CHAPTER 5

Portfolio VaR for Market Risk

The previous chapter took one step in the direction of addressing the real-world complexities of many assets, namely, the nonlinearity of their returns with respect to some underlying risk factor. This chapter deals with another source of complexity, the dependence of returns jointly on several risk factors. With these two enhancements, the simple VaR techniques we studied in Chapter 3 become applicable to a far wider range of real-world portfolios.

A simple example of a security with several risk factors, which we mentioned in Chapter 3, is foreign exchange, which is typically held either in the cash form of an interest-bearing foreign-currency bank deposit, or the over-the-counter (OTC) derivatives form of a foreign exchange forward contract. In either form, foreign exchange is generally exposed not only to an exchange rate, but to several money-market rates as well. Another example is a foreign equity. If you are, say, a dollar-based investor holding the common stock of a foreign company, you are exposed to at least two risk factors: the local currency price of the stock and the exchange rate. As yet another example, if a long domestic equity position is hedged by a short index futures position, in an effort to neutralize exposure to the stock market, a small exposure to risk-free interest rates as well as the risk of poor hedging performance are introduced. Similar issues arise for most commodity and stock index positions, which are generally established via futures and forwards.

As we set forth these simple examples, it becomes clear that exposure to a single risk factor is the exception, not the rule. Many other derivatives and credit products also have joint exposure to several risk factors and nonlinearity with respect to important risk factors, so we need techniques for measuring VaR for multiple risk factors. These characteristics are closely related to the issue of how best to map exposures to risk factors.

In addition, most real-world portfolios contain several assets or positions. The techniques developed in Chapter 3 work for portfolios with several assets or positions if they are exposed to only one risk factor. One can conjure up a example, say, a long or short position in the cash equity of a single firm plus long or short positions in the stock via total return swaps (to be discussed in Chapter 12). There is still only one market risk factor, the equity price, but even this portfolio bears counterparty credit risk (to be discussed in Chapter 6). And it is hard to imagine what sort of investor would have this as their entire portfolio, rather than just one trade idea among others.

We beg in this chapter by introducing a framework that accommodates multiple risk factors, whether generated by a single security or in a portfolio. In this framework, the risk of the portfolio is driven by the volatilities of the individual risk factors and their correlation with one another. In the second part of this chapter, we discuss one important example of a single security exposed to several risk factors: Options are exposed, among other risk factors, to both the underlying asset and its implied volatility. This advances the discussion of option risk and nonlinearity begun in the previous chapter.

5.1 THE COVARIANCE AND CORRELATION MATRICES

We'll start by developing some concepts and notation for a portfolio exposed to several risk factors; we require not only the standard deviations of risk factor log returns, but also their correlations. Suppose our portfolio contains N risk factors $S_{1t}, \ldots, S_{Nt}$. We represent this list of risk factors as a vector

$$\mathbf{S}_t = (S_{1t}, \ldots, S_{Nt})$$

The vector of log returns is

$$\mathbf{r}_t = (r_{1t}, \ldots, r_{Nt}) = \left[\log\left(\frac{S_{1,t+\tau}}{S_{1,t}}\right), \ldots, \log\left(\frac{S_{N,t+\tau}}{S_{N,t}}\right)\right]$$

The vector of volatilities is $(\sigma_1, \sigma_2, \ldots, \sigma_N)$; each σ_n is the volatility of the return r_{nt} of the nth risk factor. The correlation matrix of the returns is

$$\begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{12} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1N} & \rho_{2N} & \cdots & 1 \end{pmatrix}$$

where ρ_{mn} is the correlation coefficient of log returns to risk factors m and n. This matrix is symmetric, since $\rho_{nm} = \rho_{mn}$.

The *covariance matrix* is computed from the vector of volatilities and the correlation matrix as the following *quadratic form*:

$$\boldsymbol{\Sigma} = (\sigma_{mn})_{m,n=1,\ldots,N} = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \cdots & \sigma_1\sigma_N\rho_{1N} \\ \sigma_1\sigma_2\rho_{12} & \sigma_2^2 & \cdots & \sigma_2\sigma_N\rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1\sigma_N\rho_{1N} & \sigma_2\sigma_N\rho_{2N} & \cdots & \sigma_N^2 \end{pmatrix}$$

$$= \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_N) \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{12} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1N} & \rho_{2N} & \cdots & 1 \end{pmatrix} \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_N).$$

The notation $\mathrm{diag}(x)$ means a square matrix with the vector x along the diagonal and zeroes in all the off-diagonal positions, so $\mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_N)$ represents:

$$\begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N \end{pmatrix}$$

It is often easier in programming to use matrices rather than summation, and this notation lets us express the covariance matrix as a product of matrices.

