

## Chapter 4

# Understanding Forward Exchange Rates for Currency

In this chapter, we discuss forward contracts in perfect financial markets. Specifically, we assume that there are no transactions costs; there are no taxes, or at least they are non-discriminatory: there is but one overall income number, with all capital gains and interest earned being taxable and all capital losses and interest paid deductible; there is no default risk; and people act as price takers in free and open markets for currency and loans or deposits. Most of the implications of market imperfections will be discussed in later chapters; in this chapter we provide the fundamental insights that need to be mildly qualified later.

In Section 1, we describe the characteristics of a forward contract and how forward rates are quoted in the market. In Section 2, we show, with a simple diagram, the relationship between the money markets, spot markets, and forward markets. Using the mechanisms that enforce the Law of One Price, Section 3 then presents the Covered Interest Parity Theorem. Two ostensibly unconnected issues are dealt with in Section 4: how do we determine the market value of an outstanding forward contract, and how does the forward price relate to the expected future spot price. We wrap up in Section 5.

### 4.1 Introduction to Forward Contracts

#### Basics

Let us recall, from the first chapter, the definition of a forward contract. Like a spot transaction, a forward contract stipulates how many units of foreign currency are to be bought or sold and at what exchange rate. The difference with a spot deal, of course, is that delivery and payment for a forward contract take place in the future (for example, one month from now) rather than one or two working days from now,

as in a spot contract. The rate that is used for all contracts initiated at time  $t$  and maturing at some future moment  $T$  is called the time- $t$  forward rate for delivery date  $T$ . We denote it as  $F_{t,T}$ .

Like spot markets, forward markets are not organized exchanges, but over-the-counter (OTC) markets, where banks act as market makers or look for counterparts via electronic auction systems or brokers. The most active forward markets are the markets for 30 and 90 days, and contracts for 180, 270, and 360 days are also quite common. Bankers nowadays quote rates up to ten years forward, and occasionally even beyond that, but the very long-term markets are quite thin. Recall, lastly, that any multiple of thirty days means that, relative to a spot contract, one extra calendar month has to be added to the spot delivery date, and that the delivery date must be a working day. Thus, if day  $[t + 2 \text{ plus } n \text{ months}]$  is not a working day, we may move forward to the nearest working day, unless this would make us change months: then we'd move back.

#### Example 4.1

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A 180-day contract signed on Thursday March 2, 2006 is normally settled on September 6. Why? The initiation day being a Thursday, the “spot” settlement date is Monday, March 6. Add 6 months; September 6, being a working day (Wednesday), then is the settlement date.

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### Market Conventions for Quoting Forward Rates

Forward exchange rates can be quoted in two ways. The most natural and simple quote is to give the actual rate, sometimes called the *outright* rate. This convention is used in, for instance, *The Wall Street Journal*, the *Frankfurter Allgemeine*, and the Canadian *Globe and Mail*. The *Globe and Mail* is one of the few newspapers also quoting long-term rates, as Table 4.1 shows.

In Table 4.1, the CAD/USD forward rate exceeds the spot rate for all maturities. Traders would say that the USD trades *at a premium*. Obviously, if the CAD/USD rate is at a premium, the USD/CAD forward rates must be below the USD/CAD spot rate; that is, the CAD must trade *at a discount*.

The second way of expressing a forward rate is to quote the difference between the outright forward rate and the spot rate—that is, quote the premium or discount. A forward rate quoted this way is called a swap rate.<sup>1</sup> Antwerp's *De Tijd*, or the London *Financial Times*, for example, used to follow this convention. Since both newspapers actually showed bid and ask quotes, we will postpone actual excerpts from these newspapers until the next chapter where spreads are taken into consid-

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<sup>1</sup>Confusingly, the terms swap contract and swap rate can have other meanings, as we shall explain in Chapter 7.

Table 4.1: **Spot and Forward Quotes, Mid-market rates in Toronto at noon**

	(Outright)		(Swap rates)	
	CAD per USD	USD per CAD	Premium or discount, in cents	
	CAD per USD	USD per CAD	CAD per USD	USD per CAD
U.S. Canada spot	1.3211	0.7569		
1monthforward	1.3218	0.7565	+0.07	-0.05
2 months forward	1.3224	0.7562	+0.13	-0.07
3 months forward	1.3229	0.7559	+0.18	-0.10
6 months forward	1.3246	0.7549	+0.35	-0.20
12 months forward	1.3266	0.7538	+0.55	-0.31
3 years forward	1.3316	0.7510	+1.05	-0.59
5 years forward	1.3579	0.7364	+3.68	-0.05
7 years forward	1.3921	0.7183	+7.10	-3.86
10 years forward	1.4546	0.6875	+13.36	-6.94

**Source** Globe and Mail .

eration. The rightmost two columns in Table 4.1 shows how *The Globe and Mail* quotes would have looked in swap-rate form. In that table, the sign of the swap rate is indicated by a plus sign or a minus sign. The *Financial Times* used to denote the sign as pm (premium) or dis (discount).

The origin of the term swap rate is the swap contract. In the context of the forward market, a swap contract is a spot contract immediately combined with a forward contract in the opposite direction.

#### Example 4.2

To invest in the US stock market for a few months, a Portuguese investor buys USD 100,000 at EUR/USD 1.10. In order to reduce the exchange risk, she immediately sells forward USD 100,000 for ninety days, at EUR/USD 1.101. The combined spot and forward contract—in opposite directions—is a swap contract. The swap rate, EUR/USD 0.1(cent), is the difference between the rate at which the investor buys and the rate at which she sells.

To emphasize the difference between a stand-alone forward contract and a swap contract, a stand-alone forward contract is sometimes called an outright contract. Thus, the two quoting conventions described above have their roots in the two types of contracts. Today, the outright rate and the swap rate are simply ways of quoting, used whether or not you combine the forward trade with a reverse spot trade.<sup>2</sup>

One key result of this chapter is that there is a one-to-one link between the swap

<sup>2</sup>Sometimes the swap rate is called the cost of the swap, but to financial economists that is a very dubious concept: at the moment the contract is initiated, both the spot and the forward part are zero-NPV deals, that is, their market value is zero. So the swap rate is not the cost of the swap in the same way a stock price measures the cost of a stock. It is more like an accounting concept of cost, in the style of the interest being the cost of a loan.

rate and the interest rates for the two currencies. To explain this relation, we first show how the spot market and the forward market are linked to each other by the money markets for each of the two currencies. But first we need to agree on a convention for denoting risk-free returns.

### Our Convention for Expressing Risk-free Returns

We adopt the following terminology: the (effective) risk-free (rate of) return is the simple percentage difference between the initial, time- $t$  value and the final, time- $T$  value of a nominally risk-free asset over that holding period.

#### Example 4.3

Suppose that you deposit CLP 100,000 for four years and that the deposit will be worth CLP 121,000 at maturity. The four-year effective (rate of) return is:

$$r_{t,T} = \frac{121,000 - 100,000}{100,000} = 0.21 = 21 \text{ percent.} \quad (4.1)$$

You can also invest for nine months. Suppose that the value of this deposit after nine months is 104,200. Then the nine-month effective return is:

$$r_{t,T} = \frac{104,200 - 100,000}{100,000} = 0.042 = 4.2 \text{ percent.} \quad (4.2)$$

Of course, at any moment in time, the rate of return you can get depends on the time to maturity, which equals  $T - t = 4$  years in the first example. Thus, as in the above examples, we always equip the rate of return,  $r$ , with two subscripts:  $r_{t,T}$ . In addition, we need to distinguish between the domestic and the foreign rate of return. We do this by denoting the domestic and the foreign return by  $r_{t,T}$  and  $r_{t,T}^*$ , respectively.

It is important to understand that the above returns, 21 percent for four years and 4.2 percent for nine months, are not expressed on an annual basis. This is a deviation from actual practice: bankers always quote rates that are expressed on an annual basis. We shall call such a *per annum* (*p.a.*) percentage an *interest rate*. If the time to maturity of the investment or loan is less than one year, your banker will typically quote you a simple *p.a.* interest rate. Given the simple *p.a.* interest rate, you can then compute the effective return as:

$$r_{t,T} = [\text{time to maturity, in years}] \times [\text{simple } p.a. \text{ interest rate for that maturity}]. \quad (4.3)$$

#### Example 4.4

Suppose that the *p.a.* simple interest rate for a three-month investment is 10 percent. The time to maturity,  $T - t$ , is  $1/4$  years. The effective return, then, is:

$$r_{t,T} = (1/4) \times 0.10 = 0.025. \quad (4.4)$$

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The convention that we adopt in this text is to express all formulas in terms of effective returns, that is, simple percentage differences between end values and initial values. One alternative would be to express returns in terms of *per annum* simple interest rates—that is, we could have written, for instance,  $(T - t)R_{t,T}$ , (where capital  $R$  would be the simple interest on a *p.a.* basis) instead of  $r_{t,T}$ . Unfortunately, then all formulas would look more complicated. Worse, there are many other ways of quoting an interest rate in *p.a.* terms, such as interest with annual, or monthly, or weekly, or even daily compounding; or banker's discount; or continuously compounded interest. To keep from having to present each formula in many versions (depending on whether you start from a simple rate, or a compound rate, etc.), we assume that you have already done your homework and have computed the effective return from your *p.a.* interest rate. Appendix 4.6 shows how effective returns can be computed if the *p.a.* rate you start from is not a simple interest rate. That appendix also shows how returns should *not* be computed.

Thus, in this section, we will consider four related markets—the spot market, the forward market, and the home and foreign money markets. One crucial insight we want to convey is that any transaction in one of these markets can be replicated by a combination of transactions in the other three. Let us look at the details.

## 4.2 The Relation Between Exchange and Money Markets

We have already seen how, using the spot market, one type of currency can be transformed into another at time  $t$ . For instance, you pay home currency to a bank and you receive foreign currency. Think of one wad of HC bank notes being exchanged for another wad of FC notes. Or even better, since spot deals are settled second working days: think of a spot transaction as an exchange of two cheques that will clear two working days from now. As of now, we denote the amounts by  $HC$  and  $FC$ . To make clear that we mean amounts, not names, they are written as math symbols (full-sized and slanting), not as FC and FC, our notation for names of currency. Another notational difference between currency names and amounts is that  $FC$  and  $HC$  always get a time subscript. To emphasize the fact that, in the above example, the amounts are delivered (almost) immediately, we add the  $t$  (= current time) subscript: you pay an amount  $HC_t$  in home currency and you receive an amount  $FC_t$  of foreign currency.

By analogy to our exchange-of-cheques idea for a spot deal, then, we can picture a forward contract as an exchange of two promissory notes, with face values  $HC_T$  and  $FC_T$ , respectively:

### Example 4.5

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Suppose you sell forward USD 100,000 at EUR/USD 0.75 for December 31. (Note that the quote defines the euro as the HC.) Then

- you commit to deliver USD 100,000, which is similar to signing a promissory note (PN) with face value  $FC_T = \text{USD } 100,000$  on Dec 31, and handing it over to the bank;
- the bank promises to pay you EUR 75,000, which is similar to giving you a signed PN with face value  $HC_T = \text{EUR } 75,000$  for that date. \_\_\_\_\_

Intimately linked to the exchange markets are the money markets for the home and foreign country, that is, the markets for short-term deposits and loans. A home-currency deposit of GBP 1m “spot”<sup>3</sup> for one year at 4 percent means that you pay an amount of GBP 1m to the bank now, and the bank pays you an amount GBP 1.04m at time  $T$ . This is similar to handing over the spot money amount of  $HC_t = 1\text{m}$  in return for a PN with face value  $HC_T = 1.04\text{m}$ . Likewise, if you borrow GBP 10m at 6 percent over one year, this is tantamount to you receiving a cheque with face value  $HC_t = \text{GBP } 10\text{m}$  in return for a promissory note with face value  $HC_T = \text{GBP } 10.6\text{m}$ .

### Graphical Representation of Chains of Transactions: an Example

For the remainder of this section, we take the Chilean Peso (CLP) as our home currency and the Norwegian Crown (NOK) as the foreign one. Suppose the spot rate is  $S_t = \text{CLP}/\text{NOK } 100$ , the four-year forward rate  $F_{t,T} = \text{CLP}/\text{NOK } 110$ , the CLP four-year risk-free rate of return is  $r_{t,T} = 21$  percent effective, and the NOK one equals  $r_{t,T}^* = 10$  percent. Very often we will discuss sequences of deals, or combinations of deals. Consider, for example, an Chilean investor who has CLP 100,000 to invest. He goes for a NOK deposit “swapped into CLP”, that is, a NOK deposit combined with a spot purchase and a forward sale. Let us see what the final outcome is:

#### Example 4.6

The investor converts his CLP 100,000 into an amount  $NOK_t$ , deposits these for four years, and sells forward the proceeds  $NOK_T$  in order to obtain a risk-free amount of CLP four years from now. The outcome is computed as follows:

1. Buy spot NOK: the input given to the bank is CLP 100,000, so the output of the spot deal, received from the bank, is  $100,000 \times 1/100 = 1,000$  Crown.
2. Invest these NOK at 10 percent: the input into the money market operation is  $NOK_t = 1,000$ , so after four years you will receive from the bank an output equal to  $1,000 \times 1.10 = 1,100$  Crowns.

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<sup>3</sup>A spot deposit or loan starts the second working day. For one-day deposits, one can also define the starting date as today (“overnight”), or tomorrow (“tomorrow/next”), but this must be made explicit, then. In all our examples, the deals are spot—the default option in real life, too.

3. This future NOK outcome is already being sold forward at  $t$ ; that is, right now you immediately cover or hedge the NOK deposit in the forward market so as to make its time- $T$  value risk free rather than contingent on the time- $T$  spot rate. The input for this transaction is  $NOK_T = 1,100$ , and the output in CLP at time  $T$  will be  $1,100 \times 110 = 121,000$ .

