm{P}}{\partial \mathrm{r}}\right)=-\tau \mathrm{K}e^{-r\tau}\Phi(-d_2)=\tau \mathrm{K}e^{-r\tau}\left[\Phi(d_2)-1\right](=rh\hat{o}_C-\tau \mathrm{K}e^{-r\tau})<0$$

$$5)\ \text{theta}\left(=\frac{\partial \mathrm{P}}{\partial \tau}\right)=\frac{\sigma S}{2\sqrt{\tau}}\Phi'(d_1)-r\mathrm{Ke}^{-\mathrm{r}\tau}\left[1-\Phi(d_2)\right](=\theta_C-rKe^{-r\tau})$$

$$6)\ \frac{\partial P}{\partial K}=e^{-r\tau}(-\Phi(d_2)+1)=e^{-r\tau}\Phi(-d_2)\left(=\frac{\partial P}{\partial K}_C+e^{-r\tau}\right)>0$$

These values give interesting results concerning the influence of the considered parameters of the call and put values.

For example, we deduce that the call and put values are increasing functions of the volatility, and the call increases as S increases but the put decreases as S increases.

14.6.3. *Exercises*

Exercise 14.2 Let us consider an asset of value €1,700 and having a weekly variance of 0.000433.

(i) What is the value of a call of exercise price €1,750 with maturity 30 weeks under a non-risky rate of 6%?

(ii) Under the anticipation of a rise of €100 of the underlying asset and of a rise of 0.000018 of the weekly variance, what will be the consequences of the call and put values?

Solutions

(i) The values of the parameters necessary to calculate the call value using the Black and Scholes formula are:

$$\sigma^2_{week} = 0.00043 \Rightarrow \sigma^2_{year} = 52 \times 0.00043 = 0.2236,\ \sigma_{year} = 0.47286,$$
$$\tau = 30\ weeks = 0.576923\ year,\ K = 1750, S = 1700,$$
$$i = 6\% \Rightarrow r = \ln(1+i) = 0.05827.$$

It follows that:

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{S}{K} + \tau\left(r + \frac{\sigma^2}{2}\right)\right] \Rightarrow d_1 = 0.09760272,$$
$$\Phi(d_1) = 0.5388762,$$
$$\Rightarrow d_2 = d_1 - \sigma\sqrt{\tau} = -0.01637096, \Phi(d_2) = 0.4934692,$$
$$C(S,\tau) = S\Phi(d_1) - Ke^{-r\tau} = 81.07 Euro.$$

Using call put parity relation; we obtain for the put value

$$P = Ke^{-r\tau} + C - S \Rightarrow P = 73.07 Euro.$$

(ii) Rise of the underlying asset. We know that:

$$C(S+\Delta S,\tau) = C(S,\tau) + \frac{\partial C}{\partial S}(S,\tau)\Delta S,$$
$$\frac{\partial C}{\partial S}(S,\tau) = \Phi(d_1),$$
so :
$$C(1700+100,\tau) = 81.07 + 100 \times 0.5388762 = 135.95 Euro.$$

For the put, we obtain:

$$P(S+\Delta S,\tau)=Ke^{-r\tau}+C(S+\Delta S)-(S+\Delta S)=27.1 Euro.$$

(iii) Rise of the volatility. The value of the new weekly variance is now given by:

$$0.000433+0.00018=0.000613$$

and, so the new yearly variance and volatility are given by

$$\sqrt{0.031876}=0.1785385,$$

and consequently, the variation of the yearly volatility is given by:

$$\Delta\sigma=0.1785385-0.1500533=0.284852.$$

As the increase in volatility comes after that of the asset value, we have

$$C(S+\Delta S,\sigma+\Delta\sigma,\tau)=C(S+\Delta S,\sigma,\tau)+\frac{\partial C}{\partial\sigma}\Delta\sigma,$$

with:

$$\frac{\partial C}{\partial\sigma}=\sqrt{\tau}\Phi'(d_1).$$

However:

$$\Phi'(d_1)=\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}=0.39704658,$$

and so:

$$\frac{\partial C}{\partial\sigma}=542.84.$$

Finally, we obtain:

$$C(S+\Delta S,\sigma+\Delta\sigma,\tau)=C(S+\Delta S,\sigma,\tau)+\frac{\partial C}{\partial\sigma}\Delta\sigma=150.41F.$$

For the variation for the put, we use the call put parity relation and so:

$$P(S+\Delta S,\sigma+\Delta\sigma,\tau)=C(S+\Delta S,\sigma+\Delta\sigma,\tau)+Ke^{-r\tau}-(S+\Delta S)=42.56F.$$

Exercise 14.3 For the following data, calculate the values of the call, the put and the Greek parameters

$$S=100, K=98, \tau=30 \text{ days, } \sigma_{week}=0,01664, i=8\%.$$

Solution

Yearly vol.	0.12	
Maturity	0.08219	
R=ln(1+i)	0.076962	
Results	Call	Put
Price	3.04721	0.42926
Delta	0.7847	–0.2153
Vega	8.3826	8.3826
Theta	11.924	4.334
Gamma	0.08499	0.08499
Rho	6.199	–1.805

Table 14.1. *Example option calculation*

14.7. The impact of dividend repartition

If, between t and T, the asset distributes N dividends of amounts $D_1,\ldots,D_N$ at times:

$$(0<t<)t_1<t_2<\cdots<t_N(<T), \qquad (14.94)$$

the impact of the value of a European call is the following: as the buyer of the call cannot receive these dividends, it suffices to calculate the present value at time t of these dividends and to subtract the sum from the asset value at time t so that the call value is now:

$$C(S,\tau;D_1,\ldots,D_N)=C\left(S-\sum_{j=1}^{N}D_je^{-r\tau_j},\tau\right), \qquad (14.95)$$

$$\tau_j=t_j-t,\ j=1,\ldots,N.$$

Of course, the most usual case is N=1.

