Chapter 13

Stochastic or Itô Calculus

This chapter presents the basic results concerning the *Itô calculus* also called *stochastic calculus*, one of the main tools used in stochastic finance particularly for building stochastic models used in option theory, developed in Chapter 14 and in bond evaluation, developed in Chapter 15.

13.1. Problem of stochastic integration

In traditional analysis, it is well known that the Riemann-Stieltjes integral noted

$$\int_a^b f d\alpha \tag{13.1}$$

is well defined if for example *f* is continuous and α of bounded variation on $[a,b]$, or inversely if α is continuous and f of bounded variation on $[a,b]$. From integration by parts, we obtain:

$$\int_a^b f d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha df. \tag{13.2}$$

Let us work now on a filtered probability space $\left(\Omega, \Im, \left(\Im_t, t \geq 0\right), P\right)$ on which we define two adapted stochastic processes:

$$f = \left(f(t), t \geq 0\right), X = \left(X(t), t \geq 0\right) \tag{13.3}$$

where the process f has its trajectories a.s. of bounded variation and the process X has its trajectories a.s. continuous on $[0,t]$.

For each trajectory ω, it is still possible to integrate "à la Riemann-Stieltjes" to obtain a new random variable Y

$$Y = \int_0^t f(s)dX(s) \tag{13.4}$$

or

$$Y(w) = \int_0^t f(s,\omega)dX(s,\omega)\ . \tag{13.5}$$

The process f is called the *integrand* process and the process X the *integrator* process.

So, if process f has its trajectories a.s. of bounded variation and process X has its trajectories a.s. continued on $[0,T]$, the stochastic process $Y = (Y(t), t \in [0,T])$ is also represented by $\int fdX$ or:

$$\int fdX = \left\{ \int_0^t f(s,\omega)dX(s,\omega), t \in [0,T] \right\} \tag{13.6}$$

Nevertheless, this approach of stochastic integration is completely unsatisfactory if, for example, we are considering a standard Brownian motion, as defined in Chapter 10, $W = (W(t), t \geq 0)$ as indeed, we cannot define the following integral

$$\int_0^t W(s,\omega)dW(s,\omega) \tag{13.7}$$

as these trajectories of a Brownian motion are p.s. not of bounded variation on any interval [*0, t*]. That is why it is necessary to construct a new theory of integration called *the stochastic* or *Itô integration*.

In particular, we will see that in this new theory, the “natural” result in traditional analysis:

$$\int_0^t W(s,\omega)dW(s,\omega) = \frac{1}{2}\left[W(t,\omega)\right]^2 ; \tag{13.8}$$

is here false!

More generally, the traditional formula of derivation and differentiation will no longer be systematically true.

13.2. Stochastic integration of simple predictable processes and semi-martingales

Let $\left(\Omega,\Im,\left(\Im_t, t \geq 0\right),P\right)$ be a filtered complete probability space, T a stopping time and $\Im_T$ the σ-algebra of all the events anterior to T and introduce the following definitions.

Definition 13.1 *A stochastic process*

$$H = \left(H_t, t \geq 0\right) \tag{13.9}$$

is predictable simple if $H = \left(H_t, t \geq 0\right)$ *if:*

(i) $H_t = H_{-1}1_{\{0\}}(t) + \sum_{i=0}^{n} H_i 1_{(T_i,T_{i+1}]}$,

(ii) $T_0 = 0, \left(T_i, i = 1,\dots,n\right)$ is an increasing sequence of a.s. finite stopping times, (13.10)

(iii) $\forall i = 1,\dots,n: \; \left|H_i\right| < \infty, p.s., \; H_i \in \Im_{t_i}$.

Definition 13.2 *On* $\left(\Omega,\Im,\left(\Im_t, t \geq 0\right),P\right)$*, the set of all predictable simple stochastic processes is called S and* S_u *if it is topologized with the uniform convergence in* (t,ω).

The basic idea is to define, with a given integrator process *X*, the stochastic integral process noted $\int_0^\infty HdX$ of a simple predictable process *H*

$$\int_0^\infty HdX = \int_0^\infty H_s(\omega)dX_s(\omega) \tag{13.11}$$

eventually with the completion of *H* for $t > T_{n+1}$ as

$$H_t(\omega) = 0, p.s. \ \forall t > T_{n+1} \tag{13.12}$$

as the *operator* $I_X : S \to L^0$, this last set being the set of all r.v. with the convergence in probability, defined by relation (13.10) such that this operator has the following properties:

(i) I_X is linear:

$$H_1, H_2 \in S : \int_0^\infty (H_1 + H_2)dX = \int_0^\infty H_1 dX + \int_0^\infty H_2 dX, \tag{13.13}$$

(ii) I_X is continuous:

$$(H_n) \overset{c.u.}{\to} H \Rightarrow \int_0^\infty H_n dX \overset{c.pr.}{\to} \int_0^\infty HdX . \tag{13.14}$$

We see that the continuity property is well related to the two modes of convergence introduced before: the uniform convergence on *S* and the convergence in probability on L^0.

To define now the operator I_X for simple predictable processes, we will follow the traditional definition as follows.

Definition 13.3 *The operator* $I_X : S \to L^0$*, is defined as follows:*

$$I_X(H) = H_{-1}1\{0\} + \sum_{i=0}^{n} H_i (X_{T_{i+1}} - X_{T_i}) . \tag{13.15}$$

The new problem now is to see what the “good” integrator processes are so that this definition has a meaning and satisfies properties (13.13) and (13.14).

As from Definition 13.3, it is clear that the linearity property is always fulfilled for simple predictable processes. To see for what classes of process it remains true, it suffices to obtain property (13.14), justifying the introduction of a large class of stochastic processes called *semi-martingales*.

Definition 13.4 *The stochastic process X is a total semi-martingale if:*

(i) X is càdlàg;

(ii) X is adapted;

(iii) *operator* $I_X : S \to L^0$ is continuous.

For the restriction of the integration on the interval [*0*,*t*], we give the next definition.

Definition 13.5 *The stochastic process X is a total semi-martingale if for all* $t \in [0, \infty)$ *, the stopped process at t,* X^t *defined by*

$$X_s^t = \begin{cases} X_s, s < t, \\ X_t, s \geq t. \end{cases} \tag{13.16}$$

is a total semi-martingale.

It is now possible to prove that this class of stochastic processes is good enough for stochastic integration with the following theorem proved by Protter (1990).

Proposition 13.1

(i) Every adapted càdlàg process of bounded variation on all compacts is a semi-martingale.

(ii) Every càdlàg square integrable martingale g is a semi-martingale.

(iii) Every standard Brownian motion is a semi-martingale.

Proof Let us prove (ii) and (iii).

(ii) From Definition 13.3 and relation (13.15), we obtain:

$$E\left[\left(I_X(H) \right)^2 \right] = E\left[\left(\sum_{i=0}^{n} H_i \left(X_{T_{i+1}} - X_{T_i} \right) \right)^2 \right]. \tag{13.17}$$

As the double products have a zero expectation, we obtain:

$$E\left[\left(I_X(H)\right)^2\right]=E\left[\left(\sum_{i=0}^{n}H_i^2\left(X_{T_{i+1}}-X_{T_i}\right)^2\right)\right] \tag{13.18}$$

and so

$$E\left[\left(I_X(H)\right)^2\right]\le \sup_i\left|H_i^2\right|E\left[\left(\sum_{i=0}^{n}\left(X_{T_{i+1}}-X_{T_i}\right)^2\right)\right]. \tag{13.19}$$