There is nothing difficult about this, except perhaps that by the time you finish reading Step 3 you've already half forgotten the previous steps. We need a way to make clear at one glance what this deal is about, how it relates to other deals and what the alternatives are. One step in the right direction is to adopt a notation like

$$\begin{array}{ccccccc}
 & \text{buy spot:} & & \text{deposit:} & & \text{sell frwd:} & \\
 & \times 1/100 & & \times 1.10 & & \times 110 & \\
 HC_t = 100,000 & \xrightarrow{\quad} & FC_t = 1,000 & \xrightarrow{\quad} & FC_T = 1,100 & \xrightarrow{\quad} & HC_T = 121,000
 \end{array}$$

So the arrows show how you go from a spot CLP position into a spot NOK one (the spot deal), and so on. We can further improve upon this by arranging the amounts in a diagram, where each kind of position has a fixed location. There are four kinds of money in play: foreign and domestic, each coming in a day- $t$  and a day- $T$  version. Let's show these on a diagram, with HC on the left and FC on the right, and with time  $t$  on top and time  $T$  below. Figure 4.1 shows the result for the above example.

We can now generalize. Suppose the spot rate is still CLP/NOK 100, the four-year forward rate CLP/NOK 110, the CLP risk-free is 21 percent effective, and the NOK one 10 percent. The diagram in Figure 4.2 summarises all transactions open to the treasurer. It is to be read as follows:

Figure 4.1: **Spot/Forward/Money Market Diagram: Example 4.6**

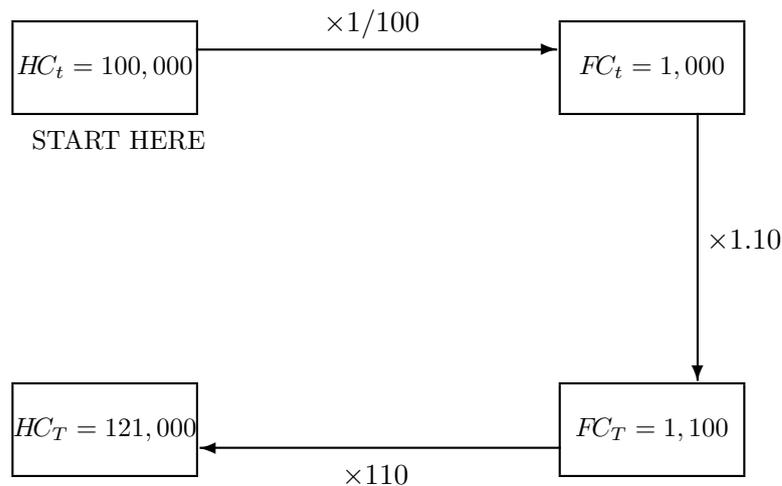
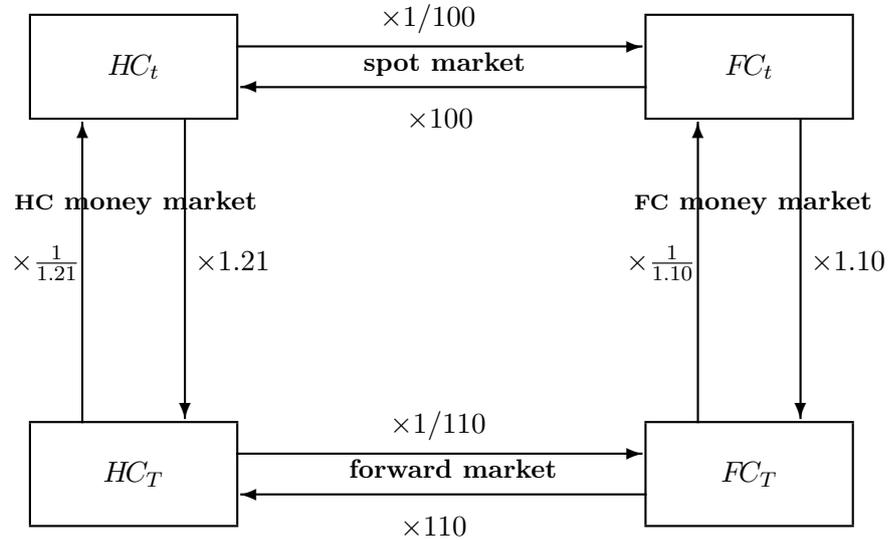


Figure 4.2: Spot/Forward/Money Market Diagram: the general picture



- Any  $t$ -subscripted symbol  $HC_t$  ( $FC_t$ ) refers to an *amount* of spot money; and any  $T$ -subscripted symbol  $HC_T$  ( $FC_T$ ) refers to a  $T$ -dated known amount of money, e.g. promised under a PN, A/P, A/R or deposit, or forward contract.
- Any possible transaction (spot or forward sale or purchase; home or foreign money-market deal) is shown as an arrow. A transaction is characterized by two numbers: (a) your position before the transaction, an input amount you surrender to the bank, and (b) your position after the transaction, the output amount you receive from the bank. The arrow starts from the (a) part and ends in the (b) part. For example,
  - a move  $HC_t \rightarrow FC_t$  refers to *buying* FC—spot (see “ $t$ ”)
  - a move  $FC_T \rightarrow HC_T$  refers to *selling* FC—forward (see “ $T$ ”)
  - a move  $HC_t \rightarrow HC_T$  refers to *investing* or *lending* HC
  - a move  $FC_T \rightarrow FC_t$  refers to *borrowing* against a FC income—*e.g.* discounting a FC PN.
- Next to each arrow we write the factor by which its “input” amount has to be multiplied to compute the “output” amount. Again: “input” is what you give to the bank (at either  $t$  or  $T$ ), “output” is what you receive from it.

### The General Spot/Forward/Money Market Diagram

To use the diagram, first identify the starting position. This is where you have money right now—like  $FC_T$  (: a customer will pay you FC in future, or a deposit will expire). Then determine the desired end point, like  $HC_T$  (: you want future HC instead; that is, you want to eliminate the exchange risk). Third, determine by which route you want to go from START to END. Lastly, follow the chosen route, sequentially multiplying the starting amount by all the numbers you see along the path.

#### Example 4.7

In Example 4.6, the path is  $HC_t \rightarrow FC_t \rightarrow FC_T \rightarrow HC_T$ , and the end outcome, starting from  $HC_t = 100,000$  is immediately computed as

$$HC_T = 100,000 \times \frac{1}{100} \times 1.10 \times 110 = 121,000. \quad (4.5)$$

The alert reader will already have noted that this is a synthetic HC deposit, constructed out of a FC deposit and a swap, and that (here) it has exactly the same return as the direct solution. Indeed, the alternative route,  $HC_t \rightarrow HC_T$ , yields  $100,000 \times 1.21 = 121,000$ . (In *im*perfect markets this equivalence of both paths will no longer be generally true, as we shall see in the next chapter.)

#### Example 4.8

Suppose that a customer of yours will pay NOK 6.5m at time T, four years from now, but you need cash Pesos to pay your suppliers and workers. You decide to sell forward, and take out a CLP loan with a time-T value that, including interest, exactly matches the proceeds of the forward sale. How much can you borrow on the basis of this invoice without taking any exchange risk?

The path chosen is  $FC_T (= \text{NOK}6,500,000) \rightarrow HC_T \rightarrow HC_t$ , and it yields

$$6.5m \times 110 \times \frac{1}{1.21} = \text{CLP } 590,909,090.91. \quad (4.6)$$

The clever reader will again eagerly point out that there is an alternative: borrow NOK against the future inflow (that is, borrow such that the loan *cum* interest is serviced by the NOK inflow), and convert the proceeds of the loan into CLP. Again, in our assumedly perfect market, the outcome is identical:  $6.5m/1.10 \times 100 = \text{CLP } 590,909,090.91$ . Thus, the diagram allows us to quickly understand the purpose, and see the outcome of, a sequence of transactions. It also shows there are always two routes that lead from a given starting point to a given end point—a useful insight for shopping-around purposes. The advantage of using the diagram will be

even more marked when we add bid-ask spreads in all markets (next chapter) or when we study forward forwards or forward rate agreements and their relationship to forward contracts (Appendix 4.7),<sup>4</sup> or when we explain forward forward swaps (Chapter 5).

### 4.3 The Law of One Price and Covered Interest Parity

The sequences of transactions that can be undertaken in the exchange and money markets, as summarized in Figure 4.2, can be classified into two types.

1. You could do a sequence of transactions that forms a roundtrip. In terms of Figure 4.2, a roundtrip means that you start in a particular box, and then make four transactions that bring you back to the starting point. For example, you may consider the sequence  $HC_T \rightarrow HC_t \rightarrow FC_t \rightarrow FC_T \rightarrow HC_T$ . In terms of the underlying transactions, this means that you borrow CLP, convert the proceeds of the CLP loan into NOK, and invest these NOK; the proceeds of the investment are then immediately sold forward, back into CLP. The question that interests you is whether the CLP proceeds of the forward sale are more than enough to pay off the original CLP loan. If so, you have identified a way to make a sure profit without using any of your own capital. Thus, the idea behind a round-trip transaction is *arbitrage*, as defined in Chapter 3.
2. Alternatively, you could consider a sequence of transactions where you end up in a box that is not the same as the box from which you start. The two examples 4.7 and 4.8 describe such non-round-trip sequences. Trips like that have an economic rationale. In the first example, for instance, the investor wants to invest CLP, and the question here is whether the swapped NOK investment ( $CLP_t \rightarrow NOK_t \rightarrow NOK_T \rightarrow CLP_T$ ) yields more than a direct CLP investment ( $CLP_t \rightarrow CLP_T$ ). Using the terminology of Chapter 3, this would be an example of *shopping around* for the best alternative.

In what follows, we want to establish the following two key results:

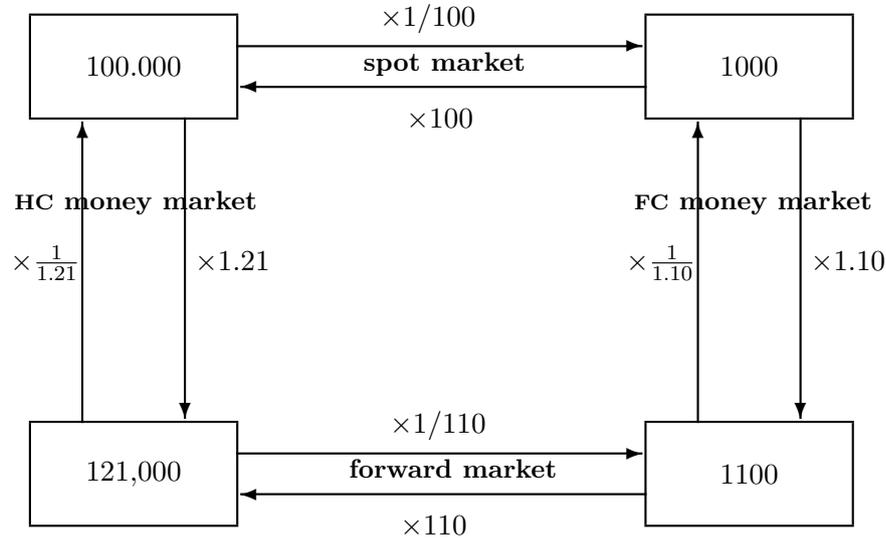
1. To rule out arbitrage in perfect markets, the following equality must hold:

$$F_{t,T} = S_t \frac{1 + r_{t,T}}{1 + r_{t,T}^*}. \quad (4.7)$$

[In *imperfect* markets, this sharp equality will be watered down to a zone of admissible values, but the zone is quite narrow.]

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<sup>4</sup>Forward Forwards and forward rate agreements (FRAs) are contracts that fix the interest rate for a deposit or loan that will be made (say) six months from now, for (say) three months. This can be viewed as a six-month forward deal on a (then) three-month interest rate. See the Appendix on forward interest rates.

Figure 4.3: **Spot/Forward/Money Market Diagram: Arbitrage Computations**

2. If equation (4.7) holds, shopping-around computations are a waste of time since the two routes that lead from a given initial position A to a desired end position B produce exactly the same. Stated positively, shopping around can (and will) be useful only because of imperfections.

### 4.3.1 Arbitrage and Covered Interest Parity

In this section, we use an arbitrage argument to verify equation (4.7), a relationship called the *Covered Interest Parity* (CIP) Theorem. The theorem is evidently satisfied in our example:

$$110 = F_{t,T} = S_t \frac{1 + r_{t,T}}{1 + r_{t,T}^*} = 100 \frac{1.21}{1.10} = 110. \quad (4.8)$$

Arbitrage, we know, means full-circle roundtrips through the diagram. There are two ways to go around the entire diagram: clockwise, and counterclockwise. Follow the trips on Figure 4.3, where the symbols for amounts have been replaced by the specific numbers used in the numerical examples. We should not make any profit if the rate is 110, and we should make free money as soon as the rate does deviate.

**Clockwise roundtrip** The starting point of a roundtrip is evidently immaterial, but let's commence with a HC loan: this makes it eminently clear that no own capital is being used. Also the starting amount is immaterial, so let's pick an amount that produces conveniently round numbers all around: we write a PN with face value

$CLP_T = 121,000$ . We discount this,<sup>5</sup> and convert the proceeds of this loan into Crowns, which are invested. At the same moment we already sell forward the future Crown balance. The final outcome is:

$$121,000 \times \frac{1}{1.21} \times \frac{1}{100} \times 1.10 \times 110 = 121,000. \quad (4.9)$$

So we break exactly even: the forward sale nets us exactly what we need to pay back the loan.

**DoItYourself problem 4.1** \_\_\_\_\_

Show, similarly, that also the counterclockwise roundtrip exactly breaks even. For your convenience, start by writing a PN with face value  $NOK_T = 1100$ . What is the path? What is the outcome? \_\_\_\_\_

**What if  $F_{t,T} = \text{too low, say } 109$ ?** If one price is too low relative to another price (or set of other prices), we can make money by *buying* at this too-low rate. The trip where we buy forward is the counterclockwise one. We start as before, except for the new price in the last step:

$$1100 \times \frac{1}{1.10} \times 100 \times 1.21 \times \frac{1}{109} = 1110.09 > 1100. \quad (4.10)$$

So the forward purchase nets us 1110.09 Pesos, 10.09 more than the 1100 we need to pay back loan.