If we assume that the distribution of dividends is given with a continuous payout at rate D per unit of time,

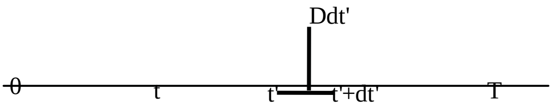

Figure 14.8. *Continuous "payout"*

the capitalized value is $e^{D\tau}$ and so the value of the call is given by:

$$C(S,\tau;D)=C(Se^{-Dt},\tau). \tag{14.96}$$

14.8. Estimation of the volatility

14.8.1 *Historic method*

This method is based on the data of the underlying asset evolution in the past, for example the n daily values

$$\left(S_0,S_1,...,S_n\right). \tag{14.97}$$

Let us consider the following sample of the consecutive ratios:

$$\left(R_1,...,R_n\right)=\left(\frac{S_1}{S_0},...,\frac{S_n}{S_{n-1}}\right). \tag{14.98}$$

From the lognormal distribution property, we have:

$$\frac{\ln R_t-\left(\mu-\dfrac{\sigma^2}{2}\right)}{\sigma}\succ N(0,1), \tag{14.99}$$

with $R_t=\dfrac{S_t}{S_{t-1}},t=1,...,n.$

It follows that the random sample $(\ln R_1,...,\ln R_n)$ can be seen as extracted from a normal population (μ',σ^2) with:

$$\mu' = \mu - \frac{\sigma^2}{2}. \tag{14.100}$$

The traditional results of mathematical statistics give as best estimators:

$$\begin{aligned}\hat{\mu}' &= \frac{1}{n}\sum_{k=1}^{n}\ln\frac{R_k}{R_{k-1}},\\ \hat{\sigma}^2 &= \frac{1}{n}\sum_{k=1}^{n}\left(\ln\frac{R_k}{R_{k-1}} - \hat{\mu}'\right)^2.\end{aligned} \tag{14.101}$$

To obtain an unbiased estimator of the variance, we have to use:

$$\hat{\hat{\sigma}}^2 = \frac{n}{n-1}\hat{\sigma}^2 \tag{14.102}$$

or:

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(\ln\frac{R_k}{R_{k-1}}\right)^2 - \frac{n}{n-1}(\hat{\mu}')^2. \tag{14.103}$$

Example 14.1 On the basis of a sample of 27 weekly values of an asset starting from the initial value of €26.375, the following *weekly* estimations are found:

$$\begin{aligned}\hat{\mu} &= 0.016732\\ \hat{\sigma}^2 &= 0.005216.\end{aligned}$$

Consequently, as the parameters of the Black and Scholes model must be evaluated on a yearly basis, we obtain

$$\begin{aligned}\hat{\mu}_{year} &= 52\times 0.016732 = 0.870064 \cong 0.87,\\ \hat{\sigma}^2_{year} &= 52\times 0.005216 = 0.271232,\\ \hat{\sigma}_{year} &= \sqrt{0.271232} = 0.520799 \cong 0.52.\end{aligned}$$

14.8.2. *Implicit volatility method*

This method assumes that the Black and Scholes calibrates the market values of the observed calls well.

Theoretically, an inversion of the Black and Scholes formula gives the value of the volatility σ.

On the basis of several observations of the calls for the same underlying asset, we can use the least square statistical method to refine the estimation.

Example 14.2 Using the data of Exercise 14.3, we assume that we have an observed value of the call 3.04715, but without knowing the volatility.

The next table gives the results using a step by step approximation method.

Weekly vol.	Annual vol.	Call value
0.02	0.144	3.26
0.015	0.1081	2.95
0.017	0.1225	3.069
0.016	0.1153	3.008
0.0165	0.1189	3.038
0.01664	0.1199	3.04713

Table 14.2. *Volatility calculation*

So, we find the correct volatility value to be 0.12.

Remark 14.5 The main difficulty is to select the historical data.

The set must not be too long or too short in order to avoid disrupted periods introducing strong biases in the results.

Moreover, we always work with the assumption of a constant volatility that we will overtake in section 14.10.

14.9. Black-Scholes on the market

14.9.1. *Empirical studies*

Since the opening of the CBOT in Chicago in 1972, numerous studies have been carried out for testing the results of the Black and Scholes formula.

In the case of efficient markets, the conclusions are as follows:

(i) the non-risky interest rate has little influence on the option values;

(ii) the Black and Scholes formula *underestimates* the market values for calls with short maturity times, for calls "deep out of the money" ($S/K<0.75$) and for calls with weak volatility;

(iii) the Black and Scholes formula *overestimates* the market values for calls "deep in the money" ($S/K<1.25$) and for calls with high volatility. The put values are often underestimated particularly in the out of the money ($S>>K$) case;

(iv) the puts are often underestimated particularly when they are out of money ($S<<K$).

14.9.2. *Smile effect*

If we calculate the volatility values with the implicit method in different times, in general, the results show that the volatility is *not constant*, thus invalidating one of the basic assumptions of the considered Black and Scholes model.

The graph of the volatility as a function of the exercise price often gives a graph with a convex curve, a result commonly called the "*smile effect*".

However, sometimes, concave functions are also observed.

Although, theoretically, volatilities for the pricing of calls and puts are identical, in practice, some differences are observed; they are assigned to differences of "bid-offer spread" and to the methodology of the implicit method used at different times.

The fact that it is important to consider option pricing models with non-constant volatility is one of the approaches of the next model.

14.10. Exotic options

14.10.1. *Introduction*

The derivative products of first generation concern the traditional calls and puts also called *plain vanilla options* and furthermore the anticipation of the investor leads to the construction of *strategies* for hedging or eventually for speculation.

However, these traditional options and the derived strategies generally have high costs and their exercise prices only depend of the value of the underlying assets at maturity. In particular, they do not work for some markets such as foreign currency and commodities markets.