Using the smoothing property of conditional expectation and the stopping time theorem of Doob (see Chapter 10), we can successively write:

$$E\left[X_{T_i}X_{T_{i+1}}\right]=E\left[E\left[X_{T_i}X_{T_{i+1}}\right]\middle|\Im_{T_i}\right], \tag{13.20}$$

$$E\left[X_{T_i}X_{T_{i+1}}\right]=E\left[X_{T_i}E\left[X_{T_{i+1}}\right]\middle|\Im_{T_i}\right], \tag{13.21}$$

$$E\left[X_{T_i}X_{T_{i+1}}\right]=E\left[X_{T_i}{}^2\right], \tag{13.22}$$

and so from relation (13.15):

$$E\left[\left(I_X(H)\right)^2\right]\le \sup_i\left|H_i^2\right|\sum_{i=0}^{n}\left(E\left[X_{T_{i+1}}^2\right]+E\left[X_{T_i}^2\right]-2E\left[X_{T_i}^2\right]\right) \tag{13.23}$$

or:

$$E\left[\left(I_X(H)\right)^2\right]\le \sup_i\left|H_i^2\right|\sum_{i=0}^{n}\left(E\left[X_{T_{i+1}}^2\right]-E\left[X_{T_i}^2\right]\right). \tag{13.24}$$

This last result finally gives:

$$E\left[\left(I_X(H)\right)^2\right]\le \sup_i\left|H_i^2\right|\left(E\left[X_{T_{i+1}}^2\right]-E\left[X_0^2\right]\right) \tag{13.25}$$

which proves the continuity property of operator I_X .

(iii) This result is a direct consequence of the property that every standard Brownian motion is a square integrable and càdlàg martingale (see Chapter 10) with trajectories a.s. continuous.

13.3. General definition of the stochastic integral

Let us now go to the last step of stochastic integration, that is, to define this concept for more general processes than the predictable simple processes. To do so, we must introduce a class of stochastic processes we can obtain using an adequate convergence using a technique similar to the construction of real numbers from the rational numbers or the construction of the integral of measurable functions starting from the integral of simple functions.

The basic idea, fully developed in Protter (1990), is always the same one. Firstly, we define a larger class of integrable functions on which the initial class is *dense*. Secondly, we approach each element of the new class with a sequence of elements of the initial class using an *adequate* mode of convergence, i.e. the punctual convergence in number theory, the uniform convergence in traditional integration and here the *uniform convergence in probability on every compact set.*

Definition 13.6 *(The uniform convergence in probability on every compact set) A sequence of stochastic processes* $\left(H^n, n \geq 1\right)$ *where* $H^n = \left(H_t^n, t > 0\right)$ *converges uniformly in probability on the compacts towards the process* $H = \left(H_t, t \geq 0\right)$ *if, for all t>0, we have:*

$$\sup_{0 \leq s \leq t} \left| H_n^s - H_s \right| \xrightarrow{pr} 0 . \tag{13.26}$$

So, we now have four basic spaces of topologized stochastic processes:

D: the space of càdlàg simple adapted processes;

L: the space of adapted càdlàg processes;

S_u : the space of predictable simple processes with the uniform convergence;

L^0 : the space of finite random variables with the convergence in probability.

The spaces of stochastic processes *D*, *L* and *S* with the *uniform* convergence in probability on the compacts are noted respectively $D_{ucp}, L_{ucp}, S_{ucp}$.

We have now the following result.

Proposition 13.2 (Protter (1990)) *With the uniform convergence in probability on the compacts, space S of predictable simple processes is dense on L.*

This result leads to the extension of the definition of stochastic integral from *S* to *L*.

Firstly, let us recall that the application

$$I_X : S_u \mapsto L^0 \tag{13.27}$$

defined from relation (13.10) is written in the following form:

$$I_X(H) = \int_0^\infty H_s dX_s \tag{13.28}$$

and with the stopped process X^t:

$$I_{X^t}(H) = \int_0^t H_s dX_s \tag{13.29}$$

For a given stochastic process *H*, this last relation defines a new stochastic process J_X:

$$J_X(H)_t = I_{X^t}(H) \tag{13.30}$$

such that for each process $H = \left(H_t, t \geq 0\right)$, the corresponding associated process is $\left(\int_0^t H_s dX_s,\ t \geq 0\right)$ and so

$$J_X(H))_t = I_{X^t}(H)\ . \tag{13.31}$$

Protter (1990) proved the two following propositions.

Proposition 13.3 *If process X is a semi-martingale, then the application*

$$J_X : S_{ucp} \mapsto D_{ucp} \tag{13.32}$$

is continuous.

Proposition 13.4 *The continuous linear operator* $J_X : S_{ucp} \mapsto D_{ucp}$ *can be extended to a continuous linear operator* $J_X : L_{ucp} \mapsto D_{ucp}$.

This last proposition is a special case of the fundamental result that every linear operator on a sub-vector space can be extended in a unique way to the whole vector space.

Definition 13.7 *If X is a semi-martingale, the continuous linear application:*

$$J_X : L_{ucp} \mapsto D_{ucp} \tag{13.33}$$

is called a stochastic integral.

Of course, we will use the same notations as for simple processes:

$$I_X(H) = \int_0^\infty H_s dX_s \tag{13.34}$$

$$I_{X^t}(H) = \int_0^t H_s dX_s,$$

$$(J_X(H))_t = I_{X^t}(H) \tag{13.35}$$

Thus, the main conclusion is that it is possible to define the stochastic integral on [0,t] *for every adapted càdlàg process as integrand process and for every semi-martingale integrator process*.

Example 13.1 Let us consider a standard Brownian motion $B = (B_t \geq 0)$ on the filtered probability space $(\Omega, \Im, (\Im_t, t \geq 0), P)$.

From Proposition 13.1, process B is a semi-martingale and moreover continuous (see Chapter 10); it follows that the following stochastic integral $\int_0^t B_s dB_s$ exists.

To calculate its value, let us introduce the following sequence of nested partitions $(\Pi_n, n \geq 1)$ of [0,t] such that the sequence of these norms $(\nu_n, n \geq 1)$ tends to 0.

For every partition Π_n, we introduce the following simple function B^n defined as follows:

$$B_s^n = \sum_{k=0}^{n-1} B_{t_k} 1_{(t_k, t_{k+1}]}, \tag{13.36}$$

with

$$\Pi_n = (t_0, ..., t_k, ..., t_n), t_0 = 0, t_n = t. \tag{13.37}$$

From the definition of the stochastic integral of simple functions, we obtain:

$$\int_0^t B_s^n dB_s = \sum_{k=0}^{n-1} B_{t_k} \left(B_{t_{k+1}} - B_{t_k} \right). \tag{13.38}$$

Using the theorem of the approximation of every continuous function by a uniformly convergent sequence of step functions, we have on [0,t]:

$$B^n \overset{ucp}{\to} B \tag{13.39}$$

and so:

$$\int_0^t B_s dB_s = \lim_{\nu_n \to 0} \sum_{k=0}^{n-1} B_{t_k} \left(B_{t_{k+1}} - B_{t_k} \right). \tag{13.40}$$

As

$$\sum_{k=0}^{n-1} B_{t_k} \left(B_{t_{k+1}} - B_{t_k} \right) = \frac{1}{2} \sum_{k=0}^{n-1} \left\{ \left(B_{t_{k+1}} + B_{t_k} \right) \left(B_{t_{k+1}} - B_{t_k} \right) - \left(B_{t_{k+1}} - B_{t_k} \right) \right\}^2, \tag{13.41}$$

or even

$$\sum_{k=0}^{n-1} B_{k} \left(B_{k+1} - B_{k} \right) = \frac{1}{2} B_t^2 - \frac{1}{2} \sum_{k=0}^{n-1} \left(B_{k+1} - B_{k} \right)^2, \tag{13.42}$$

we obtain:

$$\int_0^t B_s dB_s = \frac{1}{2} B_t^2 - \frac{1}{2} \lim_{\nu_n \to 0} \sum_{k=0}^{n-1} (B_{t_{k+1}} - B_{t_k})^2 \tag{13.43}$$

The final result comes from the application of the next proposition showing that the second term of this last relation tends towards $t/2$ and so:

$$\int_0^t B_s dB_s = \frac{1}{2} B_t^2 - \frac{1}{2} t. \tag{13.44}$$

This last result illustrates well the fact that the traditional formula of differential analysis is, in general, no more true for the Itô calculus; here, in result (13.44), there is a supplementary term -t/2 with respect to the traditional formula, called the *drift*.