**DoItYourself problem 4.2** : \_\_\_\_\_

What if  $F_{t,T}$  is too high, say 111? Indicate the path and calculate the arbitrage profit. \_\_\_\_\_

**DoItYourself problem 4.3** \_\_\_\_\_

To generalize these numerical results, we now start with PN's with face value 1, and replace all rates by their symbols. One no-arb condition is that the proceeds of the clockwise trip should not exceed the starting amount, unity. Explain how this leads to the following expression:

$$\frac{1}{1 + r_{t,T}} \times \frac{1}{S_t} \times (1 + r_{t,T}^*) \times F_{t,T} \leq 1. \quad (4.11)$$

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<sup>5</sup>Discounting a PN or a T-bill or a trade bill not only means computing its PV; it often means borrowing against the claim. In practice, under such a loan the borrower would typically also cede the claim to the financier, as security. This lowers the lender's risk and makes the loan cheaper.

This produces an inequality constraint,  $F_{t,T} \leq S_t \frac{1+r_{t,T}}{1+r_{t,T}^*}$ . Write the no-arbitrage-profit condition for the counterclockwise trip and express it as another inequality constraint. Lastly, derive CIP. \_\_\_\_\_

### 4.3.2 Shopping Around (The Pointlessness of —)

The diagram in Figure 4.2 also tells us that any non-round-trip sequence of transactions can be routed two ways. For instance, you can go directly from  $CLP_t$  to  $CLP_T$ , or you can go via  $NOK_t$  and  $NOK_T$ . In two earlier examples, 4.7 and 4.8, we already illustrated our claim that, in perfect markets where CIP holds, both ways to implement a trip produce exactly the same outcome. It is simple to show that this holds for all of the ten other possible trips one could think of, in this diagram; but it would also be so tedious that we leave this as an exercise to any non-believer in the audience. It would also be a bit pointless, because in reality shopping around does matter. As we show in the next chapter, the route you choose for your trip may matter because of imperfections like bid-ask spreads, taxes (if asymmetric), information costs (if leading to inconsistent risk spreads asked by home and foreign banks), and legal subtleties associated with swaps.

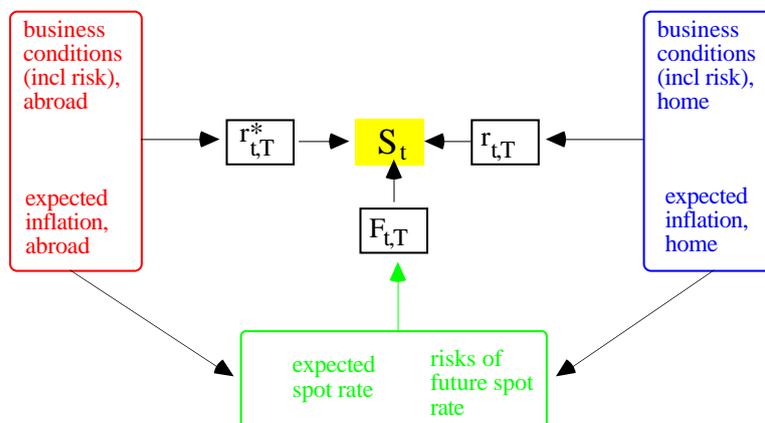
### 4.3.3 Unfrequently Asked Questions on CIP

Before we move on to new challenges like the market value of a forward contract and the relation of the forward rate with expected future spot rates, a few crucial comments are in order. We first talk about causality, then about why pros always quote the swap rate rather than the outright, and lastly about taxes.

#### Covered Interest Parity and Causality

As we have seen, in perfect markets the forward rate is linked to the spot rate by pure arbitrage. Such an arbitrage argument, however, does not imply any causality. CIP is merely an application of the Law of One Price, and the statement that two perfect substitutes should have the same price does not tell us where that “one price” comes from. Stated differently, showing  $F_{t,T}$  as the left-hand-side variable (as we did in Equation [4.7]), does not imply that the forward rate is a “dependent” variable, determined by the spot rate and the two interest rates. Rather, what Covered Interest Parity says is that the four variables (the spot rate, the forward rate, and the two interest rates) are determined jointly, and that the equilibrium outcome should satisfy Equation [4.7]. The fact that the spot market represents less than 50 percent of the total turnover likewise suggests that the forward market is not just an appendage to the spot market. Thus, it is impossible to say, either in theory or in practice, which is the tail and which is the dog, here.

Figure 4.4: Appending Underlying Stories to the Variables in CIP



Although CIP itself does not say which term causes which, many economists and practitioners do have theories about one or more terms that appear in the Covered Interest Parity Theorem. One such theory is the Fisher equation, which says that interest rates reflect expected inflation and the real return that investors require. Another theory suggests that the forward rate reflects the market's expectation about the (unknown) future spot rate,  $\tilde{S}_T$ .<sup>6</sup> We shall argue in Section 4.4 that the latter theory is true in a risk-adjusted sense. In short, while there is no causality in CIP itself, one can append stories and theories to items in the formula. Then CIP becomes an ingredient in a richer economic model with causality relations galore—but  $S$ ,  $F$ ,  $r$  and  $r^*$  would all be endogenous, determined by outside forces and circumstances. Figure 4.4 outlines a plausible causal story of how interest rates and the forward rate are set and, together, imply the spot rate.

### CIP and the Swap Rate

When the forward rate exceeds the spot rate, the foreign currency is said to be *at a premium*. Otherwise, the currency is *at a discount* ( $F_{t,T} < S_t$ ), or *at par* ( $F_{t,T} = S_t$ ). In this text, we often use the word premium irrespective of its sign; that is, we treat the discount as a negative premium. From [4.7], the sign of the premium uniquely

<sup>6</sup>We use a tilde ( $\tilde{\cdot}$ ) above a symbol to indicate that the variable is random or uncertain.

depends on the sign of  $r_{t,T} - r_{t,T}^*$ :

$$\begin{aligned}
 [\text{swap rate}]_{t,T} &\stackrel{\text{def}}{=} F_{t,T} - S_t, \\
 &= S_t \left[ \frac{1 + r_{t,T}}{1 + r_{t,T}^*} - 1 \right], \\
 &= S_t \left[ \frac{1 + r_{t,T}}{1 + r_{t,T}^*} - \frac{1 + r_{t,T}^*}{1 + r_{t,T}^*} \right], \\
 &= S_t \left[ \frac{r_{t,T} - r_{t,T}^*}{1 + r_{t,T}^*} \right]; \tag{4.12}
 \end{aligned}$$

$$\Rightarrow \frac{\partial \cdot}{\partial S_t} = \left[ \frac{r_{t,T} - r_{t,T}^*}{1 + r_{t,T}^*} \right] \approx r_{t,T} - r_{t,T}^*. \tag{4.13}$$

Thus, a higher domestic return means that the forward rate is at a premium, and vice versa. To a close approximation (with low foreign interest rates and/or short maturities), the percentage swap rate even simply is the effective return differential.

To easily remember this, think of the following. If there would be a pronounced premium, we would tend to believe that this signals an expected appreciation for the foreign currency.<sup>7</sup> That is, the foreign currency is “strong”. But strong currencies are also associated with low interest rates: it’s the weak moneys that have to offer high rates to shore up their current value. In short, a positive forward premium goes together with a low interest rate because both are traditionally associated with a strong currency.

A second corollary from the CIP theorem is that, whenever the spot rate changes, all forward rates must change in lockstep. In old, pre-computer days, this meant quite a burden to traders/market makers, who would have to manually recompute all their forward quotes. Fortunately, traders soon noticed that the swap rate is relatively insensitive to changes in the spot rate. That is, when you quote a spot rate and a swap rate, then you make only a small error if you do not change the swap rate every time  $S$  changes.

#### Example 4.9

Let the *p.a.* simple interest rates be 4 and 3 percent (HC and FC, respectively). If  $S_t$  changes from 100 to 100.5—a huge change—the theoretical one-month forward changes too, and so does the swap rate, but the latter effect is minute:

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<sup>7</sup>Empirically, the strength of a currency is predicted by the swap rate only in the case of pronounced premia. When interest are quite similar and expectations rather diffuse, as is typically the case among OECD mainstream countries, the effects risk premia and transaction costs appear to swamp any expectation effect. See Chapter 10.

	spot	forward	swap rate
level <sub>0</sub>	100.0	$100.0 \times \frac{1.003333}{1.002500} = 100.0831$	0.0831
level <sub>1</sub>	100.5	$100.5 \times \frac{1.003333}{1.002500} = 100.5835$	0.0835
change	0.5	0.5004	0.0004

The rule of thumb of not updating the swap rate all the time used to work reasonably well because, in olden days, interest rates were low<sup>8</sup> and rather similar across currencies (the gold standard, remember?), and maturities short. This makes the fraction on the right hand side of [4.12] a very small number. In addition, interest rates used to vary far less often than spot exchange rates. Nowadays, of course, computers make it very easy to adjust all rates simultaneously without creating arbitrage opportunities, so we no longer need the trick with the swap rates. But while the motivation for using swap rates is gone, the habit has stuck.

#### DoItYourself problem 4.4

Use the numbers of Example 4.9 to numerically evaluate the partial derivative in Equation [4.13],

$$\frac{\partial(F_{t,T} - S_t)}{\partial S_t} = \left[ \frac{r_{t,T} - r_{t,T}^*}{1 + r_{t,T}^*} \right] \approx r_{t,T} - r_{t,T}^* .$$

Check whether this is a small number, when interest rates are low (and rather similar across currencies) and maturities short. (If so, it means that the swap rate hardly changes when the spot rate moves.) Also check that the analytical result matches the calculations in the Example.

We now bring up an issue we have been utterly silent about thus far: taxes.

#### CIP: Capital Gains *v* Interest Income, and Taxes

When comparing the direct and synthetic HC deposits, in Example 4.7, we ignored taxes. This, we now show, is fine as long as the tax law does not discriminate between interest income and capital gains.

<sup>8</sup>During the Napoleonic Wars, for instance, the UK issued perpetual (!) debt (the consolidated war debt, or *consol*) with an interest rate of 3.25 percent. Toward the end of the nineteenth century, Belgium issued perpetual debt with a 2.75 percent coupon (to pay off a Dutch toll on ships plying for Antwerp). Rates crept up in the inflationary 70s to, in some countries, 20 percent short-term or 15 percent long-term around 1982. They then fell slowly to quite low levels, as a result of falling inflation, lower government deficits and, in the first years of the 21st century, high uncertainty and a recession—the “flight for safety” effect.

Table 4.2: HC and swapped FC investments with nondiscriminatory taxes

	Invest CLP 100		Invest NOK 1 and hedge	
initial investment		100	$1 \times 100 =$	100
final value	$100 \times 1.21 =$	121	$[1 \times 1.10] \times 110 =$	121
income		21		21
interest		21	$[1 \times 0.10] \times 110 =$	11
capgain		0	$110 - 100 =$	10
taxable		21		21
tax (33.33 %)		7		7
after-tax income		14		14

The first point you should be aware of is that, by going for a swapped FC deposit instead of a HC one, the total return is in principle unaffected but the relative weight of the interest and capital-gain components is changed. Consider our Chilean investor who compares an investment in NOK to one in CLP. Given the spot rate of 100, we consider investments of 100 CLP or 1 NOK. In Table 4.2 you see that the CLP investment yields interest income only, while the NOK deposit earns interest (10 pence, exchanged at the forward rate 110) and a capital gain (you buy the principal at 100, and sell later at 110). But in both cases, total income is 21. (This, indeed, is the origin of the name CIP: the return, covered, is the same.)

#### DoItYourself problem 4.5

Verify that the expression below follows almost immediately from CIP, Equation [4.7]:

$$F_{t,T} r_{t,T}^* + (F_{t,T} - S_t) = S_t r_{t,T}. \quad (4.14)$$

Then trace each symbol in the formula to the numbers we used in the numerical example. Identify the interest on the Peso and Crown deposits, and the capital gain or loss.

So we know that total pre-tax income is the same in both cases. If all income is equally taxable, the tax is the same too, and so must be the aftertax income. It also follows that if, because of *e.g.* spreads, there is a small advantage to, say, the Peso investment, then taxes will reduce the gain but not eliminate it. That is, if Pesos would yield more before taxes, then they would also yield more after taxes.

In most countries, corporate taxes are neutral between interest income and capital gains, especially short-term capital gains. But there are exceptions. The UK used to treat capital gains on FC loans differently from capital losses and interest received. Under personal taxation, taxation of capital gains is far from universal, and/or long-term capital gains often receive beneficial treatment. In cases like this, the ranking

of outcomes on the basis of after-tax returns could be very different from the ranking on pre-tax outcomes. Beware!

## 4.4 The Market Value of an Outstanding Forward Contract

In this and the next section, we discuss the market value of a forward contract at its inception, during its life, and at expiration. As is the case for any asset or portfolio, the market value of a forward contract is the price at which it can be bought or sold in a normally-functioning market. The focus, in this section, is on the value of a forward contract that was written in the past but that has not yet matured. For instance, one year ago (at time  $t_0$ ), we may have bought a five-year forward contract for NOK at  $F_{t_0,T} = \text{CLP/NOK } 115$ . This means that we now have an outstanding four-year contract, initiated at the rate of CLP/NOK 115. This outstanding contract differs from a newly signed four-year forward purchase because the latter would have been initiated at the now-prevailing four-year forward rate, CLP/NOK 110. The question then is, how should we value the outstanding forward contract?

This value may be relevant for a number of reasons. At the theoretical level, the market value of a forward contract comes in quite handy in the theory of options, as we shall see later on. In day-to-day business, the value of an outstanding contract can be relevant in, for example, the following circumstances:

- If we want to negotiate early settlement of the contract, for instance to stop losses on a speculative position, or because the underlying position that was being hedged has disappeared.
- If there is default and the injured party wants to file a claim.
- If a firm wishes to “mark to market” the book value of its foreign-exchange positions in its financial reports.