That is why the market of derivative products has been enlarged with the *second generation options* or *exotic options*.

Their main characteristics are as follows:

a) a *prime reduction*, essentially for barrier, binary mean and compound options defined after;

b) introduction of a diversification with the use of options of the types *out performance, best of, worst of*;

c) use of options of the types lookback, option on the mean, etc.;

d) use of options linked to the exchange market like a quanto option.

All these options are in fact two types following the way on which the exercise price is defined:

(i) *"non-path dependent" options:* the exercise price is defined at the time of the conclusion of the option contract;

(ii) *"path dependent" options*: the exercise price is not known at the time of the conclusion of the option contract but the way to calculate it at the maturity time is given in the contract.

In practice, the market of such options is less liquid than the traditional market and also has a lack of organization due to the lack of standardized contracts.

However, the foreign currency options are available on organized markets as the PHLX (Philadelphia Options Exchange) created in 1983 with a clearing room, yet approximately 80% of the transactions are over the counter and in this last case, intermediaries are big banks supporting the counterpart for bid and ask for their customers.

14.10.2. *Garman-Kohlhagen formula*

For foreign currency options, we use the Black and Scholes model:

$$\begin{aligned} &dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \\ &S(0) = S_0. \end{aligned} \tag{14.104}$$

with the usual assumptions, but here $S(t)$ is the value of the *spot exchange rate* at time t. The domestic and foreign instantaneous interest rates, respectively noted r_d , r_f , are constant over the life of the considered option.

Under these assumptions, it is possible to calculate the value of a call with the following formula:

$$\begin{aligned} &C(S,t) = Se^{-r_f(T-t)}\Phi(d_1) - Ke^{-r_d(T-t)}\Phi(d_2), \\ &d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S}{K} + \left(r_d - r_f + \frac{\sigma^2}{2}\right)(T-t)\right], \\ &d_2 = d_1 - \sigma\sqrt{T-t}, \\ &S = S(t). \end{aligned} \tag{14.105}$$

The calculation of the put value is done with the following *call put parity relation*:

$$\begin{aligned} &P_{fin}(S_0,0) = C_{fin}(S_0,0) - e^{-r_f T}S_0 + e^{-r_d T}K \\ &\text{or} \\ &P_{fin}(S\ ,t) = C_{fin}(S\ ,t) - e^{-r_f(T-t)}S\ + e^{-r_d(T-t)}K, \end{aligned} \tag{14.106}$$

so that:

$$\begin{aligned} &P(S,t) = Ke^{-r_d(T-t)}\Phi(-d_2) - Se^{-r_f(T-t)}\Phi(-d_1), \\ &d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S}{K} + (r_d - r_f + \frac{\sigma^2}{2})(T-t)\right], \\ &d_2 = d_1 - \sigma\sqrt{T-t}, \\ &S = S(t). \end{aligned} \tag{14.107}$$

Remark 14.6 Some empirical studies show that the G-K formula overestimates the observed market values.

14.10.3. *Greek parameters*

The values of the Greek parameters for calls and puts obtained by calculation as for the traditional Black and Scholes model are given below.

I. For the call:

1) $\text{delta}\left(=\dfrac{\partial C}{\partial S}\right)=e^{-r^*\tau}\Phi(d_1)>0$

2) $\text{gamma}\left(=\dfrac{\partial \Delta}{\partial S}\right)=e^{-r^*\tau}\dfrac{\Phi'(d_1)}{S\sigma\sqrt{\tau}}>0$

3) $\text{vega}\left(=\dfrac{\partial C}{\partial \sigma}\right)=e^{-r\tau}K\sqrt{\tau}\Phi'(d_1)>0$

4) $\text{rho}\left(=\dfrac{\partial C}{\partial r}\right)=K\tau e^{-r\tau}\Phi(d_2)>0$

4') $\text{rho'}\left(=\dfrac{\partial C}{\partial r^*}\right)=-K\tau e^{-r^*\tau}\Phi(d_1)<0$

5) $\text{theta}\left(=\dfrac{\partial C}{\partial \tau}\right)=-r^*e^{-r^*\tau}S\Phi(d_1)+rKe^{-r\tau}\Phi(d_2)+\dfrac{\sigma K}{2\sqrt{\tau}}\Phi'(d_2)>0$

6) $\dfrac{\partial C}{\partial K}=-e^{-r\tau}\Phi(d_2)<0$

II. For the put:

1) $\text{delta}\left(=\dfrac{\partial P}{\partial S}\right)=e^{-r^*\tau}(\Phi(d_1)-1)=-e^{-r^*\tau}\Phi(-d_1)<0$

2) $\text{gamma}\left(=\dfrac{\partial \Delta}{\partial S}\right)=e^{-r^*\tau}\dfrac{\Phi'(d_1)}{S\sigma\sqrt{\tau}}(=gamma_P)>0$

3) $\text{vega}\left(=\dfrac{\partial P}{\partial \sigma}\right)=Ke^{-r^*\tau}\sqrt{\tau}\Phi'(d_1)(=vega_C)>0$

4) $\text{rho}(=\dfrac{\partial P}{\partial r})=\tau Ke^{-r\tau}\Phi(-d_2)=\tau Ke^{-r\tau}\left[1-\Phi(d_2)\right]>0$

5) $\text{theta}\left(=\dfrac{\partial P}{\partial \tau}\right)=-r^*Se^{-r^*\tau}\Phi(-d_1)+\dfrac{\sigma Ke^{-r\tau}}{\sqrt{\tau}}\Phi'(d_2)+rKe^{-r\tau}\Phi(-d_2)$

14.10.4. *Theoretical models*

We know that there exist two ways for pricing derivative products:

(i) the resolution of a partial differential equation (PDE) with eventually a numerical solution of the risk neutral measure method;

(ii) the calculation of the present value of gain at maturity under the risk neutral measure.