In order to construct a covariance matrix, the correlation matrix must be *positive semi-definite*. A matrix $\mathbf{A}$ is positive semi-definite if for any vector x, we have $x'\mathbf{A}x \geq 0$. The covariance matrix is then positive semi-definite, too. This means that for any portfolio of exposures, the variance of portfolio returns can't be negative.

To see how all this notation fits together, take the simple case of two risk factors $\mathbf{S}_t = (S_{1t}, S_{2t})$. The covariance matrix is

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} \\ \sigma_1\sigma_2\rho_{12} & \sigma_2^2 \end{pmatrix}$$

5.2 MAPPING AND TREATMENT OF BONDS AND OPTIONS

Mapping is the process of assigning risk factors to positions. Chapter 3 alluded to these issues in the context of short positions, as did the last chapter in the context of options and fixed-income securities. In order to compute risk measures, we have to assign risk factors to securities. How we carry out this mapping depends on many things and is in the first instance a modeling decision.

In the last chapter, we gave an example of such a modeling choice: Bond risk can be measured using a duration-convexity approximation, or by treating a coupon bond as a portfolio of zero-coupon bonds. This modeling choice corresponds to a choice of mappings. In carrying out the duration-convexity approximation, we mapped the bond to a single risk factor, the yield. Bond prices can also be seen as depending jointly on several risk factors, namely, those determining the term structure of interest rates, by treating the bond as a portfolio of zero-coupon bonds. We then have to map it to multiple risk factors, a set of zero-coupon interest rates. The modeling choice brings with it a mapping, which in turn brings with it the need to adopt a portfolio approach to risk measurement. A related, but more complex example is corporate bond returns, which are driven not only by default-free interest rates or yields, but also depend on the additional risk factors that determine the credit spread.

Single domestic common equities are among the few assets that can readily be represented by a single risk factor, the time series of prices of the stock. But for portfolios of equities, one often uses factor models, rather than the individual stock returns. This is again a modeling decision. The value of the asset may be assumed to depend on some "fundamental" factors, in the sense of the arbitrage pricing model. Equities are often modeled as a function of the market factor, firm size, and valuation, to name only the classic Fama-French factors. The fundamental factors will capture common aspects of risk, and thus more accurately model return correlation than treating each equity's return stream as a risk factor in its own right.

Questions about mapping often boil down to what data are available from which we can draw inferences about risk. The data we have are, as a rule, not nearly as variegated as the risks. Data on individual bond spreads, for example, are hard to obtain. Some data, such as data on equity risk factors, have to be manufactured in a way that is consistent with the factor model being applied.

In Chapter 4, we expressed the value of a position with a nonlinear pricing function as $xf(S_t)$ (omitting then time argument for simplicity). We now let $\mathbf{x}$ represent a vector of M securities or positions, each a function of the N risk factors: $x_1 = f_1(\mathbf{S}_t), \ldots, x_M = f_M(\mathbf{S}_t)$.

The portfolio can then be written

$$V_t = \sum_{m=1}^{M} x_m f_m(\mathbf{S}_t)$$

where $\mathbf{S}_t$ is the time-t value of the vector of risk factors. Any one of the M securities, however, may be exposed to only a small subset of the N risk factors.

The P&L over the time interval τ is then approximately equal to the change in V_t:

$$V_{t+\tau} - V_t = \sum_{m=1}^{M} x_m \left[f_m(\mathbf{S}_{t+\tau}) - f_m(\mathbf{S}_t) \right]$$

The asset may be treated as linear: that is, an equity, an equity index, a commodity, a currency, or a default risk-free zero-coupon bond. The mapping in this case is straightforward. We treat each equity (or currency pair, etc.) as representing a risk factor. Many assets, however, have a more complicated nonlinear relationship with risk factors.

The gravity of the mapping choice in risk measurement is hard to overstate. As we see in Chapter 11, it is closely related to some of the difficult problems in risk measurement encountered during the subprime crisis.