### 4.4.1 A general formula

Let us agree that, unless otherwise specified, “a contract” refers to a forward purchase of one unit of foreign currency. (This is the standard convention in futures markets.) Today, at time  $t$ , we are considering a contract that was signed in the past, at time  $t_0$ , for delivery of one unit of foreign currency to you at  $T$ , against payment of the initially agreed-upon forward rate,  $F_{t_0,T}$ . Recall the convention that we have adopted for indicating time: the current date is always denoted by  $t$ , the initiation date by  $t_0$ , the future (maturity) date by  $T$ , and we have, of course,  $t_0 \leq t \leq T$ .

The way to value an outstanding contract is to interpret it as a simple portfolio that contains a FC-denominated PN with face value 1 as an asset, and a HC-denominated PN with face value  $F_{t_0,T}$  as a liability. Valuing a HC PN is easy: just

discount the face value at the risk-free rate. For the FC PN, we first compute its PV in FC (by discounting at  $r^*$ ), and then translate this FC value into HC via the spot price:

**Example 4.10**

Consider a contract that has 4 years to go, signed in the past at a historic forward price of 115. What is the market value if  $S_t = 100$ ,  $r_{t,T} = 21\%$ ,  $r_{t,T}^* = 10\%$ ?

- The asset leg is like holding a PN of FC 1, now worth  $PV^* = 1/1.10 = 0.90909$  NOK and, therefore,  $0.9090909 \times 100 = 90.909$  CLP.
- The liability leg is like having written a PN of HC 115, now worth CLP  $115/1.21 = 95.041$ .
- The net value now is, therefore, CLP  $90.909 - 95.041 = -4.132$

The generalisation is as follows:

$$\left[ \begin{array}{l} \text{Market value of} \\ \text{forward purchase} \\ \text{at } F_{t_0,T} \end{array} \right] = \underbrace{\frac{1}{1+r_{t,T}^*}}_{\substack{\text{translated value} \\ \text{of FC asset}}} \times S_t - \underbrace{\frac{F_{t_0,T}}{1+r_{t,T}}}_{\substack{\text{PV of HC} \\ \text{liability}}} . \quad (4.15)$$

There is a slightly different version that is occasionally more useful: the value is the discounted difference between the current and the historic forward rates. To find this version, multiply and divide the first term on the right of [4.15] by  $(1+r_{t,T})$ , and use CIP:

$$\begin{aligned} \left[ \begin{array}{l} \text{Market value of} \\ \text{forward purchase} \\ \text{at } F_{t_0,T} \end{array} \right] &= \frac{1}{1+r_{t,T}} \underbrace{\frac{1+r_{t,T}}{1+r_{t,T}^*} S_t}_{=F_{t,T}, \text{ (CIP)}} - \frac{F_{t_0,T}}{1+r_{t,T}} \\ &= \frac{F_{t,T} - F_{t_0,T}}{1+r_{t,T}} \end{aligned} \quad (4.16)$$

**Example 4.11**

Go back to Example 4.10. Knowing that the current forward rate is 110, we immediately find a value of  $(110 - 115)/1.21 = -4.132$  CLP for a contract with historic rate 115.

One way to interpret this variant is to note that, relative to a new contract, we're overpaying by CLP 5: last year we committed to paying 115, while we would have

gotten away with 110 if we had signed right now. This “loss”, however, is dated 4 years from now, so its PV is discounted at the risk-free rate.

The sceptical reader may object that this “loss” is very fleeting: its value changes every second; how comes, then, that we can discount at the risk-free rate? One answer is that the value changes continuously because interest rates and (especially) the spot rate are in constant motion, But that does not invalidate the claim that we can always value each PN using the risk-free rates and the spot exchange rate prevailing at that moment. Relatedly, the future loss relative to market conditions at  $t$  can effectively be locked in at no cost, by selling forward for the same date:

**Example 4.12** \_\_\_\_\_

Consider a contract that has four years to go, signed in the past at a historic forward price of 115, for speculative purposes. Right now you see there is a loss, and you want to close out to avoid any further red ink. One way is to sell forward HC 1 at the current forward rate, 110. On the common expiry date of old and new contract we then just net the loss of 115–110:

	HC flows at $T$	FC flows at $T$
old contract: buy at $F_{t_0,T}=115$	–115	1
new contract: sell at $F_{t_0,T}=110$	110	–1
net flow	–5	0

But because this loss is realized within four years only, its PV is found by discounting. Discounting can be at the risk-free rate since, as we see, the locked-in loss is risk free. \_\_\_\_\_

We can now use the result in Equation [4.15] to determine the value of a forward contract in two special cases: at its inception and at maturity.

**4.4.2 Corollary 1: The Value of a Forward Contract at Expiration**

At its expiration time, the market value of a purchase contract equals the difference between the spot rate that prevails at time  $T$ —the value of what you get—and the forward rate  $F_{t_0,T}$  that you agreed to pay:

$$\left[ \begin{array}{l} \text{Expiration value of} \\ \text{a forward contract} \\ \text{with rate } F_{t_0,T} \end{array} \right] = S_T - F_{t_0,T}. \quad (4.17)$$

Equation [4.17] can be derived formally from Equation [4.15], using the fact that the effective return on a deposit or loan with zero time to maturity is zero (that is,  $r_{T,T} = 0 = r_{T,T}^*$ ). The result in [4.17] is quite obvious, as the following example shows:

**Example 4.13** \_\_\_\_\_

- You bought forward, at time  $t_0$ , one NOK at CLP/NOK 115. At expiry,  $T$ , the NOK spot rate turns out to be CLP/NOK 123, so you pay 115 for something you can immediately re-sell at 123. The net value is, therefore,  $123-115=8$ .
- Idem, except that  $S_T$  turns out to be CLP/NOK 110. You have to pay 115 for something worth only 110. The net value is, therefore,  $110-115=-5$ : you would be willing to *pay* 5 to get out of this contract. \_\_\_\_\_

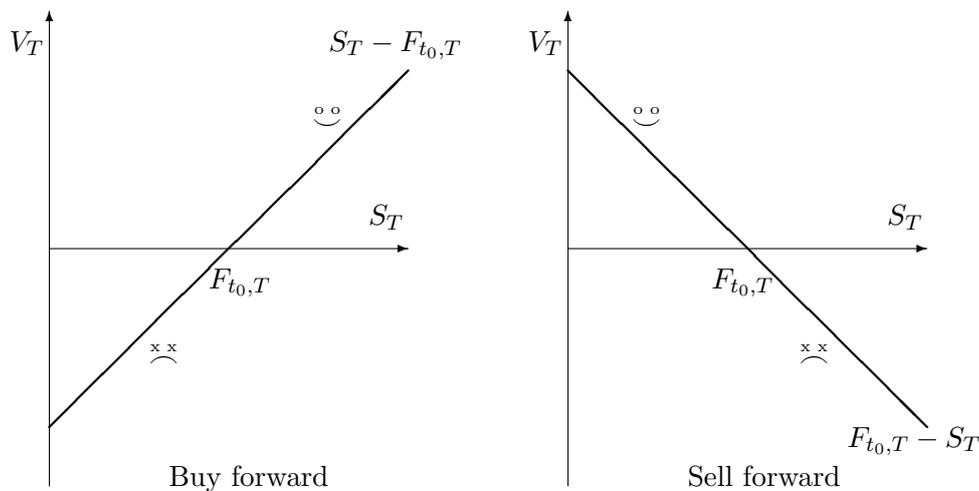
The value of a unit forward sale contract is, of course, just the negative of the value of the forward purchase: forward deals are zero-sum games. The seller wins if the spot value turns out to be below the contracted forward price, and loses if the spot value turns out to be above. Figure 4.5 pictures the formulas, with smileys and frownies indicating the positive and negative parts.

Equation [4.17] can be used to formally show how hedging works. Suppose that you have to pay one unit of foreign currency at some future time  $T$ . The foreign currency debt is risky because the cash flow at time  $T$ , in home currency, will be equal to minus the future spot rate—and, at time  $t$ , this future spot rate is uncertain, a characteristic we stress by adding a tilde ( $\sim$ ) over the variable. By adding a forward purchase, the combined cash flow becomes risk free, as the bit of arcane math shows, below:

$$\begin{array}{rcl}
 \text{Cash flow from amortizing the debt at expiration:} & & -\tilde{S}_T \\
 \text{Value of the forward purchase at expiration:} & & \tilde{S}_T - F_{t_0,T} \\
 \text{Combined cash flow:} & & -F_{t_0,T}.
 \end{array} \quad (4.18)$$

Putting this into words, we say that hedging the foreign-currency debt with a forward purchase transforms the risky debt into a risk-free debt, with a known outflow

Figure 4.5: **The Value of a Forward Purchase or Sales Contract at Expiry**



$-F_{t_0,T}$ . We shall use this result repeatedly in Chapter 5 (on uses of forward contracts), in Chapter 9 when we discuss option pricing, and in Chapters 13 where we analyse exposure and risk management.

Make sure you realize that the hedged liability may make you worse off, *ex post*, than the unhedged one. Buying at a pre-set rate  $F_{t,T}$  gives that great warm feeling, *ex post*, if the spot rate  $S_T$  turns out to be quite high; but it hurts if the spot rate turns out to be quite cheap. The same conclusion was already implicit in [4.17]: the value of the contract at expiry can be either sign. This rises the question whether hedging is really so good as it is sometimes cracked up to be. We return to the economics of hedging in Chapter 12.

#### 4.4.3 Corollary 2: The Value of a Forward Contract at Inception

The value at expiry, above, probably was so obvious that it is, in a way, just a means of proving that the general valuation formula [4.15] makes sense. The same holds for the next special case: the value at inception, *i.e.* the time the contract is initiated or signed. At inception, the market value *must* be zero. We know this because (a) when we sign a forward contract, we have to pay nothing; and (b) hard-nosed bankers would never give away a positive-value contract for free, nor accept a negative-value contract at a zero price. To show the (initial) zero-value property formally, we use the general value formula [4.16] and consider the special case where  $t_0 = t$ , implying that  $F_{t_0,T} = F_{t,T}$  (That is, the contract we are valuing is new.) Obviously,

$$\left[ \begin{array}{l} \text{Initial value of a} \\ \text{forward contract} \\ \text{with rate } F_{t,T} \end{array} \right] = \frac{F_{t,T} - F_{t,T}}{1 + r_{t,T}} = 0. \quad (4.19)$$

The value of a forward contract is zero at the moment it is signed because the contract can be replicated at zero cost. Notably, if a bank tried to charge you money for a contract at the equilibrium (Covered Interest Parity) forward rate, you would refuse, and create a synthetic forward contract through the spot and money markets:

##### Example 4.14

Let  $S_t = 100$ ,  $r_{t,T}^* = 0.10$ ,  $r_{t,T} = 0.21$ ,  $F_{t,T} = 110$ ; but a bank wants to charge you a commission of 3 for a forward purchase. You would shrug dismissively and immediately construct a synthetic forward contract at 110 at a zero cost:

- write a PN *ad* HC 110, discount it;
- convert the proceeds,  $110/1.21 = 90.909090$ , into FC: you get FC 0.90909.
- invest at 10 percent, to get HC 1 at  $T$ .

Thus, you can replicate a forward purchase contract under which your payment at  $T$  amounts to 110, just like in the genuine, direct forward contract, but it does not cost you anything now.

#### 4.4.4 Corollary 3: The Forward Rate and the Risk-Adjusted Expected Future Spot Rate

The zero-value property of forward contracts discussed above has another, and quite fundamental, interpretation. Suppose that the CLP/NOK four-year forward rate equals 110, implying that you can exchange one future NOK for 110 future CLP and *vice versa* without any up-front cash flow. This must mean that the market perceives these amounts as being *equivalent* (that is, having the same value). If this were not so, there would have been an up-front compensation to make up for the difference in value.

Since any forward contract has a zero value, the present values of CLP 1 four years and NOK 110 four years must be equal anywhere; that is, the equivalence of these amounts holds for any investor or hedger anywhere. However, the equivalence property takes on a special meaning if we pick the CLP (which is the currency in which our forward rate is expressed) as the home currency: in that particular numéraire, the CLP amount is risk-free, or certain. In terms of CLP, we can write the equal-value property as:

$$\text{PV}_t(\tilde{S}_T) = \text{PV}_t(F_{t,T}), \quad (4.20)$$

where  $\text{PV}_t(\cdot)$  is the present-value operator. In a way, equation [4.20] is just the zero-value property: the present value of the uncertain future cash inflow  $\tilde{S}_T$  generated by the contract cancels out against the PV of the known future outflow,  $F_{t,T}$ . We can lose or gain, but these prospects balance out in present-value terms, from our time- $t$  viewpoint. But the related, second interpretation stems from the fact that in home currency, the forward price on the right-hand side of Equation [4.20] is a risk-free, known number whereas the future spot rate on the left is uncertain. That is, at time  $t$  an amount of  $F_{t,T}$  Pesos payable at  $T$  is not just *equivalent* to one unit of foreign currency payable at  $T$ ; this amount of future home currency is also a *certain*, risk-free amount. For this reason, we shall say that in home currency, the forward rate is the time- $t$  *certainty equivalent* of the future spot rate,  $\tilde{S}_T$ .

##### Example 4.15

In our earlier CLP/NOK examples, the certainty equivalent of one Norwegian Crown four years out is CLP 110. You can offer the market a sure CLP 110 at  $T$  and get one Crown (with risky value  $\tilde{S}_T$ ) in return; but equally well you can offer the market one Crown (with risky value  $\tilde{S}_T$ ) and get a sure CLP 110 in return.

The notion of the certainty equivalent deserves some elaboration. Many introductory finance books discuss the concept of an investor's subjective certainty equivalent of a risky income. This is defined as the single known amount of income that is equally attractive as the entire risky distribution.