Let us recall that the first gives a PDE for the call value using Itô’s calculus and the assumption of absence of opportunity arbitrage (AOA). For non-plain vanilla options, the only way to work is to use numerical methods to obtain an approximate solution with, for example, *the finite differences* method.

It is also possible to use a discrete time model as the Cox-Rubinstein method, particularly useful for the American type.

The risk neutral measure method uses the Girsanov theorem to obtain a new probability measure Q instead of the historical probability measure P so that the call value is given with the present value of the "gain" at maturity.

Here, the new measure Q is obtained with a new trend in the SDE (14.104) given by $r_d - r_f$.

In this case, on [0, T], we obtain:

$$S(T) = S(t)e^{\left((r_d - r_f) - \frac{\sigma^2}{2}\right)(T-t) + \sigma(\tilde{B}(T) - \tilde{B}(t))}, \tag{14.108}$$

$\tilde{B} = (\tilde{B}(s), 0 \le s \le T)$ being a new standard Brownian motion standard on the filtered probability space $(\Omega, \Im, (\Im_t), Q)$.

If h represents the "gain" at maturity for the considered derivative product, the value $V(t)$ of this product at time t is given by:

$$V(t) = E_Q\left[e^{-r_d(T-t)} h \middle| \Im_t\right]. \tag{14.109}$$

For example, for a plain vanilla call of exercise price K, we obtain

$$h = (S(T) - K)_+, \tag{14.110}$$

and so:

$$V(t) = E\left[e^{-r_d(T-t)}(S(t)e^{\left((r_d - r_f) - \frac{\sigma^2}{2}\right)(T-t) + \sigma(\tilde{B}(T) - \tilde{B}(t))} - K)_+ \middle| \Im_t\right]. \tag{14.111}$$

The process S being adapted to the basic filtration, we finally obtain:

$$\begin{aligned} V(t) &= C(S(t), t) \\ &= \frac{1}{\sqrt{2\pi}} \int_R e^{-r_d(T-t)}(S(t)e^{\left((r_d - r_f) - \frac{\sigma^2}{2}\right)(T-t) + \sigma z\sqrt{T-t}} - K)_+ e^{-\frac{z^2}{2}})dz, \end{aligned} \tag{14.112}$$

An expression resulting in, after the change of variable $\frac{y}{\sqrt{T-t}} = x,$ the Garman-Kohlhagen formula.

Similarly, for the put, we obtain:

$$V(t) = E\left[e^{-r_d(T-t)} \left(K - S(t) e^{\left((r_d - r_f) - \frac{\sigma^2}{2} \right)(T-t) + \sigma\left(\tilde{B}(T) - \tilde{B}(t)\right)} \right)_+ \middle| \mathfrak{I}_t \right]. \qquad (14.113)$$

Remark 14.7 (options on shares and options on foreign currency options) Formally, the Garman-Kohlhagen formula is an simple extension of the Black and Scholes formula; indeed, setting $r_f = 0$ in the first formula, we obtain the second.

In particular, this means that all the results on exotic foreign currency options contain similar results for share options.

14.10.5. *Binary or digital options*

14.10.5.1. *Definition*

We will present the *"cash or nothing"* and *"asset or nothing"* options. In this case, the gain at maturity depends on the fact that, at maturity time, the underlying asset goes beyond a barrier called the *exercise price* and if so, the exercise of the option gives as gain a fixed amount mentioned in the contract signed at time of purchase of the considered option and independent of $S(T)$.

In other words, the purchaser of the option receives a coupon if the underlying asset is above the barrier and nothing in the other case.

Example 14.3: ***a standard cash or nothing call***

– *option type*: all or nothing call;

– *underlying asset*: CAC 40 index;

– *nominal*: 100,000;

– *device*: €;

– *index value at the issuing of the option*: 3,000 points;

– *exercise price*: 3,100 points;

– *coupon*: 10%;

– *issuing date*: 5/1/07;

– *maturity date*: 5/1/08;

– *premium option*: 2.9%.

So, if the CAC 40 index is larger than 3,100 points at maturity time, the counter part will pay an amount of €10,000.

The initial premium is €2,900, and the net return, excluding transaction costs, is 245%. On the other hand, if the CAC 40 index is smaller than 3,100 points at maturity time, the premium is lost.

14.10.5.2. *Pricing of a call cash or nothing*

Let N be the coupon of the option and K the exercise price. From the definition of the type of this option, we have, under the risk neutral measure Q:

$$\begin{aligned} C_{cn}(S(t),N,K,t) &= e^{-r_d(T-t)}E_Q\left[N.1_{\{S(T)\geq K\}}\right], \\ &= Ne^{-r_d(T-t)}E_Q\left[1_{S(t)e^{\left((r_d-r_f)-\frac{\sigma^2}{2}\right)(T-t)+\sigma(\tilde{B}(T)-\tilde{B}(t))}\geq K}\right], \\ &= Ne^{-r_d(T-t)}\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\ln\frac{S(t)}{K}+\left(\frac{r_d-r_f}{2}-\frac{\sigma}{2}\right)\sqrt{T-t}\right). \end{aligned} \tag{14.114}$$

and so:

$$C_{cn}(S(t),N,K,t) = Ne^{-r_d(T-t)}\Phi\left(d_2\right), \tag{14.115}$$

where, as for the Garman-Kohlhagen model:

$$\begin{aligned} d_2 &= d_1-\sigma\sqrt{T-t}, \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S(t)}{K}+\left(r_d-r_f+\frac{\sigma^2}{2}\right)(T-t)\right]. \end{aligned} \tag{14.116}$$

14.10.5.3. *Case of the put cash or nothing*

For the put, we have:

$$N.1_{\{S(T)\leq K\}} = N - N.1_{\{K\leq S(T)\}}, \tag{14.117}$$

and so:

$$P_{cn}(S(t),N,K,t)=Ne^{-r_d(T-t)}\Phi(-d_2). \tag{14.118}$$

14.10.5.4. *Main Greek parameters for call and put cash or nothing*

We just consider the case of the delta, gamma and vega.