5.3 DELTA-NORMAL VAR

For a single position with returns driven by a single risk factor, as we saw in Chapter 3, the VaR is easy to compute via a parametric formula. But for portfolios with more than one risk factor, there is no exact closed-form solution for calculating VaR.

The *delta-normal approach* is an approximate, parametric, closed-form approach to computing VaR. One of its virtues is that it is simple to compute. That means we don't have to do simulations, with all the programming and data manipulation burden that entails, or reprice securities, which can involve expensive repeated numerical solutions, as described in Chapter 4. Instead, we compute the value of an algebraic expression, just as in parametric VaR with a single risk factor. Another advantage is that we can exploit certain properties of the closed-form solution to get information about the risk contributions of particular securities or risk factors to the overall risk of the portfolio. We explore these properties and how to use the associated "drill-downs" in Chapter 13.

In the delta-normal approach, there are two approximations, in addition to the model's joint normal distributional hypothesis. We:

1. *Linearize* exposures to risk factors. This obviates the need for repricing. However, we cannot use a quadratic approximation, only a linear one.
2. Treat *arithmetic*, not log returns, as normally distributed.

5.3.1 The Delta-Normal Approach for a Single Position Exposed to a Single Risk Factor

To see how this approach is carried out, let's start with the simplest possible example, a single security exposed to a single risk factor, so $M = N = 1$. We'll do this to illustrate the techniques: Such simple portfolios don't really benefit from the shortcuts involved in delta-normal. Among the few examples of such positions are foreign currency positions held in non–interest bearing accounts or banknotes, common stocks, if mapped to the stock price rather than using a factor model, and short-term government bond positions.

Linearization In Chapter 4, we defined the delta of a *security* as the first derivative of its value $f(S_t)$ with respect to a risk factor S_t:

$$\delta_t \equiv \frac{\partial f(S_t)}{\partial S_t}$$

The *delta equivalent* of a *position* is its delta times the number of units or shares in the position x, evaluated at the most recent realization of the risk factor:

$$xS_t\delta_t = xS_t\frac{\partial f(S_t)}{\partial S_t}$$

The delta equivalent has approximately the same dollar P&L, but not necessarily even approximately the same market value as the position itself. It shows more or less the same sensitivity to risk factor fluctuations, but does not have the same value as the position. For example, an option's delta equivalent may be a good hedge, even though it has a market value quite different from that of the option itself.

For risk factors that are identical to non-derivative assets, such as currencies and stocks, and for exposures that move linearly with some set of risk factors, this is not an approximation, but exact. When the risk factor is identical to the security, we have

$$\delta_t = \frac{\partial f(S_t)}{\partial S_t} = 1$$

and the delta equivalent has the same market value as the position.

Arithmetic Return Approximation The VaR shock $z_* \sigma \sqrt{\tau}$, where

- α is the confidence level of the VaR, e.g. 0.99 or 0.95
- z_* is the ordinate of the standard normal distribution at which $\Phi(z) = 1 - \alpha$
- σ is the time-t annual volatility estimate
- τ is the time horizon of the VaR, measured as a fraction of a year

is a basic building block of a parametric VaR estimate, as we saw in Chapter 3. In parametric VaR for a long position in a single risk factor, we model P&L as lognormally distributed, so the VaR shock is $z_* \sigma \sqrt{\tau}$. In the delta-normal approach, we treat the P&L as *normally* distributed, so the arithmetic return corresponding to the VaR shock is $z_* \sigma \sqrt{\tau}$ rather than $e^{z_* \sigma \sqrt{\tau}} - 1$, even though the underlying asset price model is one of logarithmic returns. The same caveats apply here as in Chapter 3, where we introduced this approximation of parametric VaR for a single risk factor as Equation (3.3); if the return shock is not more than a few percentage points, the difference will be small, but for combinations of a higher confidence level, a longer horizon, and a higher volatility, the difference can be large.

Putting together the linearization of the value function and the arithmetic return approximation, the P&L shocks for a single risk factor are measured by

$$V_{t+\tau} - V_t \approx r_t x \delta_t S_t$$

and the VaR is estimated as

$$\text{VaR}_t(\alpha, \tau)(x) = -z_* \sigma \sqrt{\tau} x \delta_t S_t$$

Example 5.1 (VaR of a Foreign Currency) Suppose a U.S. dollar–based investor holds a position in euros worth \$1 million. We'll compute its 1-day, 99 percent VaR via the delta normal approach as of November 10, 2006, using the root mean square estimate of volatility. The portfolio value can be represented as

$$V_t = 1{,}000{,}000 = xS_t$$

where S_t represents the dollar price of the euro, equal on November 10, 2006, to 1.2863. The number of units of the euro is $x = \frac{1000000}{S_t}$, or € 777,424, and the delta is $\delta_t = 1$. In this mapping, we ignore the interest-rate market risk that arises from fluctuations in euro and U.S. dollar money market rates if the position is held as a forward or as a euro-denominated bank deposit.