Let us now prove the previous result.

Proposition 13.5 *For any standard Brownian motion we have, with the convergence in probability:*

$$\lim_{\nu_n \to 0} \sum_{k=0}^{n-1} (B_{t_{k+1}} - B_{t_{k+1}})^2 = t. \tag{13.45}$$

Proof With

$$\Pi_n = (t_0, ..., t_k, ..., t_n), t_0 = 0, t_n = t,$$

let us define:

$$\sum_{k=0}^{n-1} (B_{t_{k+1}} - B_{t_k})^2 = S_{n.}. \tag{13.46}$$

From the identity

$$S_n - t = \sum_{k=0}^{n-1} \left[(B_{t_{k+1}} - B_{t_{k+1}})^2 - (t_{k+1} - t_k) \right], \tag{13.47}$$

we obtain:

$$E[S_n - t]^2 = E\left[\left(\sum_{k=0}^{n-1} \left[(B_{t_{k+1}} - B_{t_{k+1}})^2 - (t_{k+1} - t_k) \right] \right)^2 \right], \tag{13.48}$$

using the property that a standard Brownian motion has independent increments (see Chapter 10):

$$E[(S_n - t)^2] = E\left[\sum_{k=0}^{n-1} \left[(B_{t_{k+1}} - B_{t_{k+1}})^2 - (t_{k+1} - t_k) \right]^2 \right]. \tag{13.49}$$

Consequently, it follows that:

$$E[(S_n - t)^2] = E\left[\sum_{k=0}^{n-1} \left[\left(\frac{B_{t_{k+1}} - B_{t_{k+1}}}{\sqrt{t_{k+1} - t_k}} \right)^2 - 1 \right]^2 (t_{k+1} - t_k)^2 \right]. \tag{13.50}$$

Let us now introduce the r.v.

$$Y_k = \frac{B_{t_{k+1}} - B_{t_k}}{\sqrt{t_{k+1} - t_k}} \tag{13.51}$$

having a $N(0,1)$ distribution (see Chapter 1) to write relation (13.50) in the form:

$$\begin{aligned} E[(S_n - t)^2] &= E\left[\sum_{k=0}^{n-1}\left[Y_k^2 - 1\right]^2 (t_{k+1} - t_k)^2\right], \\ &= \sum_{k=0}^{n-1} E\left[Y_k^2 - 1\right]^2 (t_{k+1} - t_k)^2. \end{aligned} \tag{13.52}$$

As the r.v. Y_k have the same distribution, we also obtain:

$$E[(S_n - t)^2] = E\left[(Y_1^2 - 1)^2\right]\sum_{k=0}^{n-1}(t_{k+1} - t_k)^2. \tag{13.53}$$

From the following inequality:

$$\sum_{k=0}^{n-1}\left(t_{k+1} - t_k\right)^2 \le (b-a)\nu_n \tag{13.54}$$

we obtain:

$$E[(S_n - t)^2] \le E\left[(Y_1^2 - 1)^2\right](b-a)\nu_n, \tag{13.55}$$

and so the result for $\nu_n \to 0$. □

Remark 13.1 The last proposition also shows that effectively the trajectories of a standard Brownian motion are not, a.s., of bounded variation on any compact of the real set.

Indeed, from the a.s. continuity of the trajectories on $[0,t]$, there is on this interval a subdivision $\Pi_n = \left(t_0, ..., t_k, ..., t_n\right), t_0 = 0, t_n = t$ of sufficiently small norm such that:

$$\left|B_{t_{k+1}} - B_{t_{k+1}}\right| < 1, \forall k = 0, ..., n-1 \tag{13.56}$$

and so:

$$\sum_{k=0}^{n-1}(B_{t_{k+1}}-B_{t_k})^2 \le \sup_k \left|B_{t_{k+1}}-B_{t_k}\right| \sum_{k=0}^{n-1}\left|B_{t_{k+1}}-B_{t_k}\right|. \tag{13.57}$$

This last relation proves that if the trajectories of a standard Brownian motion were a.s. of bounded variation on [0,*t*], then the first member will tend to 0, which is in contradiction with Proposition 13.5.

13.4. Itô's formula

The fact that the rules of traditional differential calculus are no longer true for stochastic calculus implies finding a new tool of differentiation and integration. This tool was created by Itô (1944) who proved a lemma called *Itô's lemma* whose main result is called Itô's *formula*.

This formula became a very important basic tool for stochastic calculus and particularly in stochastic finance.

13.4.1. *Quadratic variation of a semi-martingale*

Let us recall that we use the following notations:

$$\begin{aligned} &\int_0^t H_s dX_s = \int_{[0,t]} H_s dX_s, \\ &\int_{0+}^t H_s dX_s = \int_{(0,t]} H_s dX_s \end{aligned} \tag{13.58}$$

and so:

$$\int_0^t H_s dX_s = H_0 X_0 + \int_{0+}^t H_s dX_s. \tag{13.59}$$

Definition 13.8 *If X and Y are two semi-martingales, then*:

(i) the quadratic variation of X or bracket of X noted:

$$[X,X] = \left([X,X]_t, t \ge 0\right) \tag{13.60}$$

is the stochastic process

$$[X,X]_t = X_t^2 - 2\int_0^t X_{t_-}\,dX_s, \quad (X_{0_-} = 0), \tag{13.61}$$

(ii) the quadratic covariation process of X and Y or bracket of X and Y is the stochastic process noted

$$[X,Y] = \left([X,Y]_t, t \geq 0\right) \tag{13.62}$$

where

$$[X,Y]_t = X_tY_t - \int_0^t X_{s-}\,dY_s - \int_0^t Y_{s-}\,dX_s. \tag{13.63}$$

Protter (1990) proved some interesting properties of these new processes and the most important ones for us are presented in the next proposition.

Proposition 13.6

(i) The process [X,X] is càdlàg, non-decreasing and adapted.

(ii) The process [X,Y] is càdlàg, t bilinear and symmetric and:

$$[X,Y]_t = \frac{1}{2}\left([X+Y, X+Y]_t - [X,X]_t - [Y,Y]_t\right). \tag{13.64}$$

(iii) For every sequence of partitions of stopping times:

$$T_0^n = 0, T_1^n, \ldots, T_k^n, \ldots, T_n^n = t, \tag{13.65}$$

if norm tends a.s. to 0, then:

$$X_0^2 + \sum_{k=0}^{n-1}\left(X^{T_{k+1}^n} - X^{T_k^n}\right)^2 \underset{ucp}{\rightarrow} [X,X]. \tag{13.66}$$

(iv) X and Y being two semi-martingale, so is the process [X, X].

(v) Integration by parts asserts that:

$$X_tY_t = \int_0^t X_{s-}\,dY_s + \int_0^t Y_{s-}\,dX_s + [X,Y]_t. \tag{13.67}$$

(vi) If it is a process of class *D*, then the jump process of *Y*, denoted $\Delta Y = (\Delta Y_t, t \geq 0)$, is defined as

$$\Delta Y_t = Y_t - Y_{t-}. \tag{13.68}$$

Then, for X=Y, we have:

$$\Delta[X,X]_t = (\Delta X_t)^2, \tag{13.69}$$

it follows the non-decreasing property of [X,X] and its decomposition in

$$[X,X]_t = [X,X]_t^c + \sum_{0\leq s\leq t} (\Delta X_s)^2,$$

or (13.70)

$$[X,X]_t = [X,X]_t^c + X_0^2 + \sum_{0<s\leq t} (\Delta X_s)^2,$$

the first term representing the "continuous" part of [X,X] .