14.10.5.4.1. Case of the call

a) The delta

By definition, we have

$$\Delta_C=\frac{\partial C}{\partial S}, \tag{14.119}$$

so, by relation (14.115):

$$\Delta_C=Ne^{-r_d(T-t)}\Phi'(d_2)\frac{\partial d_2}{\partial S}, \tag{14.120}$$

and by relation (14.116):

$$\Delta_C=\frac{N}{\sigma\sqrt{T-t}.S}e^{-r_d(T-t)}\Phi'(d_2). \tag{14.121}$$

Delta being always positive, it follows that the call is an increasing function of S, the value of the underlying asset at time *t*. Furthermore, it is maximum for *S*=*K* and becomes infinite at maturity.

b) The gamma

We know that:

$$\gamma_C=\frac{\partial\Delta_c}{\partial S}=\frac{\partial^2 C}{\partial S^2}. \tag{14.122}$$

so, by relation (14.115):

$$\gamma_C=\frac{Ne^{-r_d(T-t)}}{\sigma\sqrt{T-t}}\left[\frac{-1}{S^2}\Phi'(d_2)+\frac{1}{S}\Phi''(d_2)\frac{\partial d_2}{\partial S}\right]. \tag{14.123}$$

and finally:

$$\gamma_C = \frac{Ne^{-r_d(T-t)}}{\sigma\sqrt{T-t}}\left[\frac{-1}{S^2}\Phi'(d_2)+\frac{1}{S}d_2\Phi'(d_2)\frac{1}{S\sigma\sqrt{T-t}}\right], \quad (14.124)$$

or

$$\gamma_C = \frac{Ne^{-r_d(T-t)}}{S^2\sigma\sqrt{T-t}}\Phi'(d_2)\left[-1+\frac{d_2}{\sigma\sqrt{T-t}}\right]. \quad (14.125)$$

This gamma is quasi-zero for large maturities, it changes its sign, from positive to negative values, at K, and at maturity, it becomes infinite.

14.10.5.4.2. Put case

From relation (14.117), we know that

$$P_{cn} = N - C_{cn}, \quad (14.126)$$

and so, we immediately obtain the following values:

$$\Delta_{P_{cn}} = -\Delta_{C_{cn}},$$
$$\gamma_{P_{cn}} = -\gamma_{C_{cn}}, \quad (14.127)$$
$$\upsilon_{P_{cn}} = -\upsilon_{C_{cn}}.$$

14.10.6. *"Asset or nothing" options*

14.10.6.1. *Definition*

This type of option differs from the preceding one as it arrives at maturity at the money, the coupon paid is not a fixed amount *N* but a *multiple* of the underlying asset.

Example 14.4: *a standard asset or nothing*

– *option type*: call asset or nothing;
– *underlying asset*: share X;
– *nominal*: €800,000 (1,000 shares);
– *devise*: €;
– *share value at the issuing of the option*: €800;
– *exercise price*: €850;
– *percentage*: 10%;
– *payment*: in asset value at maturity;
– *issuing date*: 5/1/07;
– *maturity date*: 5/1/08;
– *option premium*: 4.25%.

So, if the asset value at maturity is above €850, for example €900, the counterpart has to pay, per share, an amount of 0.1 × 900 = 90, that is, a total amount of €90,000.

In this case, for an initial investment of €34,000, the net return, without transaction costs, is given by:

$$\frac{90,000-34,000}{340}=164.71\%.$$

Of course, if the asset value at maturity is less than €850, the holder of the call loses the premium of €34,000.

14.10.6.2. *Pricing a call asset or nothing*

Let M be the percentage of share to be paid in cash and K the exercise price.

Proceeding as before, under the risk neutral measure Q, we successively obtain:

$$\begin{aligned} C_{an}(S(t),M,K,t) &= e^{-r_d(T-t)}E_Q\left[MS(T).1_{\{S(T)\geq K\}}\middle|\Im_t\right] \\ &= Me^{-r_d(T-t)} \\ &\cdot E_Q\left[S(t)e^{\left((r_d-r_f)-\frac{\sigma^2}{2}\right)(T-t)+\sigma(\tilde{B}(T)-\tilde{B}(t))}1_{S(t)e^{\left((r_d-r_f)-\frac{\sigma^2}{2}\right)(T-t)+\sigma(\tilde{B}(T)-\tilde{B}(t))}\geq K}\right] \\ &= MS(t)e^{-r_f(T-t)}\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\ln\frac{S(t)}{K}+\left(\frac{r_d-r_f}{2}-\frac{\sigma}{2}\right)\sqrt{T-t}\right). \end{aligned} \quad (14.128)$$

The final result is:

$$C_{an}(S(t),M,K,t)=MS(t)e^{-r_f(T-t)}\Phi\left(d_1\right). \quad (14.129)$$

For a call asset or nothing on a share market, setting $r_f=0$, we obtain:

$$C_{an}(S(t),M,K,t)=MS(t)\Phi\left(d_1\right). \quad (14.130)$$

14.10.6.3. *Premium of the put asset or nothing*

From the relation:

$$S(T).1_{\{S(T)\leq K\}}=S(T)-S(T).1_{\{K\leq S(T)\}}, \quad (14.131)$$

we obtain:

$$P_{an}(S(t),M,K,t)=e^{-r_d(T-t)}E_Q\left[MS(T)\right]-P_{cn}(S(t),M,K,t). \tag{14.132}$$

As under Q, the drift of S is given by $r_d - r_f$, we can write that:

$$P_{an}(S(t),M,K,t)=e^{-r_d(T-t)}E_Q\left[MS(T)\right]-P_{cn}(S(t),M,K,t). \tag{14.133}$$

Thus,

$$P_{an}(S(t),M,K,t)=Me^{-r_f(T-t)}S(t)\Phi\left(-d_1\right). \tag{14.134}$$

On a share market, we obtain in this case $r_f = 0$:

$$P_{an}(S(t),N,K,t)=MS(t)\Phi\left(-d_1\right). \tag{14.135}$$

14.10.6.4. *Greek parameters for call and put asset or nothing*

Here too, we just consider the case of the delta, gamma and vega.