##### Example 4.16

Suppose that you are indifferent between, on the one hand, a lottery ticket that pays

out with equal probabilities either USD 100m or nothing, and on the other hand, a sure USD 35m. Then your personal certainty equivalent of the risky lottery is USD 35m. You are indifferent between 35m for sure and the risky cash flow from the lottery.

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Another way of saying this is that, when valuing the lottery ticket, you have marked down its expected value, USD 50m, by USD 15, because the lottery is risky. Thus, we can conclude that your personal certainty equivalent, USD 35m, is the expected value of the lottery ticket corrected for risk.<sup>9</sup>

In the example, the risk-adjustment is quite subjective. A *market* certainty equivalent, by analogy, is the single known amount that the market considers to be as valuable as the entire risky distribution. And market certainty equivalents are, of course, what matter if we want to price assets, or if we want to make managerial decisions that maximize the market value of the firm. We have just argued that the (CLP) market certainty equivalent of the future CLP/NOK spot rate must be the current CLP/NOK forward rate. Stated differently, the market's time- $t$  expectation of the time- $T$  CLP/NOK spot rate, corrected for risk, is revealed in the CLP/NOK forward rate,  $F_{t,T}$ . Let's express this formally as:

$$\text{CEQ}_t(\tilde{S}_T) = F_{t,T}, \quad (4.21)$$

where  $\text{CEQ}_t(\cdot)$  is called the *certainty equivalent* operator.

A certainty equivalent operator is similar to an ordinary expectations operator,  $E_t(\cdot)$ , except that it is a risk-adjusted expectation rather than an ordinary expected value. (There are good theories as to how the risk-adjusted and the "physical" densities are related, but they are beyond the scope of this text.) Like  $E_t(\cdot)$ ,  $\text{CEQ}_t(\cdot)$  is also a conditional expectation, that is, the best possible forecast given the information available at time  $t$ . We use a  $t$  subscript to emphasize this link with the information available at time  $t$ .

To make the market's risk-adjustment a bit less abstract, assume the CAPM holds. Then we could work out the left-hand side of [4.20] in the standard way: the PV of a risky cashflow  $\tilde{S}_T$  equals its expectation, discounted at the risk-adjusted rate. The risk-adjusted discount rate, in turn, consists of the risk-free rate plus a risk premium  $RP_{t,T}(\beta_S)$  which depends on market circumstances and the risk of the asset to be priced,  $\beta_S$ . Working out the right-hand side of [4.20] is straightforward: the PV of a risk-free flow  $F$  is  $F$  discounted at the risk-free rate  $r$ . Thus, we can

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<sup>9</sup>When we say that investors are risk-averse, we mean they do not like symmetric risk for their entire wealth. The amounts in the example are so huge that they would represent almost the entire wealth of most readers; so in that case, risk aversion guarantees that the risk-adjustment is downward. But for small investments with, for instance, lots of right skewness, one observes upward adjustments: real-world lottery players, for instance, are willing to pay more than the expected value because, when stakes are small, right-skewness can give quite a kick.

flesh out [4.20] into

$$\frac{E_t(\tilde{S}_T)}{1 + r_{t,T} + RP_{t,T}(\beta_S)} = \frac{F_{t,T}}{1 + r_{t,T}}. \quad (4.22)$$

After a minor rearrangement (line 1, below) we can then use the notation CEQ as in [4.21], to conclude that

$$\begin{aligned} F_{t,T} &= E_t(\tilde{S}_T) \frac{1 + r_{t,T}}{1 + r_{t,T} + RP_{t,T}} \\ \Rightarrow \text{CEQ}_t(\tilde{S}_T) &= E_t(\tilde{S}_T) \frac{1 + r_{t,T}}{1 + r_{t,T} + RP_{t,T}} \end{aligned} \quad (4.23)$$

$$\approx E_t(\tilde{S}_T) \frac{1}{1 + RP_{t,T}}. \quad (4.24)$$

The last line is only an approximation of the true relation [4.23]. We merely add it to show why the fraction on the right-hand side of [4.23] is called the risk adjustment.

#### Example 4.17

Suppose your finance professor offers you a 1-percent share in the next-year royalties from his finance textbook, with an expected value, next year, of USD 3,450,000. Given the high risk (☺), the market would discount this at 10 percent—3 risk-free plus a 7 risk premium. The CEQ would be

$$\text{CEQ} = 3,450,000 \frac{1.03}{1.10} = 3,450,000 \times 0.936363636 = 3,090,000. \quad (4.25)$$

Thus, the market would be indifferent between this proposition and USD 3,090,000 for sure. You could unload either of these in the market at a common PV,

$$\frac{3,450,000}{1.10} = 3,090,000 = \frac{3,090,000}{1.03}. \quad (4.26)$$

The risk-adjusted expected value plays a crucial role in the theory of international finance. As we shall see in the remainder of this chapter and later in the book, the risk-adjusted expectation has many important implications for asset pricing as well as for corporate financial decisions.

#### 4.4.5 Implications for Spot Values; the Role of Interest Rates

In principle, we can see the spot value as the expected future value of the investment—including interest earned—corrected for risk and then discounted at the appropriate risk-free rate. In this subsection we consider the role of interest rates and changes therein, hoping to clear up any confusion that might exist in your mind. Notably, we have noted that a forward discount, *i.e.* a relatively high foreign riskfree rate, signals a weak currency. Yet we see central banks increase interest rates when their

currency is under pressure, and the result often is an appreciation of the spot value. How can increasing the interest rate, a sign of weakness, strengthen the currency?

The relation to watch is, familiarly,

$$S_t = \frac{\text{CEQ}_t(\tilde{S}_T)(1 + r_{t,T}^*)}{1 + r_{t,T}}. \quad (4.27)$$

We also need to be clear about what is changing, here, and what is held constant. Let's use an example to guide our thoughts.

**Example 4.18**

Assume that the CAD (home currency) and GBP risk-free interest rates,  $r_{t,T}$  and  $r_{t,T}^*$ , are both equal to 5 percent *p.a.* Then, from Equation [4.27], initially no change in  $S$  is expected, after risk adjustments: the spot rate is set equal to the certainty equivalent future value. Now assume that bad news about the British (foreign) economy suddenly leads to a downward revision of the expected next-year spot rate from, say, CAD/GBP 2 to 1.9. From Equation [4.27], if interest rates remain unchanged, the current spot rate would immediately react by dropping from 2 to 1.9, too. Exchange rates, like any other financial price, anticipate the future.

Now if the Bank of England does not like this drop in the value of the GBP, it can prop up the current exchange rate by increasing the British interest rate. To do this, the UK interest rate will need to be increased from 5 percent to over 10.5 percent, so that  $S_t$  equals CAD/GBP 2 even though  $\text{CEQ}_t(\tilde{S}_T)$  equals 1.9:

$$\frac{1.9 \times 1.10526}{1.05} = 2. \quad (4.28)$$

Thus, the higher UK interest rate does strengthen the current GBP spot rate, all else being equal.

But this still means that the currency is weak, in the sense that the GBP is still expected to drop towards 1.9, after risk-adjustment, in the future. Actually, in this story the pound strengthens now so that it can become weak afterwards. So there is no contradiction, since “strengthening” has to do with the immediate spot rate (which perks up as soon as the UK interest rate is raised, holding constant the CEQ), while “weakening” refers to the expected movements in the future.

A second comment is that, in the example, the interest-rate hike merely postpones the fall of the pound to a risk-adjusted 1.9. In this respect, however, this partial analysis may be incomplete, because a change in interest rates may also affect expectations. For instance, if the market believes that an increase in the British interest rate also heralds a stricter monetary policy, this would increase the expected future spot rate, and reinforce the effect of the higher foreign interest rate. Thus, the BoE would get away with a lower rise in the UK interest rate than in the first version of the story.

Of course, if expectations change in the opposite direction, the current spot rate may decrease even when the foreign interest rate is increased. For example, if the

foreign interest rate rises by a smaller amount than was expected by the market, this may then lead to a downward revision of the expected future exchange rate and, ultimately, a drop in the spot value.

**Example 4.19** \_\_\_\_\_

Suppose that the current interest rates are equal to 5 percent p.a. in both Canada (the home country) and the UK, and the current and expected exchange rate are CAD/GBP 2. The Bank of England now increases its interest rate to 5.025 percent p.a. in an attempt to stem further rises in UK inflation. It is quite possible that this increase in interest rates is interpreted by the market as a negative signal about the future state of the UK economy (the BoE wants to slow things down) or as insufficient to stop inflation. So the market may revise expectations about the CAD/GBP exchange rate from 2 to 1.95. Thus, the change in the interest rate is insufficient to match the drop in the expected exchange rate. Instead of appreciating, the current exchange rate drops to CAD/GBP  $1.95 \times 1.0525/10, 5 = 1.955$ . \_\_\_\_\_

Note the difference between the two examples. In the first, there was a drop in expectations that was perfectly offset by the interest rate—for the time being, that is: the drop is just being postponed, by assumption. In the second example the interest rate change came first, and then led to a revision of expectations. So we need to be careful about expectations too when the role of interest rates is being discussed.

Let's return to more corporate-finance style issues:

#### 4.4.6 Implications for the Valuation of Foreign-Currency Assets or Liabilities

The certainty-equivalent interpretation of the forward rate implies that, for the purpose of corporate decision making, one can use the forward rate to translate foreign-currency-denominated claims or liabilities into one's domestic currency without much ado. Indeed, identifying the true expectation and then correcting for risk would just be reinventing the wheel: the market has already done this for you, and has put the result upon the Reuters screen. This makes your life much more simple. Rather than having to tackle a valuation problem involving a risky cash flow—the left-hand side of Equation [4.22]—we can simply work with the right-hand side where the cash flow is risk free. With risk-free cash flows, it suffices to use the observable domestic risk-free rate for discounting purposes.

**Example 4.20** \_\_\_\_\_

If the domestic CLP risk-free return is 21 percent, effective for 4 years, and the 4-year forward rate is CLP/NOK 110, then the (risk-adjusted) economic value of a NOK 5,000 4-year zero-coupon bond can be found as

$$\frac{\text{NOK } 5,000 \times \text{CLP/NOK } 110}{1.21} = \text{CLP } 454,545.45, \quad (4.29)$$

without any fussing and worrying about expectations or risk premia. \_\_\_\_\_

As illustrated in the example, the expected spot rate is not needed in order to value this position, and discounting can be done at the risk-free rate of return. In contrast, if you had tried to value the position using the left-hand side of Equation [4.22], you would probably have had to discount the expected future spot rate at some risk-adjusted rate. Thus, the first problem would have been to estimate the expected future spot rate. Unlike the forward rate, this expectation is not provided in the newspaper or on the Reuters screens. Second, you would have had to use some asset-pricing theory like the international Capital Asset Pricing Model (CAPM) to calculate a risk-adjusted discount rate that we use on the left-hand side of Equation [4.22]. In this second step, you would run into problems of estimating the model parameters, not the mention the issue of whether the CAPM is an appropriate model. In short, the forward rate simplifies decision making considerably. We shall use this concept time and again throughout this text.

#### 4.4.7 Implication for the Relevance of Hedging

In this mercifully short last section before the wrap-up, we briefly touch upon the implications of the zero initial value for the relevance of hedging, that is, using financial instruments to reduce or even entirely eliminate the impact of exchange rates on the cash flow. Forward contracts are a prime instrument for this purpose: if one contractually fixes the rates at which future exchanges will be made, then the future spot rate no longer affects your bank account—at least not for those transactions.

The zero-value property has been invoked by some (including me, when *very* young) as implying that such hedging does not add value, or more precisely that any value effects must stem from market imperfections. This is wrong, but it took me some time to figure out exactly what was wrong.

The argument views the firm as a bunch of cash-flow-generating activities, to which a hedge is added. The cash flow triggered by the hedge is some positive or negative multiple of  $\tilde{S}_T - F_{t,T}$ , and its PV is zero the moment the hedge contract is signed. True, it's value will become non-zero one instant later, but we have no clue whether this new value will be positive or negative; so our knowledge that the zero-value property is short-lived is of no use for hedging decisions. But does zero initial value mean that the hedge is (literally) worthless? There can be, and will be, a value effect if the firm's other cash flows are affected. For instance, the chances that adverse currency movements wipe out so much capital that R&D investments must be cut, or that banks increase their risk spreads on loans, or customers desert the company, or the best employees leave like rats from a sinking ship—the chances that all these bad things happen should be lower, after hedging. Perhaps the firm is so well off that the probability of painful bad luck—bad luck that affects operations, not just the bank account—is zero already. If so, count your blessings: hedging will probably not add any value. But many firms are not in so comfortable a position. To them hedging adds value because it improves the future cash-flow prospects from

other activities. We return to this in Chapter 12.

## 4.5 CFO's Summary

In this chapter, we have analyzed forward contracts in a perfect market. We have discovered that forward contracts are essentially packaged deals, that is, transactions that are equivalent to a combination of a loan in one currency, a spot transaction, and a deposit in the other currency. In this sense, the forward contract is a distant forerunner of financial engineering. We have also seen how exchange markets and money markets are interlinked and can be used for arbitrage transactions and for identifying & comparing the two ways to make a particular transaction.

In perfect markets, it does not matter whether one uses forward contracts as opposed to their money-market replications. This holds for any possible transaction and its replication. For instance, a German firm will neither win nor lose if it replaces a EUR deposit by a swapped USD deposit since, from Interest Rate Parity, the two are equivalent. Or, more precisely, *if* it matters, it's because of market imperfections like spreads, or because the firm's other cash flows are affected too, but not because of the pure exchange of a FC cash flow by one in HC. We turn to market imperfections in the next chapter, and to the relevance of hedging in Chapter 12.