Remark 13.2 From (ii) and Proposition (13.5), it follows that for every standard Brownian motion:

$$[B,B]_t = t. \tag{13.71}$$

13.4.2. *Itô's formula*

In traditional differential calculus, it is well-known that the *fundamental theorem* asserts that for any integrable function f on [0,*t*], we have:

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)dt. \tag{13.72}$$

From stochastic calculus, the problem is as follows: with a semi-martingale process *X* as integrator process, we seek the additional term, if it exists, such that we can extend the preceding result (13.72) to obtain the following extension:

$$f(X_t) - f(X_0) = \int_{0+}^{t} f'(X_{s-})dX_s + ... \tag{13.73}$$

For any function f of class $C_{\mathbb{R}}^2$, the solution is given by the two next propositions.

Proposition 13.7 (general Itô formula) *If X is a semi-martingale and f a function of class* $C^2_{\mathbb{R}}$*, then the composed process* $f(X) = \left(f(X_t), t \geq 0 \right)$ *is also a semi-martingale and moreover:*

$$\begin{aligned} f(X_t) - f(X_0) &= \int_{0+}^{t} f'(X_{s-}) dX_s + \frac{1}{2} \int_{0+}^{t} f'(X_{s-}) d\left[X, X\right]_s^c \\ &+ \sum_{0<s\leq t} \left\{ f(X_s) - f(X_{s-}) - f'(X_s) \Delta X_s \right\}. \end{aligned} \tag{13.74}$$

Proposition 13.8 (Itô formula: continuous case) *If X is a continuous semi-martingale and f a function of class* $C^2_{\mathbb{R}}$*, then the composed process* $f(X) = \left(f(X_t), t \geq 0 \right)$ *is also a semi-martingale and moreover:*

$$f(X_t) - f(X_0) = \int_{0+}^{t} f'(X_{s-}) dX_s + \frac{1}{2} \int_{0+}^{t} f'(X_{s-}) d\left[X, X\right]_s. \tag{13.75}$$

Proof Relation (13.75) is a direct consequence of result (13.74) as the continuity assumption on *X* implies that:

$$\forall s \geq 0 : X_s = X_{s-}, \Delta X_s = 0 \tag{13.76}$$

Remark 13.3 It is possible to show that (see Protter (1990)) the first supplementary term in the general Itô's formula is nothing other than:

$$\frac{1}{2} \left[f'(X), X \right]_t^c \tag{13.77}$$

and so we can put the Itô formula in the form:

$$\begin{aligned} f(X_t) - f(X_0) &= \int_{0+}^{t} f'(X_{s-}) dX_s + \frac{1}{2} \left[f'(X), X \right]_t^c \\ &+ \sum_{0<s\leq t} \left\{ f(X_s) - f(X_{s-}) - f'(X_s) \Delta X_s \right\}. \end{aligned} \tag{13.78}$$

13.5. Stochastic integral with standard Brownian motion as integrator process

Main applications in finance begin with stochastic integrals with a standard Brownian motion as integrator process; thus, we will now particularize the general preceding results to this special case to obtain results that are more precise.

13.5.1. *Case of predictable simple processes*

On the probability space $\left(\Omega, \Im, \left(\Im_t, t \geq 0\right), P\right)$, let us consider:

– a simple predictable process defined on [0, t]:

$$H_s = H_k, t_k < s \leq t_{k+1}, k=0,...,n-1, \tag{13.79}$$

$\left(t_0 = 0, t_1, ..., t_n = t\right)$ being a partition of [0,t];

– B, a standard Brownian motion.

From the construction of the stochastic integral, we know that:

$$\int_0^t H_s dB_s = \sum_{k=0}^{n-1} H_k (B_{t_{k+1}} - B_{t_k}) . \tag{13.80}$$

Consequently, the mean and variance of the stochastic integral are given by:

(i) *mean*:

$$E\left[\int_0^t H_s dB_s\right] = \sum_{k=0}^{n-1} E\left[H_k\left(B_{t_{k+1}} - B_{t_k})\right)\right], \tag{13.81}$$

and as the process H is adapted and B with independent increments, we obtain:

$$E\left[\int_0^t H_s dB_s\right] = \sum_{k=0}^{n-1} E\left[H_k\right] E\left[B_{t_{k+1}} - B_{t_k}\right] \tag{13.82}$$

and finally:

$$E\left[\int_0^t H_s dB_s\right] = 0. \tag{13.83}$$

(ii) variance

As from result (13.83):

$$\operatorname{var}\left(\int_0^t H_s dB_s\right) = E\left[\left(\sum_{k=0}^{n-1} H_k (B_{t_{k+1}} - B_{t_k})\right)^2\right], \tag{13.84}$$

we obtain:

$$\operatorname{var}\left(\int_0^t H_s dB_s\right) = E\left[\left(\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} H_k H_l (B_{t_{k+1}} - B_{t_k})(B_{t_{l+1}} - B_{t_l})\right)\right], \tag{13.85}$$

or:

$$\begin{aligned}\operatorname{var}\left(\int_0^t H_s dB_s\right) &= \sum_{k=0}^{n-1} E\left[H_k^2 (B_{t_{k+1}} - B_{t_k})^2\right] \\ &+2E\left[\left(\sum_{k<l} H_k H_l (B_{t_{k+1}} - B_{t_k})(B_{t_{l+1}} - B_{t_l})\right)\right],\end{aligned} \tag{13.86}$$

using the "smoothing property" from Chapter 10, we obtain:

$$E\left[H_k^2 (B_{t_{k+1}} - B_{t_k})^2\right] = E\left[H_k^2 (B_{t_{k+1}} - B_{t_k})^2 \middle| \Im_{t_k}\right], \tag{13.87}$$

and so from the fact that H is adapted to the given filtration and B with independent increments such that:

$$E\left[B_{t_{k+1}} - B_{t_k}\right] = t_{k+1} - t_k, \tag{13.88}$$

we obtain:

$$E\left[H_k^2 (B_{t_{k+1}} - B_{t_k})^2\right] = E\left[H_k^2\right](t_{k+1} - t_k),\ k = 0,\dots,n-1. \tag{13.89}$$

Using analog reasoning, we also have that all the double products in relation (13.86) have a zero expectation so that finally:

$$\operatorname{var}\left(\int_0^t H_s dB_s\right) = \sum_{k=0}^{n-1} E\left[H_k^2\right](t_{k+1} - t_k). \tag{13.90}$$

To summarize, we have the following basic results:

$$\begin{aligned}&E\left[\int_0^t H_s dB_s\right] = \int_0^t H_s dE\left[B_s\right] = 0, \\ &\operatorname{var}\left[\int_0^t H_s dB_s\right] = E\left(\int_0^t H_s dB_s\right)^2 = E\left(\int_0^t H_s^2 ds\right) = \int_0^t E\left[H_s^2\right] ds.\end{aligned} \tag{13.91}$$

Similarly, we can prove the following proposition.

Proposition 13.9 *Under the above assumptions and if moreover the process H is square integrable, then the following process*

$$\left(\int_0^t H_s dB_s, t \geq 0\right) \tag{13.92}$$

is a square integrable $(\mathfrak{I}_t)$*-martingale with a.s. continuous trajectories and moreover the process*

$$\left(\left(\int_0^t H_s dB_s\right)^2 - \int_0^t H_s^2 ds, t \geq 0\right) \tag{13.93}$$

is a $(\mathfrak{I}_t)$*-martingale with a.s. continuous trajectories.*

Let us also mention the following property: *if X and Y are two simple predictable square integrable processes, then*

$$E\left[\int_0^t X_s dB_s \int_0^t Y_s dB_s\right] = E\left[\int_0^t X_s Y_s ds\right] = \int_0^t E\left[X_s Y_s\right] ds. \tag{13.94}$$

13.5.2. *Extension to general integrand processes*

As we know from the preceding section, we will use uniform convergence in probability to extend the preceding results to the class D of square integrable adapted càdlàg processes.