Using the 91 business days (90 return observations) of data ending November 10, 2006, the annualized root mean square of the daily log changes in the euro exchange rate is 6.17 percent. The VaR is then just over 0.9 percent:

$$\begin{aligned}
\text{VaR}_t\left(0.99, \frac{1}{252}\right)(x) &= -z_*\sigma\sqrt{\tau}x\delta_t S_t \\
&= 2.33 \times 0.0617\sqrt{\frac{1}{252}} \times 777{,}424 \times 1.2863 \\
&= 2.33 \times 0.0617\sqrt{\frac{1}{252}} \times 1{,}000{,}000 \\
&= \$9{,}044
\end{aligned}$$

5.3.2 The Delta-Normal Approach for a Single Position Exposed to Several Risk Factors

In the next example, we use the delta-normal approach to measure the risk of a single position that is a function of several risk factors. The number of securities or positions is still $M = 1$, so x is still a scalar rather than a vector, but the number of risk factors is $N > 1$. So S_t now represents a vector and we also represent delta equivalents by the vector

$$\mathbf{d}_t = x\begin{pmatrix} S_{1t}\delta_{1t} \\ \dots \\ S_{nt}\delta_{Nt} \end{pmatrix}$$

where x is the number of units of the security. The VaR is

$$\mathrm{VaR}_t(\alpha, \tau)(x) = -z_* \sqrt{\tau}\sqrt{\mathbf{d}_t' \boldsymbol{\Sigma} \mathbf{d}_t} \tag{5.1}$$

For a two-factor position ($N = 2$),

$$\mathrm{VaR}_t(\alpha, \tau)(x) = -z_* \sqrt{\tau} x \sqrt{S_{1t}^2 \delta_{1t}^2 \sigma_1^2 + S_{2t}^2 \delta_{2t}^2 \sigma_2^2 + 2 S_{1t} S_{2t} \delta_{1t} \delta_{2t} \sigma_1 \sigma_2 \rho_{12}}$$

Consider, for example, a position in a foreign stock. The risk will not be the same for an overseas investor as for a local investor. For the overseas investor, value is a function of two risk factors:

$$f(\mathbf{S}_t) = S_{1t} \times S_{2t}$$

where S_{1t} is the local currency stock price and S_{2t} the exchange rate, in units of the overseas investor's currency per foreign currency unit. The position is equivalent to a portfolio consisting of a long position in the stock, denominated in the local currency, plus a long position in foreign exchange.

Example 5.2 (VaR of a Foreign Stock) Suppose a U.S. dollar–based investor holds \$1 million worth of the Istanbul Stock Exchange National 100 Index (also known as the ISE 100, Bloomberg ticker XU100). We'll consider the 1-day, 99 percent VaR of this portfolio as of November 10, 2006, using EWMA volatility and correlation estimates. We denote the local currency price of XU100 by S_1, while S_2 represents the exchange rate of the Turkish lira against the dollar, in USD per TRL; the time-t U.S. dollar value of the index is thus $S_{1t}S_{2t}$.

The portfolio value can be represented as

$$V_t = 1{,}000{,}000 = x f(\mathbf{S}_t) = x S_{1t} S_{2t}$$

Since we have set the value of the portfolio at \$1,000,000, the number of units of XU100 x is

$$x = \frac{1{,}000{,}000}{S_{1t} S_{2t}} = 3.65658 \times 10^7$$

using market data for November 10, 2006:

n	Description	S_{nt}	δ_{nt}	$d_{nt} = xS_{nt}\delta_{nt}$	σ_n
1	long XU100	39627.18	6.9013×10^{-7}	1,000,000	20.18
2	long TRL	6.9013×10^{-7}	39627.18	1,000,000	12.36

The deltas, delta equivalents, and EWMA volatilities (annualized, in percent) are also displayed. Note that both delta equivalents are equal to the \$1,000,000 value of the portfolio. The EWMA return correlation is 0.507.