We have also found that the value of a forward contract is zero. This means that, everything else being the same, our German firm will not win or lose if it replaces a EUR deposit by an *uncovered* USD deposit. Again, a big word of caution is in order, here, because the "everything else being the same" clause is crucial. The above statement is perfectly true about the pure PV of two isolated cash flows, one in EUR and one in USD. But if the firm is so levered, the USD deposit is so large, and the EUR/USD so volatile that the investment could send the firm into receivership, then the dollar deposit would still not be a good idea—not because of the deposit *per se*, but because of the repercussions it could have on the firm's legal fees and interest costs and asset values. In short, the deposit's cash flows can have interactions with the company's other business, and these interactions might affect the firm's value.

A last crucial insight is that the forward rate is the market certainty equivalent, that is, the market's expectation corrected for any risks it thinks to be relevant. This insight can save a company a lot of time. It is also fundamental for the purpose of asset pricing. For cashflows with a known FC component, the logic is of course straightforward: (a) an asset with a known FC flow  $C_T^*$  is easy to hedge: sell forward  $C_T^*$  units of FC; (b) the hedged asset is easy to value;  $(C_T^* \times F_{t,T}) / (1 + r_{t,T})$ ; and (c) the unhedged asset must have the same value because the hedge itself has zero initial value and because a risk-free FC amount  $C_T^*$  cannot be affected by the hedge. Interestingly, under some distributional assumptions we can also apply the logic to cashflows that are highly non-linear functions of the future exchange rate. We return to this issue in Chapter 9.

Forward currency contracts have been around for centuries. A more recent instrument is the forward or futures contract on interest rates. Since forward interest contracts are not intrinsically ‘international’ and many readers may already know them from other sources, I stuff them into appendices, but if they new to you be warned that we are going to use them further on in this book. A key insight is that interest rates (spot and forward interest rates, and “yields at par”) are all linked by arbitrage. Forward interest rates in various currencies are likewise linked through the forward markets.

## 4.6 Appendix: Interest Rates, Returns, and Bond Yields

### 4.6.1 Links Between Interest Rates and Effective Returns

We have defined the effective (rate of) return as the percentage difference between the initial (time- $t$ ) value and the maturity (time- $T$ ) value of a nominally risk-free asset over a certain holding period. For instance, suppose you deposit CLP 100,000 for six months, and the deposit is worth CLP 105,000 at maturity. The six-month effective return is:

$$r_{t,T} = \frac{105,000 - 100,000}{100,000} = 0.05 = 5\%. \quad (4.30)$$

In reality, bankers never quote effective rates of returns; they quote interest rates. An interest rate is an annualized return, that is, a return extrapolated to a twelve-month horizon. In the text, we emphasize this by adding an explicit *per annum* (or *p.a.*) qualification whenever we mention an interest rate. However, annualization can be done in many ways. It is also true that, for any system, there is a corresponding way to de-annualize the interest rate into the effective return—the number you need.

1. Annualization can be “simple” (*i.e.* linear): 5 percent for six months is extrapolated linearly, to 10 percent *per annum* (*p.a.*). A simple interest rate is the standard method for term deposits and straight loans when the time to maturity is less than one year. Conversely, the effective return is computed from the quoted simple interest rate as

$$1 + r_{t,T} = 1 + (T - t) \times [\text{simple interest rate}]. \quad (4.31)$$

---

#### Example 4.21

Let  $(T - t) = 1/2$  year, and the simple interest rate 10 percent *p.a.* Then:

$$1 + r_{t,T} = 1 + 1/2 \times 0.10 = 1.05. \quad (4.32)$$


---

2. Annualization can also be compounded, with a hypothetical reinvestment of the interest. Using this convention, an increase from 100 to 105 in six months would lead to a constant-growth-extrapolated value of  $105 \times 1.05 = 110.25$  after another six months. Thus, under this convention, 5 percent over six months corresponds to 10.25 percent *p.a.*. Conversely, the return is computed from the quoted compound interest rate as:

$$1 + r_{t,T} = (1 + [\text{compound interest rate}])^{T-t}. \quad (4.33)$$

**Example 4.22**

Let  $(T - t) = 1/2$ , and the compound interest rate 10.25 percent *p.a.*; then

$$1 + r_{t,T} = 1.1025^{1/2} = \sqrt{1.1025} = 1.05. \quad (4.34)$$

---

Compound interest is the standard method for zero-coupon loans and investments (without interim interest payments) exceeding one year.

3. Banks may also compound the interest every quarter, every month, or even every day. The result is an odd mixture of linear and exponential methods. If the interest rate for a six-month investment is  $i$  *p.a.*, compounded  $m$  times per year, the bank awards you  $i/m$  per subperiod of  $1/m$  year. For instance, the *p.a.* interest rate may be  $i = 6$  percent, compounded four times per year. This means you get  $6/4 = 1.5$  percent per quarter. Your investment has a maturity of six months, which corresponds to two capitalization periods of one quarter each. After compounding over these two quarters, an initial investment of 100 grows to  $100 \times (1.015)^2 = 103.0225$ , implying an effective rate of return of 3.0225 percent. Thus, the effective return is computed from the quoted interest rate as:

$$1 + r_{t,T} = \left( 1 + \frac{[\text{quoted interest rate}]}{m} \right)^{(T-t)m}. \quad (4.35)$$

**Example 4.23**

Let  $(T - t) = 1/2$ , and the compound interest rate 9.878% with quarterly compounding; then

$$1 + r_{t,T} = (1 + 0.098781/4)^{1/2 \times 4} = 1.05. \quad (4.36)$$

---

You may wonder why this byzantine mixture of linear and exponential is used at all. In the real world it is used when the bank has a good reason to understate the effective interest rate. This is generally the case for loans. For deposits, the reason may be that the quoted rate is capped (by law, like the U.S.' former Regulations Q and M; or because of a cartel agreement amongst banks). In finance theory, the mixture of linear and exponential is popular in its limit form, the continuously compounded rate:

4. In the theoretical literature, the frequency of compounding is often carried to the limit ("continuous compounding", *i.e.*  $m \rightarrow \infty$ ). From your basic math course, you may remember that:

$$\lim_{m \rightarrow \infty} (1 + x/m)^m = e^x, \quad (4.37)$$

where  $e = 2.7182818$  is the base of the natural (Neperian) logarithm. Conversely, the return is computed from the quoted interest rate  $\rho$  as:

$$1 + r_{t,T} = \lim_{m \rightarrow \infty} \left(1 + \frac{\rho}{m}\right)^{(T-t)m} = e^{\rho(T-t)} \quad (4.38)$$

---

**Example 4.24**

Let  $(T - t) = 1/2$ , and assume the continuously compounded interest rate equals 9.75803 percent. Then:

$$1 + r_{t,T} = e^{0.0975803/2} = 1.05. \quad (4.39)$$


---

Note the following link between the continuously and the annually compounded rates  $i$  and  $\rho$ :

$$(1 + i) = e^{\ln(1+i)} \Rightarrow (1 + i)^{T-t} = e^{\ln(1+i) \cdot (T-t)} \Rightarrow \ln(1 + i) = \rho. \quad (4.40)$$

5. Bankers' discount is yet another way of annualizing a return. This is often used when the present value is to be computed for T-bills, promissory notes, and so on—instruments where the time- $T$  value (or “face value”) is the known variable, not the PV like in the case of a deposit or a loan. Suppose the time- $T$  value is 100, the time to maturity is 0.5 years, and the p.a. discount rate is 5 percent. The present value will then be computed as

$$PV = 100 \times (1 - 0.05/2) = 97.5. \quad (4.41)$$

Conversely, the return is found from the quoted bankers' discount rate as:

$$1 + r_{t,T} = \frac{1}{1 - (T - t) \times [\text{banker's discount rate}]}. \quad (4.42)$$

---

**Example 4.25**

Let  $(T - t) = 1/2$  and the p.a. bankers' discount rate 9.5238 percent. Then:

$$1 + r_{t,T} = \frac{1}{1 - 1/2 \times 0.095238} = 1.05. \quad (4.43)$$


---

In summary, there are many ways in which a bank can tell its customer that the effective return is, for instance, 5 percent. It should be obvious that what matters is the effective return, not the stated *p.a.* interest rate or the method used to annualize the effective return. For this reason, in most of this text, we use effective returns. This allows us to write simply  $(1 + r_{t,T})$ . If we had used annualized interest rates, all formulas would look somewhat more complicated, and would consist of many versions, one for each possible way of quoting a rate.

### 4.6.2 Common Pitfalls in Computing Effective Returns

To conclude this Appendix we describe the most common mistakes when computing effective returns. The first is forgetting to de-annualize the return. Always convert the bank's quoted interest rate into the effective return over the period  $(T - t)$ . And use the correct formula:

#### Example 4.26

Let  $T - t = 3/4$  years. What are the effective rates of return when a banker quotes a 4 percent *p.a.* rate, to be understood as, alternatively, (1) simple interest, (2) standard compound interest, (3) interest compounded quarterly, (4) interest compounded monthly, (5) interest compounded daily, (6) interest compounded a million times a year, (7) interest compounded continuously, and (7) bankers' discount rate?

convention	formula	result (1+r)
simple	$1 + 3/4 \times 0.04$	1.030000000
compound, $m = 1$	$(1 + 0.04)^{3/4}$	1.029852445
compound, $m = 4$	$(1 + 0.04/4)^{4 \times 3/4}$	1.030301000
compound, $m = 12$	$(1 + 0.04/12)^{12 \times 3/4}$	1.030403127
compound, $m = 360$	$(1 + 0.04/360)^{360 \times 3/4}$	1.030452817
compound, $m = 1,000,000$	$(1 + 0.04/10^6)^{10^6 \times 3/4}$	1.030454533
continuous compounding	$e^{0.04 \times 3/4}$	1.030454533
banker's discount	$1/(1 - 3/4 * 0.04)$	1.030927835

Second, it is important to remember that there is an interest rate (or a discount rate) for every maturity,  $(T - t)$ . For instance, if you make a twelve-month deposit, the *p.a.* rate offered is likely to differ from the *p.a.* rate on a six-month deposit. Students sometimes forget this, because basic finance courses occasionally assume, for expository purposes, that the *p.a.* compound interest rate is the same for all maturities. Thus, there is a second pitfall to be avoided—using the wrong rate for a given maturity.

The third pitfall is confusing an interest rate with an internal rate of return on a complex investment. Recall that the return is the simple percentage difference between the maturity value and the initial value. This assumes that there is only one future cash flow. But many investments and loans carry numerous future cash flows, like quarterly interest payments and gradual amortisations of the principal. We shall discuss interest rates on multiple-payment instruments in the next appendix. For now, simply remember that the interest rate on, say, a five-year loan with annual interest payments should not be confused with the interest rate on a five-year instrument with no intermediate interest payments (zero-coupon bond).

#### Example 4.27

If a newspaper says the 10-year bond rate is 6%, this means that a bond with an annual coupon of 6% can be issued at par. That is, the 6 % is a “yield at par” on

bullet bonds with annual coupons. What we need, in this chapter, are zero-coupon rates rather than yields at par. \_\_\_\_\_

## 4.7 Appendix: The Forward Forward and the Forward Rate Agreement

### 4.7.1 Forward Contracts on Interest Rates

You may know that loans often contain options on interest rates (caps and floors; see Chapter 16). Besides interest-rate options, there are also forward contracts on interest rates. Such forward contracts come under two guises: the Forward Forward contract (FF), and the Forward Rate Agreement (FRA).

A *Forward Forward contract* is just a forward deposit or loan: it fixes an interest rate today ( $=t$ ) for a deposit or loan starting at a future time  $T_1(>t)$  and expiring at  $T_2(>T_1)$ .

#### Example 4.28

Consider a six-to-nine-month Forward Forward contract for 10m Brazilian Real at 10 percent *p.a.* (simple interest). This contract guarantees that the return on a three-month deposit of BRL 10m, to be made six months from now, will be  $10\%/4 = 2.5\%$ . At time  $T_1$  (six months from now), the BRL 10m will be deposited, and the principal plus the agreed-upon interest of 2.5 percent will be received at time  $T_2$  (nine months from now).

A more recent, and more popular, variant is the *Forward Rate Agreement*. Under an FRA, the deposit is *notional*—that is, the contract is about a hypothetical deposit rather than an actual deposit. Instead of effectively making the deposit, the holder of the contract will settle the gain or loss in cash, and pay or receive the present value of the difference between the contracted forward interest rate and market rate that is actually prevailing at time  $T_1$ .

#### Example 4.29

Consider a nine-to-twelve-month CAD 5m notional deposit at a forward interest rate of 4 percent *p.a.* (that is, a forward return of 1 percent effective). If the Interbank Offer Rate after nine months ( $T_1$ ) turns out to be 3.6 percent *p.a.* (implying a return of 0.9 percent), the FRA has a positive value equal to the difference between the promised interest (1 percent on CAD 5m) and the interest in the absence of the FRA, 0.9 percent on CAD 5m. Thus, the investor will receive the present value of this contract, which amounts to:

$$\text{market value FRA} = \frac{5m \times (0.01 - 0.009)}{1.009} = 4955.40. \quad (4.44)$$

In practice, the reference interest rate on which the cash settlement is based is computed as an average of many banks' quotes, two days before  $T_1$ . The contract stipulates how many banks will be called, from what list, and how the averaging is done. In the early eighties, FRAs were quoted for short-term maturities only. Currently, quotes extend up to ten years.

### 4.7.2 Why FRAs Exist

Like any forward contract, an FRA can be used either for hedging or for speculation purposes. Hedging may be desirable in order to facilitate budget projections in an enterprise or to reduce uncertainty and the associated costs of financial distress. Banks, for example, use FRAs, along with T-bill futures and bond futures, to reduce maturity mismatches between their assets and liabilities. For instance, a bank with average duration of three months on the liability side and twelve months on the asset side, can use a three-to-twelve month FRA to eliminate most of the interest risk. An FRA can, of course, serve as a speculative instrument too.

As we shall show in the next paragraph, FFS (or FRAs) can be replicated from term deposits and loans. For financial institutions, and even for other firms, FRAs and interest futures are preferred over such synthetic FRAs in the sense that they do not inflate the balance sheet.