For such a process X, there exists a sequence adapted simple square integrable processes $(H_n, n \geq 0)$ ucp converging to X such that in particular:

$$\int_0^t X_s dB_s = \lim_{L^2} \int_0^t H_s^n dB_s. \tag{13.95}$$

From this result, it follows that all the properties of section 13.5.1 remain valid in this general case.

13.6. Stochastic differentiation

13.6.1. *Definition*

On the probability space $\left(\Omega,\Im,\left(\Im_t,t\geq 0\right),P\right)$, let us consider an adapted standard Brownian motion B and two sufficiently smooth adapted processes a and b.

Definition 13.9 *The stochastic process*

$$\xi=\left(\xi(t),t\geq 0\right) \tag{13.96}$$

has as stochastic differential on $[0, T]$

$$d\xi(t)=a(t)dt+b(t)dB(t) \tag{13.97}$$

if and only if:

$$\begin{aligned}&\forall t_1,t_2: 0\leq t_1<t_2\leq T:\\ &\xi(t_2)-\xi(t_1)=\int_{t_1}^{t_2}a(t)dt+\int_{t_1}^{t_2}b(t)dB(t).\end{aligned} \tag{13.98}$$

13.6.2. *Examples*

1) Result (13.44) gives:

$$\int_0^t B_s dB_s=\frac{1}{2}B_t^2-\frac{1}{2}t. \tag{13.99}$$

Consequently, we also have:

$$\int_{t_1}^{t_2} B_s dB_s=\frac{1}{2}(B_{t_2}^2-B_{t_1}^2)-\frac{1}{2}(t_2-t_1) \tag{13.100}$$

and from our new definition, it follows that:

$$dB_t^2=dt+2B_t dB_t. \tag{13.101}$$

2) From the definition of the stochastic integral, we know that:

$$\int_{t_1}^{t_2} tdB_t=\lim_n\sum_{k=1}^{n-1}t_{n,k}\left[B_{t_{n,k+1}}-B_{t_{n,k}}\right], \tag{13.102}$$

$\left(t_{n,1}=t_1,...,t_{n,k},...,t_{n,n}=t_2\right)$ being a subdivision of order n of the interval $\left[t_1,t_2\right]$.

Moreover, from the definition of the traditional Lebesgue integral, we obtain:

$$\int_{t_1}^{t_2} B_t dt = \lim_n \sum_{k=0}^{n-1} B_{t_{n,k+1}}(t_{n,k+1}-t_{n,k}). \tag{13.103}$$

Adding member-to-member relations (13.102) and (13.103), we obtain:

$$\int_{t_1}^{t_2} B_t dt + \int_{t_1}^{t_2} tdB_t + = \lim_n \sum_{k=1}^{n-1}\left[t_{n,k+1}B_{t_{n,k+1}} - t_{n,k}B_{t_{n,k}}\right] \tag{13.104}$$

and so:

$$\int_{t_1}^{t_2} B_t dt + \int_{t_1}^{t_2} tdB_t = t_2 B_{t_2} - t_1 B_{t_1} \tag{13.105}$$

or in terms of stochastic differential:

$$d(tB_t) = B_t dt + tdB_t, \tag{13.106}$$

this formula also being different from the one of the traditional calculus.

13.7. Back to Itô's formula

Using now the concept of stochastic differential, we will have a supplementary look to Itô's formula.

13.7.1. *Stochastic differential of a product*

On the probability space $\left(\Omega,\Im,\left(\Im_t,t\geq 0\right),P\right)$, let us consider an adapted standard Brownian motion B and four càdlàg adapted processes a_1,a_2,b_1,b_2 of class D and sufficiently smooth defining the two following stochastic differentials:

$$d\xi_i(t) = a_i(t)dt + b_i(t)dB(t), i=1,2. \tag{13.107}$$

Then, we have as next result.

Proposition 13.10 (A. Friedman (1975)) *The process* $\xi_1\xi_2$ *is differentiable (in Itô's sense) and*

$$d\left(\xi_1(t)\xi_2(t)\right) = \xi_1(t)d\xi_2(t) + \xi_2(t)d\xi_1(t) + b_1(t)b_2(t)dt. \tag{13.108}$$

Examples

1) With $\xi_1(t) = \xi_2(t) = B(t)$, we find back this known result (see relation (13.101)):

$$d\left(B^2(t)\right) = 2B(t)dB(t) + dt\ . \tag{13.109}$$

2) Similarly, we can find result (13.106) concerning

$$d(tB(t)) = tdB(t) + B(t)dt, \tag{13.110}$$

with

$$\begin{aligned} &\xi_1(t) = t \Rightarrow a_1(t) = 1, b_1(t) = 0, \\ &\xi_2(t) = B(t) \Rightarrow a_1(t) = 0, b_1(t) = 1. \end{aligned} \tag{13.111}$$

13.7.2. *Itô's formula with time dependence*

For our applications, the main result is *Itô's lemma* or the *Itô formula*, which is equivalent to the rule of derivatives for composed functions in traditional differential calculus, but now with a function f of two variables.

Starting with

$$d\xi(t) = a(t)dt + b(t)dB(t), \tag{13.112}$$

let f be a function of two non-negative real variables x, t such that

$$f \in C^0_{\mathbb{R}\times\mathbb{R}^+},\ f_x,\ f_{xx},\ f_t \in C^0_{\mathbb{R}\times\mathbb{R}^+}. \tag{13.113}$$

Then *the composed stochastic process*

$$\left(f(\xi(t),t), t \geq 0\right) \tag{13.114}$$

is also Itô differentiable and its stochastic differential is given by:

$$d\big(f(\xi(t),t)\big) = \left[\frac{\partial f}{\partial x}(\xi(t),t)a(t) + \frac{\partial f}{\partial t}(\xi(t),t) + \frac{1}{2}\frac{\partial^2}{\partial^2 x^2} f(\xi(t),t)b^2(t)\right]dt + \frac{\partial f}{\partial x}(\xi(t),t)b(t)dB(t). \quad (13.115)$$

Remark 13.4 Compared with traditional differential calculus, we know that in this case, we should have:

$$d\big(f(\xi(t),t)\big) = \left[\frac{\partial f}{\partial x}(\xi(t),t)a(t) + \frac{\partial f}{\partial t}(\xi(t),t)\right]dt + \frac{\partial f}{\partial x}(\xi(t),t)b(t)dB(t). \quad (13.116)$$

Therefore, the difference between relations (13.115) and (13.116) is the *supplementary term*

$$\frac{1}{2}\frac{\partial^2}{\partial^2 x^2} f(\xi(t),t)b^2(t) \quad (13.117)$$

appearing in (13.115) and which is zero if and only if in two cases:

1) f is a linear function of x,

2) b is identically equal to 0.

Example 13.1

1) For ξ given by:

$$d\xi(t) = dB(t), \quad \xi(0) = 0. \quad (13.118)$$

Using notation (13.112), we obtain:

$$a(t) = 0,\ b(t) = 1. \quad (13.119)$$

With the aid of Itô's formula, the value of $de^{B(t)}$ is thus given by

$$de^{B(t)} = \frac{1}{2}e^{B(t)}dt + e^{B(t)}dB(t). \tag{13.120}$$

As we can see, the first term is the supplementary term with respect to the traditional formula and is called the *drift*.