14.10.6.4.1. Case of the call

a) The delta

As

$$\Delta_C=\frac{\partial C}{\partial S}, \tag{14.136}$$

we obtain from relation (14.129):

$$\Delta_C=MS(t)e^{-r_f(T-t)}\Phi'\left(d_1\right)\frac{\partial d_1}{\partial S}+Me^{-r_f(T-t)}\Phi\left(d_1\right), \tag{14.137}$$

or

$$\Delta_C=\frac{M}{\sigma\sqrt{T-t}}e^{-r_d(T-t)}\Phi'\left(d_1\right)+Me^{-r_d(T-t)}\Phi\left(d_1\right). \tag{14.138}$$

The delta being always positive, it follows that the call is an increasing function of S, the value of the underlying asset at time *t*. Furthermore, it is maximum for S = K and, at maturity, it has the value M in the case of being in the money and 0 out of the money. At maturity and at the money, the delta becomes infinite.

b) The gamma

As

$$\gamma_C = \frac{\partial \Delta_c}{\partial S} = \frac{\partial^2 C}{\partial S^2}, \tag{14.139}$$

we obtain:

$$\gamma_C = Me^{-r_d(T-t)}\left[\frac{1}{\sigma\sqrt{T-t}}\Phi''(d_1) + \Phi'(d_1)\right]\frac{\partial d_1}{\partial S}, \tag{14.140}$$

and

$$\gamma_C = \frac{Me^{-r_d(T-t)}}{S(t)\sigma\sqrt{T-t}}\left[\frac{1}{\sigma\sqrt{T-t}}\Phi''(d_1) + \Phi'(d_1)\right], \tag{14.141}$$

or finally

$$\gamma_C = \frac{Me^{-r_d(T-t)}}{S(t)\sigma\sqrt{T-t}}\left[-\frac{d_1}{\sigma\sqrt{T-t}} + 1\right]\Phi'(d_1). \tag{14.142}$$

This gamma changes sign, from positive to negative values, at *K*, and at maturity, it becomes infinite.

14.10.6.4.2. Case of the put

As

$$P_{an} = Me^{-r_f(T-t)}S - C_{an}, \tag{14.143}$$

we immediately have:

$$\begin{aligned} \Delta_{P_{an}} &= Me^{-r_f(T-t)} - \Delta_{C_{an}}, \\ \gamma_{P_{cn}} &= -\gamma_{C_{cn}}, \\ \upsilon_{P_{cn}} &= -\upsilon_{C_{cn}}. \end{aligned} \tag{14.144}$$

14.10.7. *The barrier options*

14.10.7.1. *Definitions*

Let us assume that a French enterprise has to pay one of its American furnishers in dollars and in three months.

If the exchange rate $/€ is 1.27, this enterprise *e* can be hedged against an increase of the exchange rate with a call in $ or a put in € in the money. However, if this enterprise anticipates that the rate will not be higher than 1.31, for example, it is possible to add a supplementary condition to the standard option contract as follows: if on [0, *T*], *the rate goes beyond this value, then the option disappears and arrives at maturity without any value.*

This means that we introduce the concept of a *barrier*, here at a value of 1.31, and so this new type of option has a final value which depends on all the paths of the underlying asset and not only on its final value.

It is clear that this new type of options, called *barrier options*, will find a liquid enough market as their premiums are lower than the plain vanilla options. So, we have the following definition.

Definition 14.2 *A barrier option is a path-dependent option, the payoff of which depends on the payoff of a traditional option and whether a pre-specified barrier has been crossed.*

Most popular types are: *down-and-in options, down-and-out options, up-and-in options* and *up-and-down options.*

The definitions are as follows:

(i) *down-and-out options*: a lower barrier (i.e. smaller than *S*(0)) is specified. If the spot exchange rate falls below this barrier during the life of the option, that is on [0, *T*], the option ceases to exist and if not, the option remains traditional;

(ii) *down-and-in options*: the option becomes active only if the spot exchange rate goes below a given barrier; otherwise, the contract gives no right;

(iii) *up-and-out-options*: with a given specified upper barrier, if the spot exchange rate goes above the barrier on [*0, T*], the option ceases to exist; otherwise, it remains a traditional option;

(iv) *up-and-it-options*: with a given specified upper barrier, if the spot exchange rate does not go above the barrier on [*0, T*], the option is worthless; otherwise, it remains a traditional option.

14.10.7.2. *Examples of pricing*

Let us consider the case of a *down-and-in* call. It is clear that the value of the call is given by

$$C_{di}(S,t) = e^{-r_d(T-t)} E_Q\left[(S(T)-K)^+ 1_{\{T_H<T\}} \middle| \mathfrak{I}_t \right],$$

T_H being the hitting time of the barrier H for the process S:

$$T_H = \inf\{s > 0 : S(s) \le H\}$$

Thus, we have:

$$K \le H:$$

$$C_{di}(S,t) = Se^{-r_f(T-t)}\left[\left(\frac{H}{S}\right)^{2\lambda}\Phi(y_1) + \Phi(x) - \Phi(x_1)\right] - Ke^{-r_d(T-t)}$$

$$\cdot\left[\left(\frac{H}{S}\right)^{2\lambda-2}\Phi(y_1 - \sigma\sqrt{T-t}) + \Phi(x - \sigma\sqrt{T-t}) - \Phi(x_1 - \sigma\sqrt{T-t})\right],$$

$$K > H:$$

$$C_{di}(S,t) = Se^{-r_f(T-t)}$$

$$\cdot\left[\left(\frac{H}{S}\right)^{2\lambda}\Phi(y)\right] - Ke^{-r_d(T-t)}\left[\left(\frac{H}{S}\right)^{2\lambda-2}\Phi(y - \sigma\sqrt{T-t})\right].$$