#### Example 4.30

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Suppose that you need a three-to-six month forward loan for JPY 1b. Replication would mean that you borrow (somewhat less than) JPY 1b for six months and invest the proceeds for three months, until you actually need the money. Thus, your balance sheet would have increased by JPY 1b, without any increase in profits or cash flows compared to the case where you used a Forward Forward or an FRA. —

The drawback of using an FF or FRA is that there is no organized secondary market. However, as in the case of forward contracts on foreign currency, long-term FRA contracts are sometimes collateralized or periodically recontracted. This reduces credit risk. Thus, a fairly active over-the-counter market for FRAs is emerging.

### 4.7.3 The Valuation of FFS (or FRAs)

We now discuss the pricing of FFS (or FRAs—both have the same value): How should one value an outstanding contract, and how should the market set the normal forward interest rate at a given point in time? In this section, we adopt the following notation:

$t_0$  : the date on which the contract was initiated

$t (\geq t_0)$  : the moment the contract is valued

$T_1$  : the expiration date of the forward contract (that is, the date that the gains or losses on the FRA are settled, and the date at which the notional deposit starts)

$T_2 (> T_1)$  : the expiration date of the notional deposit

$r_{t_0, T_1, T_2}^f$  : the effective return between  $T_1$  and  $T_2$ , without annualization, promised on the notional deposit at the date the FRA was signed,  $t_0$ .

First consider a numerical example:

**DoItYourself problem 4.6**

Consider a FF under which you will deposit JPY 1b in nine months and receive 1.005b in twelve. The effective risk-free rates for these maturities are  $r_{t,T_1}=0.6\%$  and  $r_{t,T_2}=0.81$ , respectively. Value each of the PN's that replicate the two legs of the FF. Compute the net value.

The generalisation is obvious. Below, we take a notional deposit amount of 1 (at  $T_1$ ):

$$\begin{aligned} \text{PV, at } t, \text{ of a unit FF} &= \underbrace{\frac{\text{promised in-}}{\text{flow at } T_2}}_{\text{"asset" PN}} - \underbrace{\frac{\text{promised out-}}{\text{flow at } T_1}}_{\text{"liability" PN}} \\ &= \frac{1 + r_{t_0, T_1, T_2}^f}{1 + r_{t, T_2}} - \frac{1}{1 + r_{t, T_1}}. \end{aligned} \quad (4.45)$$

In one special case we can consider the expiry moment ( $t = T_1$ ):

**DoItYourself problem 4.7**

Derive, from this general formula, our earlier cash-settlement equation,

$$\text{PV, at } T_1, \text{ of a unit FF} = \frac{r_{t_0, T_1, T_2}^f - r_{T_1, T_2}}{1 + r_{T_1, T_2}}. \quad (4.46)$$

The other special case worth considering is the value at initiation ( $t_0 = t$ ). We know that this value must be zero, like for any standard forward contract, so this provides a way to relate the forward rate to the two spot rates, all at  $t$ :

**DoItYourself problem 4.8**

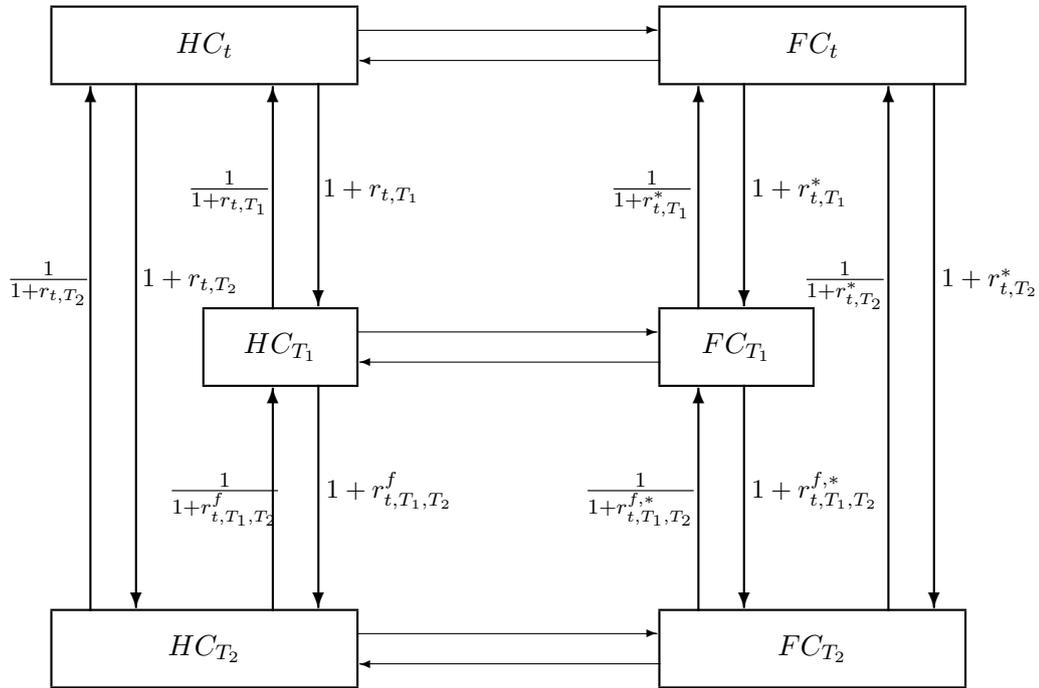
Derive, from the general formula, the relation between the time- $t$  spot and forward rates:

$$(1 + r_{t, T_1})(1 + r_{t, T_1, T_2}^f) = 1 + r_{t, T_2}; \quad (4.47)$$

$$\Leftrightarrow r_{t, T_1, T_2}^f = \frac{1 + r_{t, T_2}}{1 + r_{t, T_1}} - 1. \quad (4.48)$$

The left-hand side of the first equality, Equation [4.47], has an obvious interpretation: it shows the gross return from a synthetic deposit started right now ( $t$ ) and expiring at  $T_2$ , made not directly, but replicated by making a  $t$ -to- $T_1$  spot deposit which is rolled over (i.e. re-invested, here including the interest earned) via a  $T_1$ -to- $T_2$  forward deposit. So the money is contractually committed for the total  $t$ -to- $T_2$  period, and the total return is fixed right now—two ingredients that also characterize a  $t$ -to- $T_2$  deposit. In that light, Equation [4.47] just says that the direct and the synthetic  $t$ -to- $T_2$  deposits should have the same return.

Figure 4.6: Spot and Forward Money Markets (with International Links)



As in the case of currency forwards, no causality is implied by our way of expressing Equation [4.48]. The three rates are set jointly and have to satisfy Equation [4.48], that's all. As in Chapter 4, one could argue that causality, if any, may run from the forward interest rate towards the spot rate because the forward rate reflects the risk-adjusted expectations about the future interest rate. We shall use Equation [4.48] when we discuss eurocurrency futures, in the Appendix to Chapter 6.

There is an obvious no-arbitrage version of this. In Figure 4.6 we combine two of our familiar spot-forward currency diagrams, one for future date  $T_1$  and the other for date  $T_2$ . The focus, this time, is not on the exchange markets, so the horizontal lines that refer to currency deals are made thinner. The forward deposits and loans are shown as transactions that transform  $T_1$ -dated money into  $T_2$  money (the deposit) or vice versa (the loan), and the multiplication factors needed to compute the output from a transaction, shown next to the arrows, are  $(1+r^f)$  and  $1/(1+r^f)$ , respectively. This diagram shows that every spot or forward money-market deal can be replicated, which helps you in shopping-around problems. The diagram also helps identifying the no-arb constraints.

**DoItYourself problem 4.9**

We have already shown how to replicate the  $t$ -to- $T_2$  deposit. In the table below, add

the replications for the other transactions and check that they generate the gross returns shown in the rightmost column.

replicand	replication	output value <sup>#</sup>
$t$ -to- $T_2$ deposit	spot deposit $t$ to $T_1$ , rolled over forward $T_1$ to $T_2$	$(1 + r_{t,T_1})(1 + r_{t,T_1,T_2}^f)$
forward deposit $T_1$ to $T_2$		$\frac{1+r_{t,T_2}}{1+r_{t,T_1}}$
spot deposit $t$ to $T_1$		$\frac{1+r_{t,T_2}}{1+r_{t,T_1,T_2}^f}$
$t$ -to- $T_2$ loan		$\frac{1}{(1+r_{t,T_1})(1+r_{t,T_1,T_2}^f)}$
forward loan $T_1$ to $T_2$		$\frac{1+r_{t,T_1}}{1+r_{t,T_2}}$
spot loan $t$ to $T_1$		$\frac{1+r_{t,T_1,T_2}^f}{1+r_{t,T_2}}$

<sup>#</sup>: output is computed for input value equal to unity.

But the diagram shows not just the replication possibilities: there are also two no-arb constraints inside each money market, corresponding to, respectively, the clockwise and counterclockwise roundtrips. You start for instance in the time- $T_2$  box, issuing a PN note dated  $T_2$ . You discount it immediately and make a synthetic deposit  $t$  to  $T_2$ . The constraint is that the proceeds of this deposit be no higher than 1, the amount you owe to the holder of the PN:

$$HC_{T_2} \rightarrow HC_t \rightarrow HC_{T_1} \rightarrow HC_{T_2}$$

$$1 \times \frac{1}{1 + r_{t,T_2}} \times (1 + r_{t,T_1}) \times (1 + r_{t,T_1,T_2}^f) \leq 1. \quad (4.49)$$

#### DoItYourself problem 4.10

Identify the other no-arb trip in the home money market and write the corresponding constraint. Combine it with constraint [4.49] and check that you find back Equation [4.47].

To seasoned arbitrageurs like you it is easy to add bid- and ask-superscripts to the rates of return (and to the exchange rates, while you are at it). The no-arb constraints still are [synthetic bid]  $\leq$  [ask] and [bid]  $\leq$  [synthetic ask], with the synthetics computed from the worst-possible-combination versions of the perfect-market replication that you just worked out yourself.

Before we move to other markets, there is another set of no-arb constraints and shopping-around opportunities to be discussed, namely those created via international linkages rather than relations within each money market. Remember one can

replicate a currency- $X$  spot deposit or loan by swapping a currency- $Y$  spot deposit or loan into currency  $X$ . Well, the same holds for forward deposits and loans. For instance, in the few years when USD or GBP had FRA markets but minor European currencies had not (yet), pros replicated the missing FRA's by swapping USD or GBP FRA's into, say, NLG via a forward-forward currency swap, in or out. Such swaps are described in Chapter 5, and consist of a currency forward in one direction combined with a second currency forward, in the other direction. In short, when the starting date of a deposit or loan is not spot but  $n$  days forward, we just replace the spot leg of the swap by the appropriate forward leg.

#### 4.7.4 Forward Interest Rates as the Core of the Term Structure(s)

Remember that forward exchange rates, being the risk-adjusted expectations, are central in any theory of exchange rates. In the same way, forward interest rates can be viewed as the core of every theory of interest rates. The standard *expectations theory* hypothesizes that forward interest rates are equal to expected future spot rates, and Hicks added a *risk premium*, arguing—to use a post-Hicksian terminology—that the beta risk of a bond is higher the longer its time to maturity. Modern versions would rather state everything in terms of PN prices rather than interest rates, but would agree with the basic intuition of the old theories: forward rates reflect expectations corrected for risks.

Various theories or models differ as to how expectations evolve and risk premia are set, but once the forward rates are set, the entire term structure follows. We illustrate this with a numerical example, and meanwhile initiate you to the various interest-rate concepts: spot rates, yields at par for bullet bonds, and other yields at par.

We start from the first row in Table 4.3, which shows a set of forward rates. For simplicity of notation, current time  $t$  is taken to be zero, so that a one-period forward rate looks like  $r_{0,n-1,n}^f$  rather than the more laborious  $r_{t,t+n-1,t+n}^f$ . For some reason—mainly expectations, one would presume—there is a strong “hump” in the forward rates: they peak at the 3-to-4 year horizon. (A period is of unspecified length, in the theories; but let's agree they are years).<sup>10</sup> The initial spot rate and the forward rate with starting date 0 are, of course, the same. Below we show you the formulas to be used in a spreadsheet to generate all possible term structures (TS).

**The TS of spot rates** is obtained in two steps. First we cumulate the forward

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<sup>10</sup>For this reason the only non-arbitrary theory is one that works with continuous time, where a period lasts  $dt$  years. But for intro courses this has obvious drawbacks.