13.7.3. *Interpretation of Itô's formula*

Itô's formula simply means that the composed stochastic process

$$\left((f(\xi(t),t) - f(\xi(0),0), t \geq 0 \right) \tag{13.121}$$

is stochastically equivalent to the following stochastic process:

$$\begin{aligned} &\left(\int_0^t \left[f_t(\xi(s),s)ds + f_x(\xi(s),s)a(s) + \frac{1}{2} f_{xx}(\xi(s),s)b^2(s) \right] ds \right) \\ &+ \int_0^t f_x(\xi(s),s)b(s)dB(s), t \geq 0). \end{aligned} \tag{13.122}$$

13.7.4. *Other extensions of Itô's formula*

13.7.4.1. *First extension*

It is possible to extend Itô's formula in the following way. Let $\boldsymbol{\xi} = \left(\boldsymbol{\xi}(t), t \geq 0 \right)$ be an *m*-dimensional stochastic process:

$$\boldsymbol{\xi}(t) = \left(\xi_1(t), \ldots, \xi_n(t) \right)' \tag{13.123}$$

with every component having a stochastic differential given by:

$$d\xi_i(t) = a_i(t)dt + b_i(t)dB(t), i = 1, \ldots, m\ . \tag{13.124}$$

Then, it can be shown that the stochastic differential of the one-dimensional process:

$$\left(f\left(\boldsymbol{\xi}(t), t \right), t \geq 0 \right), \tag{13.125}$$

with f being a real function of m+1 variables:

$$f(\mathbf{x},t) = f(x_1,\ldots,x_n,t) \tag{13.126}$$

satisfying the following assumptions:

$$f \in C^0_{\mathbb{R}^m\times\mathbb{R}^+},\ f_{x_i}, i=1,...,m,\ f_{x_i x_j}, i,j=1,...,m,\ f_t \in C^0_{\mathbb{R}^m\times\mathbb{R}^+} \tag{13.127}$$

exists and is given by

$$\begin{aligned} &d\left(f\left(\boldsymbol{\xi}(t),t\right)\right) = \\ &\left[\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\left(\boldsymbol{\xi}(t),t\right)a_i(t) + \frac{\partial f}{\partial t}\left(\boldsymbol{\xi}(t),t\right) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j} f\left(\boldsymbol{\xi}(t),t\right)b_i(t)b_j(t)\right]dt \\ &+\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\left(\boldsymbol{\xi}(t),t\right)b_i(t)dB(t) \end{aligned} \tag{13.128}$$

Here, the supplementary time is given by

$$\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j} f\left(\boldsymbol{\xi}(t),t\right)b_i(t)b_j(t) \tag{13.129}$$

13.7.4.2. *Second extension*

The second possible extension also starts with an m-dimensional stochastic process $\boldsymbol{\xi}(t) = \left(\xi_1(t),\ldots,\xi_n(t)\right)'$ such that its dynamics are governed by the following stochastic differential:

$$d\boldsymbol{\xi}(t) = \mathbf{a}(t)dt + \mathbf{b}(t)d\mathbf{B}(t), i = 1,\ldots,m \tag{13.130}$$

a being a m-dimensional random vector of class L or D and **b** a stochastic matrix mxn whose elements are stochastic processes of class L and **B** a n-vector of n independent standard Brownian motions.

As in the preceding section, we are interested in the stochastic differential of the one-dimensional process:

$$\left(f\left(\boldsymbol{\xi}(t),t\right), t \geq 0\right), \tag{13.131}$$

with f being a real function of m+1 variables:

$$f(\mathbf{x},t) = f(x_1,\ldots,x_n,t) \tag{13.132}$$

satisfying the following assumptions:

$$f \in C^0_{\mathbb{R}^m \times \mathbb{R}^+},\ f_{x_i}, i=1,\ldots,m,\ f_{x_i x_j}, i,j=1,\ldots,m,\ f_t \in C^0_{\mathbb{R}^m \times \mathbb{R}^+}. \tag{13.133}$$

Under these assumptions, it is still possible to show the composed stochastic process $\left(f\left(\xi(t),t\right), t \geq 0\right)$ is Itô differentiable and that its stochastic differential is given by:

$$\begin{aligned} &d\left(f\left(\xi(t),t\right)\right) = \\ &\left[\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(\xi(t),t\right) a_i(t) + \frac{\partial f}{\partial t}\left(\xi(t),t\right) + \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij}(t) \frac{\partial^2}{\partial x_i \partial x_j} f\left(\xi(t),t\right)\right] dt \\ &+ \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}\left(\xi(t),t\right) b_{ij}(t) dB_j(t) \\ &\sigma_{ij}(t) = \frac{1}{2}\left(bb'(t)\right)_{ij} \end{aligned} \tag{13.134}$$

Using matrix notation, we can rewrite this last expression in the form:

$$\begin{aligned} &d\left(f\left(\xi(t),t\right)\right) = \frac{\partial f}{\partial t}\left(\xi(t),t\right) dt + \mathrm{grad} f(t) d\xi(t) + \frac{1}{2}\mathrm{tr}(\mathbf{bb}')(t)\mathbf{f}_{\mathbf{xx}}(t) dt, \\ &\mathbf{f}_{\mathbf{xx}}(t) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(t)\right]. \end{aligned} \tag{13.135}$$

Here, the supplementary time is given by

$$\frac{1}{2}\mathrm{tr}(\mathbf{bb}')(t)\mathbf{f}_{\mathbf{xx}}(t) dt \tag{13.136}$$

13.7.4.3. *Third extension*

The last extension we will present now is related to the case of vector **B** whose components are *n dependent* standard Brownian motions.

This means that:

$$\forall i, j, \forall s, t(s < t) : E\left[(B_i(t) - B_i(s))(B_i(t) - B_i(s))\right] = \rho_{ij}(t - s). \tag{13.137}$$

The matrix $\mathbf{Q} = \left[\rho_{ij}\right]$ is called the *correlation matrix* of the vector Brownian motion $\mathbf{B} = \left(\mathbf{B}(t), t \geq 0\right)$.

If $\mathbf{Q} = \mathbf{I}$ and $\mathbf{B}(0) = 0$. the vector Brownian motion $\mathbf{B} = \left(\mathbf{B}(t), t \geq 0\right)$ is called *standard*.

In the case of a *n*-dimensional Brownian motion and with the same assumptions of the function *f* as above, Itô's formula becomes:

$$\begin{aligned} & d\left(f\left(\xi(t),t\right)\right) = \frac{\partial f}{\partial t}\left(\xi(t),t\right)dt + \operatorname{grad} f(t)d\xi(t) + \frac{1}{2}\operatorname{tr}(\mathbf{bQb}')(t)\mathbf{f}_{\mathbf{xx}}(t)dt, \\ & \mathbf{f}_{\mathbf{xx}}(t) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(t)\right]. \end{aligned} \tag{13.138}$$

13.7.4.4. *Exercises*

1) Prove the following results:

$$\begin{aligned} & dB^n(t) = nB^{n-1}(t)dB(t) + \frac{1}{2}n(n-1)B^{n-2}(t)dt, \\ & da^{B(t)} = a^{B(t)} \ln a dB + \frac{1}{2}a^{B(t)} \ln^2 a dt, a > 0. \end{aligned} \tag{13.139}$$

2) (i) Prove that:

$$\int_0^t s dB(s) = tB(t) - \int_0^t B(s)ds. \tag{13.140}$$

(ii) Generalize to the following case (partial validity of the traditional integration by parts formula)

$$\int_0^t f(s)dB(s) = f(t)B(t) - \int_0^t B(s)df(s), \tag{13.141}$$

f being a deterministic function with bounded variation.

3) Let **B** an n-dimensional standard Brownian motion and consider the following one-dimensional process

$$R = \left(R(t), t \geq 0\right),$$
$$R(t) = \sqrt{\sum_{k:=1}^{n} B_k^2(t)}. \tag{13.142}$$

called the *Bessel process of order n*.

Prove that:

$$dR = \frac{1}{R}\sum_{i=1}^{n} B_i(t)dB(t) + \frac{n-1}{2R}dt. \tag{13.143}$$

4) Calculate $E\left[e^{B(t)}\right]$.