In these results, we have:

$$\begin{aligned} x &= \frac{1}{\sigma\sqrt{T-t}}\ln\frac{S}{K} + \lambda\sigma\sqrt{T-t},\ y \\ &= \frac{1}{\sigma\sqrt{T-t}}\ln\frac{H^2}{SK} + \lambda\sigma\sqrt{T-t}, \\ x_1 &= \frac{1}{\sigma\sqrt{T-t}}\ln\frac{S}{H} + \lambda\sigma\sqrt{T-t},\ y_1 \\ &= \frac{1}{\sigma\sqrt{T-t}}\ln\frac{H}{S} + \lambda\sigma\sqrt{T-t}, \\ \lambda &= \frac{1}{2} + \frac{r_d - r_f}{\sigma^2}. \end{aligned} \tag{14.145}$$

14.10.8. *Lookback options*

These are also called "no regrets options" and are path-dependent options favorable to the holder as they are generally expansive.

The two main types are:

– the "*standard lookback*" option: in the case of a call, the payoff is given by:

$$S(T) - \inf_{0 \le t \le T} S(t);$$

– the "*option on extrema*" with a given exercise price K has as payoff for a *call on maximum*:

$$\left(\sup_{[0,T]} S(s) - K \right)_+ . \tag{14.146}$$

They are only interesting if the underlying asset is highly increasing or decreasing on $[0,T]$ and with a high volatility.

14.10.9. *Asiatic (or average) options*

Such an option has as final payoff determined by the average price of the asset during a specified period, say $[a,b]$, included in $[0, T]$.

For a *fixed strike average option*, the payoff depends on the difference of the average and a fixed striking price; for a *floating strike average option*, the payoff at maturity depends on the difference of the spot price and the average.

Sometimes, the *geometric mean* is used instead of the arithmetic mean. The evaluation of such options is complicated and, in general, there is no explicit formula for the pricing except in the last case for which Vorst (1990) proved that it suffices to use the Garman-Kohlhagen formula with

$$\sigma' = \frac{\sigma}{\sqrt{3}}, r'_f = \frac{1}{2}\left(r_d + r_f + \frac{\sigma^2}{6} \right).$$

The "arithmetic mean" case was studied by Geman and Yor who gave the explicit form of the Laplace transform of the premium.

14.10.10. *Rainbow options*

They depend on at least two underlying assets in the same device and have generally lower prices; this is due to the correlation between the considered assets.

As example, let us present the *outperformance* or "*Margrabe option*" giving the right to the holder to receive the difference of returns between two assets if it is positive.

This means that the holder receives the outperformance of asset *A* on asset *B* at maturity time T, that is, $\left(S_2(T)-S_1(T)\right)_+$ where S_1 and S_2 are two foreign currency rates expressed in the same device.

The model to be considered is the following one:

$$\begin{aligned} dS_i(t) &= S_i(t)\left[\mu_i dt + \sigma_i dW_i(t)\right],\ i=1,2, \\ E\left[dW_1(t)dW_2(t)\right] &= \rho dt. \end{aligned} \tag{14.147}$$

so that

$$d(\ln\frac{S_2(t)}{S_1(t)}) = (\mu_2-\mu_1-\frac{1}{2}(\sigma_2^2-\sigma_1^2))dt + \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\,dW(t) \tag{14.148}$$

It is possible to prove the following result

$$\begin{aligned} &O_M(S_1(t),S_2(t),t) = S_2(t)e^{-r_2(T-t)}\Phi(d_1) - S_1(t)e^{-r_1(T-t)}\Phi(d_2), \\ &d_1 = \frac{1}{\sigma\sqrt{T-t}}(\ln(\frac{S_2}{S_1}) + (r_1 - r_2 + \frac{\sigma^2}{2})(T-t) \\ &d_2 = d_1 - \sigma\sqrt{T-t} \\ &\sigma = \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}. \end{aligned}$$

which is in fact the Garman-Kohlhagen formula with: $K = S_1, S = S_2$,r_d=r_1,r_f=r_2

$$\eta = (\mu_2-\mu_1-\frac{1}{2}(\sigma_2^2-\sigma_1^2)), \sigma = \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}.$$

14.11. The formula of Barone-Adesi and Whaley (1987): formula for American options

Using the PDE approach for pricing American options giving a continuous dividend at rate y and an approximation by solving an ordinary differential equation, Barone-Adesi and Whaley (1987) obtained the following good approximations for the American call and put:

1) For the call

$$C_{am}(S,T,K)=\begin{cases} S-K, & S\geq S^*, \\ C_{eur}(S,T,K)+A_2\left(\dfrac{S}{S^*}\right)^{\gamma_2}, & S<S^*, \end{cases} \tag{14.149}$$

where A_2 and $d_1(S^*)$ are given by:

$$A_2=\left(\frac{S^*}{\gamma_2}\right)[1-e^{-yT}\Phi(d_1(S^*)],$$
$$d_1(S^*)=\frac{\ln\dfrac{S^*}{K}+\left(r-y+\dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \tag{14.150}$$

S* being the solution of the following algebraic equation to be solved by iteration:

$$S^*-K=C_{eur}(S^*,T,K)+\left(\frac{S^*}{\gamma_2}\right)(1-e^{-yT}\Phi(d_1(S^*)),$$
$$\gamma_2=\frac{1}{2}[-(\beta-1)+\sqrt{(\beta-1)^2+4\frac{\alpha}{1-e^{-rT}}}(<0), \tag{14.151}$$
$$\alpha=\frac{2r}{\sigma^2},\beta=\frac{2(r-y)}{\sigma^2}.$$