In matrix notation, we have:

$$\begin{aligned}\boldsymbol{\Sigma} &= \begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}\begin{pmatrix}1 & \rho_{12}\\ \rho_{12} & 1\end{pmatrix}\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}\\ &= \begin{pmatrix}0.2018 & 0\\ 0 & 0.1236\end{pmatrix}\begin{pmatrix}1 & 0.5066\\ 0.5066 & 1\end{pmatrix}\begin{pmatrix}0.2018 & 0\\ 0 & 0.1236\end{pmatrix}\\ &= \begin{pmatrix}0.04074 & 0.012633\\ 0.01263 & 0.015269\end{pmatrix}\end{aligned}$$

and

$$\mathbf{d}_t = 1000000\begin{pmatrix}1\\ 1\end{pmatrix}$$

The VaR is then about 4.2 percent:

$$\begin{aligned}\text{VaR}_t\left(0.99, \frac{1}{252}\right)(x) &= -z_*\sqrt{\tau}\sqrt{\mathbf{d}_t'\boldsymbol{\Sigma}\mathbf{d}_t}\\ &= 2.33 \times 10^6\sqrt{\frac{1}{252}}\sqrt{\begin{pmatrix}1\\ 1\end{pmatrix}'\begin{pmatrix}0.04074 & 0.01263\\ 0.01263 & 0.01527\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}}\\ &= \$41{,}779\end{aligned}$$

5.3.3 The Delta-Normal Approach for a Portfolio of Securities

Now let's apply the delta-normal approach to general portfolios of $M > 1$ securities, exposed to $N > 1$ risk factors. Recall that the value of a portfolio can be represented

$$V_t = \sum_{m=1}^{M} x_m f_m(\mathbf{S}_t)$$

where $f_m(\mathbf{S}_t)$ is the pricing function for the m-th security. The delta equivalent of the portfolio is the sum of the delta equivalents of the positions. The number of units of each of the M positions is now a vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix}$$

In measuring portfolio risk, we are interested in the portfolio's total exposure to each risk factor. So first, we need to add up exposures of different securities to each risk factor. The delta equivalent for each risk factor is

$$\sum_{m=1}^{M} x_m S_{nt} \frac{\partial f_m(\mathbf{S}_t)}{\partial S_{nt}} = \sum_{m=1}^{M} x_m S_{nt} \delta_{mnt} \qquad n = 1, \ldots, N$$

The vector of delta equivalents of the portfolio is thus

$$\mathbf{d}_t = \begin{pmatrix} \sum_m^M x_m S_{1t} \delta_{m1t} \\ \cdots \\ \sum_m^M x_m S_{Nt} \delta_{mNt} \end{pmatrix}$$

We want to express this in matrix notation, again because it is less cumbersome in print and because it gets us closer to the way the procedure might be programmed. We now have a $N \times M$ matrix of deltas:

$$\Delta_t = \begin{pmatrix} \delta_{11t} & \delta_{21t} & \cdots & \delta_{M1t} \\ \delta_{12t} & \delta_{22t} & \cdots & \delta_{M2t} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1Nt} & \delta_{2Nt} & \cdots & \delta_{MNt} \end{pmatrix}$$

If security m is not exposed to all the risk factors in S_t, then some of its first derivatives will be zero, that is, $\delta_{nmt} = 0$, and make no contribution to the delta equivalent of the portfolio to that risk factor.

Using this convenient notation, we can now express the vector of delta equivalents as

$$\mathbf{d}_t = \text{diag}(\mathbf{S}_t)\Delta_t\mathbf{x}$$

Since each of the risk factors is assumed to be lognormal, the portfolio returns, too, are normal.

Once we have aggregated the exposures to the risk factors, we can use the variance-covariance matrix of the risk factor returns to measure portfolio risk. Letting $\mathbf{\Sigma}$ now denote an estimate of the covariance matrix, the τ-period P&L variance in dollars is estimated as

$$\tau\mathbf{d}_t'\mathbf{\Sigma}\mathbf{d}_t$$

and the P&L volatility in dollars as

$$\sqrt{\tau}\sqrt{\mathbf{d}_t'\mathbf{\Sigma}\mathbf{d}_t}$$

At this point, the simplifying assumption of treating returns as arithmetic rather than logarithmic comes into play: This simple matrix expression isn't possible if we measure P&L shocks exponentially. The VaR is

$$\text{VaR}_t(\alpha,\tau)(\mathbf{x}) = -z_*\sqrt{\tau}\sqrt{\mathbf{d}_t'\mathbf{\Sigma}\mathbf{d}_t}$$

This is the same expression as in Equation (5.1), for the case of one security with multiple risk factors. What has changed here is only the way we compute the vector of delta equivalents.