Solution

The integral form of the Itô's formula leads to

$$e^{B(t)} - 1 = \int_0^t e^{B(t)}dB(s) + \frac{1}{2}\int_0^t e^{B(t)}ds. \tag{13.144}$$

Then, if:

$$X(t) = E\left[e^{B(t)}\right], \tag{13.145}$$

we get:

$$X(t) - 1 = \frac{1}{2}\int_0^t X(s)ds. \tag{13.146}$$

By derivation, we obtain:

$$X'(t) = \frac{1}{2}X(t). \tag{13.147}$$

Moreover, as $X(0)=1$, the traditional differential equation has as unique solution:

$$X(t) = e^{\frac{t}{2}}. \tag{13.148}$$

5) a and b being two deterministic functions of bounded variation, calculate the mean and the variance of the process X defined by

$$dX(t) = a(t)dt + b(t)dB(t), \tag{13.149}$$

B being a standard Brownian motion.

6) If the stochastic process $\lambda = (\lambda(t), t \geq 0)$ has the following stochastic differential:

$$d\lambda(t) = a(t)dt + b(t)dB(t), \tag{13.150}$$

calculate Itô's differential of $e^{\lambda(t)}$

Answer

$$de^{\lambda(t)} = e^{\lambda(t)}\left[\left(a(t) + \frac{1}{2}b^2(t)\right)dt + b(t)dB(t)\right]. \tag{13.151}$$

13.8. Stochastic differential equations

13.8.1. *Existence and unicity general theorem (Gikhman and Skorokhod (1969))*

The problem is, in the deterministic case, as follows: given the following stochastic differential:

$$\begin{aligned} &d\xi(t) = \mu(\xi(t),t)dt + \sigma(\xi(t),t)dB(t), \\ &\xi(0) = \xi_0, a.s. \end{aligned} \tag{13.152}$$

$B = (B(t), t \geq 0)$ being a standard Brownian motion on the complete filtered probability space $(\Omega, \Im, (\Im_t, t \geq 0), P)$, find, if possible, a stochastic process

$$\xi = (\xi(t), t \in [0,T]) \tag{13.153}$$

satisfying in the interval [0, T] relations (13.152), under minimal assumptions on the two functions μ, σ from $\mathbb{R} \times [0,T] \mapsto \mathbb{R}$.

Relation (13.152) is called a stochastic differential equation (SDE). Gikhman and Skorokhod (1969) proved a general *theorem of existence and unicity* also given, in a more modern form, by Protter (1990).

Under a relatively simple form, the main result is as follows.

Proposition 13.11 (general theorem of existence and unicity) *Let us consider the following SDE:*

$$\begin{aligned} &d\xi(t) = \mu(\xi(t),t)dt + \sigma(\xi(t),t)dB(t), \\ &\xi(0) = \xi_0, a.s. \end{aligned} \tag{13.154}$$

under the following assumptions:

(i) the functions μ,σ are measurable functions from $\mathbb{R}\times[0,T]\mapsto\mathbb{R}$ verifying a Lipschitz condition in the first variable:

$$\begin{aligned} &\forall(x_1,t),(x_2,t)\in\mathbb{R}\times[0,T]: \\ &|\mu(x_1,t)-\mu(x_2,t)|\le\bar{K}|x_1-x_2|, \\ &|\sigma(x_1,t)-\sigma(x_2,t)|\le\bar{K}|x_1-x_2|, \end{aligned} \tag{13.155}$$

$\bar{K}$ *being a positve constant;*

(ii) *on* $\mathbb{R}\times[0,T]$, *the functions* μ,σ *are linearly bounded:*

$$|\mu(x,t)|\le K(1+|x|),|\sigma(x,t)|\le K(1+|x|), \tag{13.156}$$

K being a positve constant;

(iii) *the r.v.* ξ_0 belongs to $L^2(\Omega,\Im,P)$ and is independent *of the* σ*-algebra* $\sigma(B(t),t\in[0,T])$, *then, there exists a solution belonging for all* $t\in[0,T]$, *to* $L^2(\Omega,\Im,P)$, *continuous and a.s. unique on* $[0,T]$.

Remark 13.5

1) The initial condition:

$$\xi(0) = x_0, \in R \tag{13.157}$$

naturally satisfies assumption (iii).

2) This theorem can be extended in the case of a SDE on $[s,s+T]$, with as initial condition:

$$\xi(s) = \xi_s, \tag{13.158}$$

the r.v. now independent of the σ-algebra $\sigma\left(B(s+\tau)-B(s),\tau\in[0,T]\right)$ and belonging to $L^2\left(\Omega,\Im,P\right)$.

3) It is also possible to prove that:

$$E\left[\sup_{[0,T]}\left|\xi(t)\right|^2\right]\leq C\left(1+E\left[\xi_0^2\right]\right), \tag{13.159}$$

C being a constant depending only on K and T.

In Proposition 8.1, the coefficients μ,σ are deterministic functions but it is possible to extend it in the stochastic case. Then, formally, we have:

$$\mu(x,t)=\mu(x,t,\omega),\sigma(x,t)=\sigma(x,t,\omega),\forall x\in\mathbb{R},\forall t\in[0,T]. \tag{13.160}$$

The initial condition (13.157) becomes:

$$\xi(0)=\varphi(0), \tag{13.161}$$

where

$$\varphi=\left(\varphi(t),t\in[0,T]\right) \tag{13.162}$$

is the given initial process.

The extension of Proposition 8.1 is now given.

Proposition 13.12 (case of random coefficients) *For the SDE:*

$$\begin{aligned}&d\xi(t)=d\varphi(t)+\mu(\xi(t),t)dt+\sigma(\xi(t),t)dB(t),\\&\xi(0)=\varphi(0),\end{aligned} \tag{13.163}$$

where:

(i) the processes μ,σ are measurable as functions from $\mathbb{R}\times[0,T]\times\Omega\mapsto\mathbb{R}$, adapted and lipschitzian in the first variable, i.e. with probability 1:

$$\begin{aligned}&\forall(x_1,t),(x_2,t)\in R\times[0,T]:\\&\left|\mu(x_1,t)-\mu(x_2,t)\right|\leq\bar{K}\left|x_1-x_2\right|,\\&\left|\sigma(x_1,t)-\sigma(x_2,t)\right|\leq\bar{K}\left|x_1-x_2\right|,\end{aligned} \tag{13.164}$$

$\overline{K}$ *being a positive constant;*

(ii) the processes μ, σ are measurable as functions from $\mathbb{R} \times [0,T] \times \Omega \mapsto \mathbb{R}$, satisfy a.s. the following condition:

$$|\mu(x,t)|^2 + |\sigma(x,t)|^2 \le K^2(1+x^2), \tag{13.165}$$

K being a positive constant;

(iii) the process $\varphi = (\varphi(t), t \in [0,T])$ is of bounded variation, adapted and such that

$$E\left[\sup_{[0,T]} |\varphi(t)|^2\right] < \infty \tag{13.166}$$

then, there is a solution belonging for $t \in [0,T]$, *to* $L^2(\Omega, \Im, P)$; *moreover, if* ξ_1, ξ_2 *are two solutions, then they are stochastically equivalent, i.e.:*

$$P[\xi_1(t) = \xi_2(t)] = 1, \forall t \in [0,T]. \tag{13.167}$$

Finally, if the process φ is continuous a.s. on $[0,T]$, then there exists a.s. unicity on $[0,T]$:

$$P\left[\sup_{[0,T]} \{t : |\xi_1(t) - \xi_2(t)|\} > 0\right] = 0. \tag{13.168}$$

Remark 13.6 This theorem can be extended in the case of a SDE on $[s, s+T]$.