2) For the put

$$P_{am}(S,T,K)=\begin{cases} K-S, & S^{**}\geq S, \\ P_{eur}(S,T,K)+A_1\left(\dfrac{S}{S^{**}}\right)^{\gamma_1}, & S^{**}<S, \end{cases} \tag{14.152}$$

where

$$A_1 = -\left(\frac{S^{**}}{\gamma_1}\right)[1 - e^{-yT}\Phi(-d_1(S^{**})],$$
$$d_1(S^{**}) = \frac{\ln\frac{S^{**}}{K} + \left(r - y + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \tag{14.153}$$

S** being the solution of the following algebraic equation to be solved by iteration:

$$K - S^{**} = P_{eur}(S^*, T, K) - \left(\frac{S^{**}}{\gamma_1}\right)(1 - e^{-yT}\Phi(d_1(S^{**})),$$
$$\gamma_1 = \frac{1}{2}\left[-(\beta - 1) - \sqrt{(\beta - 1)^2 + 4\frac{\alpha}{1 - e^{-rT}}}\right](> 0), \tag{14.154}$$
$$\alpha = \frac{2r}{\sigma^2}, \beta = \frac{2(r - y)}{\sigma^2}.$$

In these formulae, quantities S^* and S^{**} represent the thresholds to exercise respectively the call and put, i.e.:

$$S^* - K = C_{am}(S^*, T, K)(K - S^{**} = P_{am}(S^{**}, T, K). \tag{14.155}$$

These values are good for $T \to 0$ or $T \to \infty$ but not so good for mean maturity values.

Remark 14.8 *Interpolation method for American puts* (Johnson (1983), Broadie and Detemple (1996))

Johnson showed the following double inequality:

$$P_{eur}(S, T - t, K) \leq P_{am}(S, T - t, K) \leq P_{eur}(S, T - t, Ke^{-r(T-t)}). \tag{14.156}$$

Then, he gave the following result:

$$P_{am}(S, T - t, K) = \alpha P_{eur}(S, T - t, K) + (1 - \alpha)P_{eur}(S, T - t, Ke^{-r(T-t)}). \tag{14.157}$$

where the value of parameter α depends on the values of $S/K, r(T - t), \sigma^2(T - t)$.

Geske and Johnson model

Discretizing [0, *T*] with the subdivision $(t_1,...,t_n)$, it is possible to approach the put value with a type of Cox-Rubinstein model.

Parity relation

Without dividend repartition, the traditional *parity relation* is replaced by the following double inequality:

$$S - K \leq C_{am}(S,T,K) - P_{am}(S,T,K) \leq S - Ke^{-rT}. \tag{14.158}$$

Furthermore, without dividend repartition, we can use the traditional parity relation for European options to obtain:

$$0 \leq P_{am}(S,T,K) - P_{eur}(S,T,K) \leq K(1 - e^{-rT}). \tag{14.159}$$

Relation of symmetry

Chesney and Gibson (1995) proved the following important result:

$$C_{am}(S,T,K,\sigma,r,y) = P_{am}(K,T,S,\sigma,y,r) \tag{14.160}$$

so that, for the American options, every result on the call (respectively put) gives a result on the put (res. call) with the permutation of *S* and *K* and *r* and *y*.

Example 14.5 Let us suppose that we have to know the value of an American put with parameters:

$$S = 100, K = 95, T = 1, \sigma = 35\%, r = 2.75\%, y = 3\%,$$

we can solve the problem of an American call with parameters:

$$S = 95, K = 100, T = 1, \sigma = 35\%, r = 3\%, y = 2.75\%.$$

Example 14.6 Let us consider an asset with a value of €100 at $t = 0$ and suppose that the European call of maturity is three months and an exercise price of €102 has the value of €5.43. The European put with the same parameters has the value of €6.22.

Knowing that the asset gives no dividend on the considered period provides:

(i) the value of the American call with the same parameters;

(ii) a double inequality for the American put of same parameters;

(iii) the value of the risky instantaneous rate.

Answers

(i) Knowing that the asset gives no dividend on the considered period, we know that the American call has the same value as the European call: $C_{am} = €5.43$.

(ii) The American is always larger than the European put so that: $€6.22 \leq P_{am}$.

From the double inequality (14.156), we obtain:

$$C_{am}(S,T,K) - S + Ke^{-rT} \leq P_{am}(S,T,K) \leq C_{am}(S,T,K) - S + K, \quad (14.161)$$

and from result (i) and the traditional parity relation, we obtain:

$$\begin{aligned} &C_{eur}(S,T,K) - S + Ke^{-rT} \leq P_{am}(S,T,K) \leq C_{am}(S,T,K) - S + K, \\ &\text{and} \\ &P_{eur}(S,T,K) \leq P_{am}(S,T,K) \leq C_{eur}(S,T,K) - S + K. \end{aligned} \quad (14.162)$$

From the second inequality, we obtain here:

$$P_{am}(S,T,K) \leq C_{eur}(S,T,K) - S + K = 5.43 - 100 + 102 = €7.43.$$

The final reply is:

$$6.22 Euro \leq P_{am} \leq 7.43 Euro. \quad (14.163)$$

(i) From the traditional parity relation for European options, we have:

$$C_{eur}(S,T,K) - S + Ke^{-rT} = P_{eur}(S,T,K), \quad (14.164)$$

and so:

$$\begin{aligned} &e^{-rT} = \frac{P_{eur}(S,T,K) - C_{eur}(S,T,K) + S}{K} \\ &\text{and} \\ &r = -\frac{1}{T}\ln\frac{P_{eur}(S,T,K) - C_{eur}(S,T,K) + S}{K}. \end{aligned} \quad (14.165)$$

We finally obtain:

$$\begin{aligned} &r = -4\ln\frac{6.22 - 5,43 + 100}{102}, \\ &r = 0.04773. \end{aligned} \quad (14.166)$$