In the case of two securities, each exposed to one risk factor, we have

$$\mathbf{d}_t = \begin{pmatrix} x_1 S_{1t}\delta_{11t} \\ x_2 S_{2t}\delta_{22t} \end{pmatrix}$$

Just as a matter of notation, we have security m mapped to risk factor n, $m, n = 1, 2$. The VaR is then

$$\begin{aligned}\text{VaR}_t(\alpha,\tau)(\mathbf{x}) &= -z_*\sqrt{\tau}\sqrt{\begin{pmatrix} x_1 S_{1t}\delta_{11t} \\ x_2 S_{2t}\delta_{22t} \end{pmatrix}' \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} \\ \sigma_1\sigma_2\rho_{12} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} x_1 S_{1t}\delta_{11t} \\ x_2 S_{2t}\delta_{22t} \end{pmatrix}} \\ &= -z_*\sqrt{\tau}\sqrt{(x_1 S_{1t}\delta_{11t})^2\sigma_1^2 + (x_2 S_{2t}\delta_{22t})^2\sigma_2^2 + 2x_1x_2S_{1t}S_{2t}\delta_{11t}\delta_{22t}\sigma_1\sigma_2\rho_{12}}\end{aligned}$$

If all the delta equivalents are positive, it can helpful to express VaR as a percent of portfolio value. For example, if $\delta_{11t} = \delta_{22t} = 1$, we can divide by $x_1 S_{1t} + x_2 S_{2t}$ to express VaR as

$$\begin{aligned}\text{VaR}_t(\alpha, \tau)(\mathbf{x}) &= -z_* \sqrt{\tau}\sqrt{\omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1^2\sigma_1\omega_2\sigma_1\sigma_2\rho_{12}} \\ &= -z_*\sqrt{\tau}\sigma_p\end{aligned}$$

where the ω_m are the shares of $x_m S_{mt}$ in V_t, $m = 1, 2$, with $\omega_1 + \omega_2 = 1$, and

$$\sigma_p = \sqrt{\omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\sigma_1\sigma_2\rho}$$

is the volatility of portfolio returns as a percent of initial market value.

Example 5.3 (A Portfolio Example of the Delta-Normal Approach) We'll calculate VaR results for a portfolio of five securities. The securities and the risk factors they are mapped to are listed in Table 5.1. We have already encountered two of the securities and three of the risk factors in our earlier examples of the delta-normal approach.

The portfolio is assumed to be U.S.–dollar-based, hence risk is measured in dollar terms. Each position has a market value of \$1,000,000. The risk factors are identified by their Bloomberg tickers. The number of shares or units of each security is

$$\mathbf{x} = \frac{1{,}000{,}000}{f(\mathbf{S}_t)} = \begin{pmatrix} 777{,}424 \\ -117{,}410{,}029 \\ -726 \\ 1{,}000{,}000 \\ 36{,}565{,}786 \end{pmatrix}$$

The matrix of deltas is

$$\Delta_t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -7.8 & 0 \\ 0 & 0 & 0 & 0 & 6.90132 \times 10^{-7} \\ 0 & 0 & 0 & 0 & 39627.2 \end{bmatrix}$$

TABLE 5.1 Portfolio Description

			Securities
m	**Description**	$f(\mathbf{S}_t)$	**Detailed Description**
1	long EUR	S_{1t}	Long spot EUR vs. USD
2	short JPY	S_{2t}	Short spot JPY vs. USD
3	short SPX	S_{3t}	Short S&P 500 via futures; mapped to SPX Index
4	long GT10	$p(S_{4t}) = 1.0000$	Price of a long on-the-run U.S. 10-year Treasury note with a duration of 7.8; duration mapping applied
5	long ISE 100	$S_{5t}S_{6t}$	Long Istanbul Stock Exchange 100 Index via futures, with no currency hedge; mapped to XU100 Index
			Risk Factors
	Ticker	**Last Price**	**Description**
S_{1t}	EUR Crncy	1.2863	Spot price of EUR 1 in USD
S_{2t}	JPY–USD Crncy	0.008517	Spot price of JPY 1 in USD
S_{3t}	SPX Index	1376.91	Closing price of S&P 500 Index
S_{4t}	GT10 Govt	0.0458	Yield of on-the-run U.S. 10-year note
S_{5t}	XU100 Index	39627.2	Closing price of Istanbul Stock Exchange 100 Index
S_{6t}	TRL Crncy	6.90132×10^{-7}	Spot price of 1 (old) TRL in USD