The proofs of these two fundamental propositions use the method of *successive approximations* used in the deterministic case under the name of *Piccard method*: on $[0,T]$, we begin to use the following very rough approximation:

$$\xi_0(t) = \xi_0 \tag{13.169}$$

and, by induction, on constructs on $[0,T]$, the following sequence of stochastic processes $\xi_n = (\xi_n(t), n > 0)$ is defined by

$$\xi_{n+1}(t) = \xi_0 + \int_0^t \mu(\xi_n(s), s)ds + \int_0^t \sigma(\xi_n(s), s)dB(s). \tag{13.170}$$

Then, it is possible to show (see, for example, Friedman (1975)) that the sequence $\xi_n = \left(\xi_n(t), n > 0\right)$ converges uniformly a.s. on $\left[0,T\right]$ towards the stochastic process $\xi = \left(\xi(t), 0 \leq t \leq T\right)$, which is a solution of the considered SDE (13.163). Using assumption (13.164), Friedman (1975) also proved the a.s. unicity.

13.8.2. *Solution of stochastic differential equations*

Let us consider the following general SDE

$$\begin{aligned} &d\xi(t) = d\varphi(t) + \mu(\xi(t),t)dt + \sigma(\xi(t),t)dB(t), \\ &\xi(0) = \varphi(0), \end{aligned} \tag{13.171}$$

where $B = \left(B(t), t \geq 0\right)$ is a standard Brownian motion *on* $\left(\Omega, \Im, \left(\Im_t, t \geq 0\right), P\right)$.

The general procedure to find the process $\xi = \left(\xi(t), t \in \left[0,T\right]\right)$ solution of this SDE under the assumptions of Proposition 13.12 is to try to put this SDE in its *canonical form*, that is to say

$$\begin{aligned} &d\xi(t) = a(t)dt + b(t)dB(t), \\ &\xi(0) = \xi_0, \end{aligned} \tag{13.172}$$

with known *a* and *b* functions or stochastic processes. If so, the unique solution of the considered SDE takes the form:

$$\xi(t) = \xi_0 + \int_0^t a(s)ds + \int_0^t b(s)dB(s). \tag{13.173}$$

More generally, we can look for a transformation *f* in two variables *x* and *t*, monotone in *t* satisfying the assumptions of Itô’s lemma and such that:

$$df\left(\xi(t),t\right) = \overline{A}(t)dt + \overline{B}(t)dB(t) \tag{13.174}$$

In this case, we obtain:

$$f\left(\xi(t),t\right) = f\left(\xi(0),0\right) + \int_0^t \overline{A}(s)ds + \int_0^t \overline{B}(s)dB(s) \tag{13.175}$$

where we find by inverse transformation in variable *x* the form of $\xi(t), t \in \left[0,T\right]$.

13.9. Diffusion processes

Let us consider the SDE:

$$d\xi(t) = \mu(\xi(t),t)dt + \sigma(\xi(t),t)dB(t),\\ \xi(0) = \xi_0, \tag{13.176}$$

under the assumptions of Proposition 13.12.

The solution $\xi = \left(\xi(t), t \in [0,T]\right)$ of this SDE is called a *diffusion process or Itô process*.

Let *s* and *t* be such that: $0 \le s < t \le T$ and suppose that $\xi(s) = x$.

From the theorem of existence and unicity, we know that on the interval $[s,T]$ there exists only one process solution, noted $\xi_{x,s}$, of the SDE (13.176) such that

$$\xi_{x,s}(s) = x . \tag{13.177}$$

So it is clear that, setting $x = \xi(t)$, we have the Markov property for the ξ-process in continuous time, which is of course generally non-homogenous.

More precisely, we have the following propositions.

Proposition 13.13 *Under the assumptions of Proposition 13.12 and if, for each t,* $\Im_t$ *represents the* σ*-algebra generated by* ξ_0 *and the set* $\left(B(s), s \le t\right)$*, then the a.s. unique stochastic process solution of (13.176), satisfies a.s.:*

$$P\left[\xi(t) \in A \middle| \Im_s\right] = P\left[\xi(t) \in A \middle| \xi(s)\right] (= p(s, \xi(s), t, A)) \tag{13.178}$$

for all t>s and for all Borel set A.

Proposition 13.14 *The function of* $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \beta \mapsto [0,1]$ *defined by relation (13.178) satisfies the following properties:*

(i) for all fixed *s, x, t, p*(*s, x, t*) is a probability measure on $\mathbb{R}$;

(ii) for all fixed *s, t, A, p*(*s, t, A*) is Borel-measurable;

(iii) the function *p* satisfies the *Chapman-Kolmogorov* equations:

$$\forall 0 \le s < t < \tau, x \in R, A \in \beta :\\ \int_R p(s,x,t,dy)\, p(t,y,\tau,A) = p(s,x,\tau,A). \tag{13.179}$$

(iv) the process solution $\xi = (\xi(s), s \geq 0)$ is a *Feller process*; i.e. for all continuous bounded function of $\mathbb{R} \mapsto \mathbb{R}$, the application

$$(s,x) \mapsto \int f(y)\, p(s,x,s+t,dy) \tag{13.180}$$

is continuous.

(v) the process solution $\xi = (\xi(s), s \geq 0)$ satisfies the *strong Markov property*, i.e. condition (13.178) but where *s* and *t* are replaced by stopping times.

Remark 13.7

a) If the drift and the diffusion coefficient are continuous functions, it can be shown that:

(i)

$$\begin{aligned} &\forall \varepsilon > 0, t \geq 0, x \in R: \\ &\lim_{h\downarrow 0} \frac{1}{h} \int_{|y-x|>\varepsilon} p(t,x,t+h,dy) = 0, \end{aligned} \tag{13.181}$$

(ii)

$$\begin{aligned} &\forall \varepsilon > 0, t \geq 0, x \in R: \\ &a) \lim_{h\downarrow 0} \frac{1}{h} \int_{|y-x|<\varepsilon} (y-x)\, p(t,x,t+h,dy) = \mu(x,t), \\ &b) \lim_{h\downarrow 0} \frac{1}{h} \int_{|y-x|<\varepsilon} (y-x)^2\, p(t,x,t+h,dy) = \sigma^2(x,t), \end{aligned} \tag{13.182}$$

For the applications of such processes in finance, it is interesting to give the interpretations of these last properties:

1) the probability for the process $\xi = (\xi(s), s \geq 0)$ to have a jump of amplitude more then ε between *t* and *t*+*h* is $o(h)$. Consequently, the process $\xi = (\xi(s), s \geq 0)$ is continuous in probability;

2) properties *a* and *b* can be rewritten as follows:

$$\begin{aligned} &a) E\left[\xi(t+h) - \xi(t) \middle| \xi(t) = x\right] = \mu(x,t)h + o(h), \\ &b) E\left[\left|\xi(t+h) - \xi(t)\right|^2 \middle| \xi(t) = x\right] = \sigma^2(x,t)h + o(h). \end{aligned} \tag{13.183}$$

Consequently, drift μ gives the rate of the conditional mean of the increment of the diffusion process on the infinitesimal time $(t,t+h)$ interval and the square of the diffusion coefficient of diffusion σ, the conditional variance of this increment as the square of the mean is of order $o(h)$.

b) If the function p has a density p', then it is a solution of *the partial differential equation of Fokker-Planck:*

$$\frac{\partial p'}{\partial t}+\frac{\partial}{\partial x}(\sigma(x,t)p')-\frac{1}{2}\frac{\partial^2}{\partial x^2}(\mu(x,t)p')=0. \tag{13.184}$$

Example 13.2 For the Ornstein-Uhlenbeck-Vasicek process defined by the SDE (see later in section 15.3.1)

$$\begin{aligned} &d\xi(t)=a(b-\xi(t))dt+\sigma dB(t),\\ &\xi(0)=\xi_0. \end{aligned} \tag{13.185}$$

it can be shown that:

$$p'(s,x,t,y)=\frac{1}{\sqrt{2\pi V_t}}e^{-\frac{1}{2V_t}(x-M_t)^2}, \tag{13.186}$$

M_t, V_t representing respectively the mean and variance of $\xi(t)$ whose explicit forms will be given in Chapter 15.