Table 4.3: Term Structures and their Linkages

	rate orPV factor for $n = \dots$						
	1	2	3	4	5	6	7
forward rate $p.p.$ , $r_{0,n-1,n}^f$	0.0300	0.0350	0.0380	0.0400	0.0360	0.0300	0.0200
$1 + r_{0,n-1,n}^f$	1.0300	1.0350	1.0380	1.0400	1.0360	1.0300	1.0200
$1 + r_{0,n} = \prod_{j=1}^n (1 + r_{0,j-1,j}^f)$	1.0300	1.0661	1.1066	1.1508	1.1923	1.2280	1.2526
$\bar{r}_{0,n} = (1 + r_{0,n})^{1/n} - 1$	0.0300	0.0325	0.0343	0.0357	0.0358	0.0348	0.0327
$PV_{0,n} = 1/(1 + r_{0,n})$	0.9709	0.9380	0.9037	0.8689	0.8387	0.8143	0.7984
PV annuity, $a_{0,n} = \sum_{j=1}^n PV_{0,j}$	0.9709	1.9089	2.8126	3.6816	4.5203	5.3346	6.1330
$R_{0,n} : \frac{1 - (1 + R_{0,n})^{-n}}{R_{0,n}} = a_{0,n}$	0.0300	0.0316	0.0330	0.0340	0.0346	0.0347	0.0342
$c_{0,n} = (1 - PV_{0,n})/a_{0,n}$	0.0300	0.0325	0.0342	0.0356	0.0357	0.0348	0.0329

**Key** Starting from an assumed set of forward rates I compute the set of ‘spot’ zero-coupon rates (lines 3 and 4) and present value factors (line 5). This allows us to find the PV of a constant unit annuity (line 6) and the corresponding yield. Finally I compute the yield at par for a bullet bond. The math is described in the text.

rates into effective spot rates, using Equation [4.47]:

$$1 + r_{0,n} = \prod_{j=1}^n (1 + r_{0,j-1,j}^f). \tag{4.50}$$

The rate on the left-hand side is the effective rate we have always used in this book. But for the theory of term structures it is useful to convert the effective rate to a per-period rate, which we denote by  $\bar{r}$ . The computation is

$$1 + \bar{r}_{0,n} := (1 + r_{0,n})^{1/n}. \tag{4.51}$$

The spot rates are the yields to maturity on zero-coupon bonds expiring at  $n$ . Note how the per-period gross rates are rolling geometric averages—numerically close to simple averages—of all gross forward rates between times 0 and  $n$ .<sup>11</sup> See how the strong hump is very much flattened out by the rolling-averaging, and the peak pushed to  $n = 5$  instead of  $n = 4$  for the forward rates. A second alternative way to work with the effective rate is to compute the PV of one unit of HC payable at time  $n$ ,

$$PV_{0,n} = 1/(1 + r_{0,n}). \tag{4.52}$$

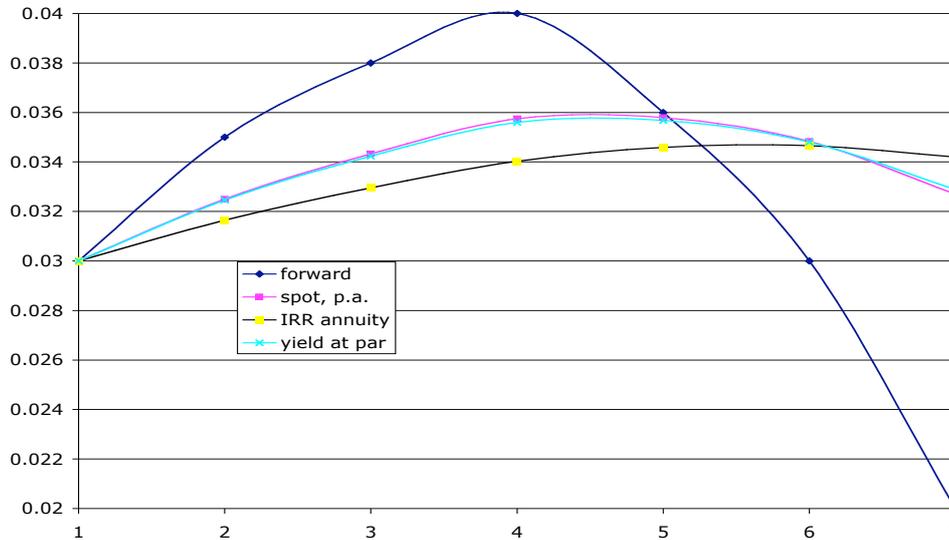
**The TS of yields for constant-annuity cash flows** is a different TS. It is not as popular as the TS of yields at par for bullet loans, see below, but it is convenient to look at this one first. Any yield or *internal rate of return* is the compound “flat” rate that equates a discounted stream of known future cash flows  $C_j$  to an observed present value:

$$y : \frac{C_1}{1 + y} + \frac{C_2}{(1 + y)^2} + \dots + \frac{C_n}{(1 + y)^n} = \text{observed PV}. \tag{4.53}$$

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<sup>11</sup>A gross rate is  $1 + r$ ,  $r$  being the net rate we always use in this text.

Figure 4.7: **Term Structures: Forward, Spot, and Two Types of Yields**



Here we look at the special case  $C_j = 1, \forall j$ , the constant unit-cash-flow stream, the right-hand side of the above equation. Let us first find the PV of the constant stream. Since we already know the PV of a single unit payment made at  $n$ , the PV of a stream paid out at times  $1, \dots, n$  is simply the sum. This special PV is denoted as  $a_{0,n}$ , from “annuity”. We compute its value for various  $n$  as

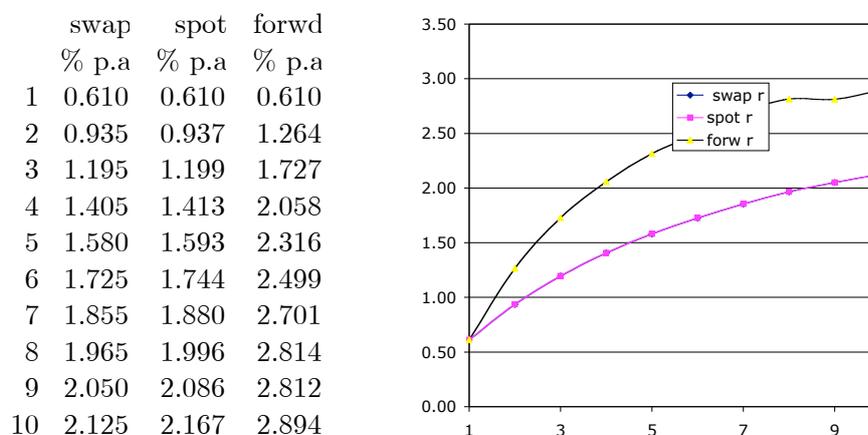
$$a_{0,n} = \sum_{j=1}^n PV_{0,j}. \tag{4.54}$$

Next we find the yield that equates this PV to the discounted cash flows. When the cash flows all equal unity, the left-hand side of Equation [4.53] is equal to  $(1 - (1 + y)^{-n})/y$ , but the  $y$  that solves the constraint must still be found numerically, using e.g. a spreadsheet tool. In the table the result is found under the label  $R_{0,n}$ . Note how this yield is an analytically non-traceable mixture of all spot rates. The hump is flattened out even more, and its peak pushed back one more period.

**The ts of yields at par for bullet loans** is defined as a yield that sets the PV of a bullet loan equal to par. But it is known that to get a unit value the yield must be set equal to the coupon rate. So we can now rephrase the question as follows: how do we set the coupon rate  $c$  such that the PV’s of the coupons and the principal sum to unity?

$$c_{0,n} : \underbrace{c_{0,n} \times a_{0,n}}_{\text{PV of coupons}} + \underbrace{PV_{0,n} \times 1}_{\text{PV of amortization}} = 1 \Rightarrow c_{0,n} = \frac{1 - PV_{0,n}}{a_{0,n}}. \tag{4.55}$$

Again, this is numerically much closer to the spot rates than the yield on constant-annuity loans, and the reason obviously is that the bullet loan is closer to a zero-

Figure 4.8: **Extracting spot and forward rates from the JPY swap rates**

**Key** Swap and spot are so close, in these figures, that on the graph you no longer spot the difference; you need to look at the numbers.

coupon loan—especially in an example where, like in ours, interest rates are generally low. In Figure 4.7, which shows the four term structures graphically, those for swap and spot rates overlap almost perfectly.

This illustrates how the TS of forward rates contains all information for pricing, so that TS theories are basically theories about forward rates. It also gives you a feeling how swap dealers set their long-term interest rates, yields-at-par for bullet bonds. Like us here, they construct them from spot rates. These spot rates, in turn, are obtained from PV factors extracted, via regression analysis, from bond prices in the secondary market. You can, of course, reverse-engineer all this and extract PV factors from swap rates, and thence forward rates. Then you may ask the question whether there seem to be good reasons for the forward rate to behave as it appears to do, and perhaps invest or disinvest accordingly. For instance, in Figure 4.8 we have taken the JPY swap rates from Chapter 7 and extracted spot and forward rates. Spot rates are familiarly close to swap rates (yields at par for bullet bonds) but the forward rate, equally familiar, moves much faster than the spot rate (a rolling average). So one can ask the question how these forward rates compare to your expectations about future spot rates.

A second insight you should remember is that there is no such thing as “the” TS. Academics would first think of the TS of spot rates or forward rates (and be precise about that). But practitioners first think about the TS of yields at par for bullet bonds, the numbers one sees in the newspaper or that are quoted by swap dealers (who call them swap rates). Many traditional practitioners would apply the yield-at-par rates for any instrument, whether it is a bullet loan or not. That can imply serious errors and inconsistencies.

Yields are funny. Even if we just consider bullet bonds, there still is a yield for

every *total* time to maturity  $n$ . So in the situation depicted in Table 4.3 a coupon paid at time 1 would be discounted at 3% if it is part of a one-period bond, 3.25% if it is part of a two-period bond, 3.42% if it is part of a three-period bond, and so on. It is much more logical to work with a discount rate for every payment horizon, regardless of what bond pays out the money, rather than a discount rate for every bond, regardless of the date of the payment.

## 4.8 Test Your Understanding

### 4.8.1 Quiz Questions

1. Which of the following statements are correct?
  - (a) A forward purchase contract can be replicated by: borrowing foreign currency, converting it to domestic currency, and investing the domestic currency.
  - (b) A forward purchase contract can be replicated by: borrowing domestic currency, converting it to foreign currency, and investing the foreign currency.
  - (c) A forward sale contract can be replicated by: borrowing foreign currency, converting it to domestic currency, and investing the domestic currency.
  - (d) A forward sale contract can be replicated by: borrowing domestic currency, converting it to foreign currency, and investing the foreign currency.
  - (e) In a perfect market you could forbid forward markets (on the basis of anti-gambling laws, for instance), and nobody would give a fig.
  - (f) The spot rate and the interest rate determine the forward price.
  - (g) No, the forward determines the spot.
  - (h) No, the forward and the spot and the foreign interest rate determine the domestic interest rate.
  - (i) No, there are just four products that are so closely related that their prices cannot be set independently.
  
2. What's wrong with the following statements?
  - (a) The forward is the expected future spot rate.
  - (b) The sign of the forward premium tells you nothing about the strength of a currency; it just reflects the difference of the interest rates.
  - (c) The difference of the interest rates tells you nothing about the strength of a currency; it just reflects the forward premium or discount.
  - (d) The forward rate is a risk-adjusted expectation but the spot rate is independent of expectations
  - (e) A certainty equivalent tends to be above the risk-adjusted expectation because of the risk correction.
  - (f) A risk-adjusted expectation is always below the true expectation because we don't like risk.
  - (g) A risk-adjusted expectation can be close to, or above the true expectation. In that case the whole world would hold very little of that currency, or would even short it.
  - (h) Adding a zero-value contract cannot change the value of the firm; therefore a forward hedge cannot make the shareholders better off.

### 4.8.2 Applications

1. Check analytically the equivalence of the two alternative ways to do the following trips:
  - (a) Financing of international trade: you currently hold a FC claim on a customer payable at  $T$ , but you want cash HC instead.
  - (b) Domestic deposits: you currently hold spot HC and you want to park that money in HC, risk-free.
  - (c) You want to borrow HC for 3 months.
  - (d) Immunizing a HC debt: you want to set aside some of your cash HC so as to take care of a future FC debt.
  - (e) Borrowing FC. You want to borrow FC but a friend tells you that swapping a HC loan is much cheaper
  
2. You hold a set of forward contracts on EUR, against USD (=HC). Below I show you the forward prices in the contract; the current forward prices (if available) or at least the current spot rate and interest rates (if no forward is available for this time to maturity). Compute the fair value of the contracts.
  - (a) Purchased: EUR 1m 60 days (remaining). Historic rate: 1.350; current rate for same date: 1.500; risk-free rates (simple per annum): 3% in USD, 4% in EUR.
  - (b) Purchased: EUR 2.5m 75 days (remaining). Historic rate: 1.300; current spot rate: 1.5025; risk-free rates (simple per annum): 3% in USD, 4% in EUR.
  - (c) Sold: EUR 0.75m 180 days (remaining). Historic rate: 1.400; current rate for same date: 1.495; risk-free rates (simple per annum): 3% in USD, 4% in EUR.
  
3. 60-day interest rate (simple, p.a.) are 3% at home (USD) and 4% abroad (EUR). The spot rate moves from 1.000 to 1.001.
  - (a) What is the return differential, and what is the corresponding prediction of the change in the forward rate?
  - (b) What is the actual change in the forward rate?
  - (c) What is the predicted change in the swap rate computed from the return differential?
  - (d) What is the actual change in the swap rate?
  
4. 60-day interest rate (simple, p.a.) are 3% at home (USD) and 4% abroad (EUR). The spot rate is 1.250.
  - (a) Check that investing EUR 1m, hedged, returns as much as USD 1.25m

- (b) Check that if taxes are neutral, and the tax rate is 30%, also the after-tax returns are equal. (Yes, this *is* trivial.)
  - (c) How much of the income from swapped EUR is legally interest income and how much is capital gain or loss?
  - (d) If you do not have to pay taxes on capital gains and cannot deduct capital losses, would you still be indifferent between USD deposits and swapped EUR?
5. 60-day interest rate (simple, p.a.) are 3% at home (USD) and 4% abroad (EUR). The spot rate is 1.250.
- (a) Check that borrowing EUR 1m (=current proceeds, not future debt), hedged, costs as much as borrowing USD 1.25m
  - (b) Check that if taxes are neutral, and the tax rate is 30%, also the after-tax costs are equal. (Yes, this *is* trivial.)
  - (c) How much of the costs of borrowing swapped EUR is legally interest paid and how much is capital gain or loss?
  - (d) If you do not have to pay taxes on capital gains and cannot deduct capital losses, would you still be indifferent between USD loans and swapped EUR?
6. Groucho Marx, as Governor of Freedonia's central bank, has problems. He sees the value of his currency, the FDK, under constant attack from Rosor, a wealthy mutual-fund manager. Apparently, Rosor believes that the FDK will soon devalue from GBP 1.000 to 0.950.
- (a) Currently, both GBP and FDK interest rates are 6% *p.a.* By how much should Groucho change the one-year interest rate so as to stabilize the spot rate even if Rosor expects a spot rate of 0.950 in one year? Ignore the risk premium—that is, take 0.950 to be the certainty equivalent.
  - (b) If the interest-rate hike also affects Rosor's expectations about the future spot rate, in which direction would this be? Taking into account also this second-round effect, would Groucho have to increase the rate by more than your first calculation, or by less?