with the m-th column representing the exposure of security m to each risk factor. The vector of delta equivalents is therefore:

$$\mathbf{d}_t = \text{diag}(\mathbf{S}_t)\Delta_t\mathbf{x} = \begin{pmatrix} 1{,}000{,}000 \\ -1{,}000{,}000 \\ -1{,}000{,}000 \\ -357{,}240 \\ 1{,}000{,}000 \\ 1{,}000{,}000 \end{pmatrix}$$

The delta equivalent exposure of the bond position to the yield risk factor per dollar's worth of bond is equal to minus one, times the yield as a decimal, times the bond's modified duration, or -0.0458×7.8.

Next, we need the statistical properties of the risk factors. In the delta-normal approach, these are the volatilities and correlations, which we calculate using the EWMA/RiskMetrics approach with a decay factor of 0.94. But the delta-normal approach can be used together with any other estimator of these parameters. The vector of annualized volatility estimates is

Position	σ_n
EUR	0.0570
JPY	0.0644
SPX	0.0780
GT10	0.1477
XU100	0.2018
TRL	0.1236

and the estimated correlation matrix is

	EUR	JPY	SPX	GT10	XU100
JPY	0.75				
SPX	−0.08	−0.05			
GT10	−0.58	−0.68	−0.09		
XU100	0.25	0.26	0.25	−0.22	
TRL	0.13	−0.09	0.00	0.18	0.51

The estimated covariance matrix is then

$$\boldsymbol{\Sigma} = \text{diag}\begin{pmatrix} 0.0570 \\ 0.0644 \\ 0.0780 \\ 0.1477 \\ 0.2018 \\ 0.1236 \end{pmatrix} \begin{pmatrix} 1.00 & 0.75 & -0.08 & -0.58 & 0.25 & 0.13 \\ 0.75 & 1.00 & -0.05 & -0.68 & 0.26 & -0.09 \\ -0.08 & -0.05 & 1.00 & -0.09 & 0.25 & 0.00 \\ -0.58 & -0.68 & -0.09 & 1.00 & -0.22 & 0.18 \\ 0.25 & 0.26 & 0.25 & -0.22 & 1.00 & 0.51 \\ 0.13 & -0.09 & 0.00 & 0.18 & 0.51 & 1.00 \end{pmatrix} \text{diag}\begin{pmatrix} 0.0570 \\ 0.0644 \\ 0.0780 \\ 0.1477 \\ 0.2018 \\ 0.1236 \end{pmatrix}$$

$$= \begin{pmatrix} 0.003246 & 0.002751 & -0.000343 & -0.004863 & 0.002887 & 0.000931 \\ 0.002751 & 0.004148 & -0.000249 & -0.006456 & 0.003415 & -0.000741 \\ -0.000343 & -0.000249 & 0.006088 & -0.001081 & 0.003876 & 0.000018 \\ -0.004863 & -0.006456 & -0.001081 & 0.021823 & -0.006609 & 0.003223 \\ 0.002887 & 0.003415 & 0.003876 & -0.006609 & 0.040738 & 0.012635 \\ 0.000931 & -0.000741 & 0.000018 & 0.003223 & 0.012635 & 0.015269 \end{pmatrix}$$

Finally, substituing the values of $\mathbf{d}_t$ and $\boldsymbol{\Sigma}$, the VaR of the portfolio is \$43,285:

$$\text{VaR}_t\left(0.99, \frac{1}{252}\right)(\mathbf{x}) = 2.33\sqrt{\frac{1}{252}}\sqrt{\mathbf{d}_t'\boldsymbol{\Sigma}\mathbf{d}_t} = 43{,}285$$

In Chapter 13, we further analyze this portfolio to identify the main drivers of risk.

5.4 PORTFOLIO VAR VIA MONTE CARLO SIMULATION

We can also compute VaR via Monte Carlo or historical simulation. The process is similar to that described in Chapter 3 for a single position. The main difference is that, instead of simulating I values of a single random variable, we require I values of a multivariate random variable, each a vector with a length equal to the number of risk factors. The i-th simulation thread, for example, would use a vector random variable $(\tilde{\epsilon}_{i1}, \ldots, \tilde{\epsilon}_{iN})$, drawn from a zero-mean normal distribution with variance-covariance matrix $\mathbf{\Sigma}$. Rather than one return shock in each simulation thread, we have a set of return shocks $\sqrt{\tau}(\tilde{\epsilon}_{i1}, \ldots, \tilde{\epsilon}_{iN})$.[1]

In parametric VaR, we are bound to treat P&L as normally distributed in order to compute the closed form expression for the annual P&L variance $\mathbf{d}_t' \mathbf{\Sigma} \mathbf{d}_t$. In estimation via simulation, in contrast, we can return to the log-normal model of the P&L distribution. In each thread i of the simulation, we would then compute the vector of asset price shocks as

$$\tilde{\mathbf{S}}_{ti} = \mathbf{S}_0 e^{\tilde{\mathbf{r}}_i} = e^{\sqrt{\tau}} \left(S_{0,1} e^{\tilde{\epsilon}_{i1}}, \ldots, S_{0,N} e^{\tilde{\epsilon}_{iN}} \right) \qquad i = 1, \ldots, I$$

The simulated portfolio-level P&L shocks are computed from the simulated asset price shocks by multiplying each by the vector of deltas to get $\delta'_t \tilde{\mathbf{S}}_{t,i}$.

Table 5.2 displays the results for our sample portfolio. The VaR at the 99 percent confidence interval is the absolute value of the 10th or 11th worst portfolio outcome. The exact results are dependent on the simulations used and will vary due to simulation noise. The simulation noise can be damped to some extent by averaging the scenarios neighboring the VaR scenario, for example, by taking the average of the 10th or 11th worst outcome as the VaR. In the example, the VaR is somewhat larger computed via Monte Carlo than via parametric VaR, due to random fluctuations more than offsetting the effect of using logarithmic rather than arithmetic shocks.

[1]Our notation here is a bit different from that of Chapter 3. In the earlier discussion of one-factor VaR, the $\tilde{\epsilon}_i$ represented simulations from $N(0, 1)$, and were subsequently multiplied by an estimate of σ to obtain an estimate of the return shock $\sqrt{\tau}\hat{\sigma}\tilde{\epsilon}_i$. Here, $(\tilde{\epsilon}_{i1}, \ldots, \tilde{\epsilon}_{iN})$ does not need to multiplied by the volatility estimate, as it is generated using $\mathbf{\Sigma}$.

TABLE 5.2 Example of Portfolio VaR via Monte Carlo Simulation

i	EUR	JPY	SPX	GT10	XU100	TRL	$\tilde{V}^i - V_0$
1	−0.0054	−0.0033	−0.0063	0.0113	−0.0342	−0.0198	−53,040
2	−0.0062	−0.0052	0.0038	0.0050	−0.0325	−0.0143	−52,841
3	−0.0052	−0.0024	−0.0043	0.0049	−0.0263	−0.0258	−51,648
4	0.0013	0.0032	−0.0042	0.0035	−0.0326	−0.0189	−49,778
5	−0.0041	−0.0041	0.0068	0.0049	−0.0214	−0.0196	−49,075
6	−0.0030	−0.0030	0.0057	0.0110	−0.0255	−0.0141	−48,852
7	−0.0063	−0.0039	0.0041	0.0137	−0.0264	−0.0109	−48,454
8	−0.0035	−0.0018	0.0016	0.0032	−0.0303	−0.0142	−48,247
9	−0.0029	−0.0013	0.0009	0.0094	−0.0265	−0.0160	−47,777
10	−0.0060	−0.0039	0.0025	0.0048	−0.0175	−0.0231	−46,492
11	−0.0031	−0.0014	−0.0056	0.0181	−0.0323	−0.0112	−45,508
12	−0.0010	0.0013	0.0007	0.0012	−0.0318	−0.0102	−44,901
⋮	⋮	⋮	⋮				
997	−0.0012	−0.0052	−0.0012	−0.0023	0.0313	0.0205	58,451
998	0.0028	0.0052	0.0026	−0.0079	0.0469	0.0155	61,518
999	0.0086	0.0020	−0.0018	−0.0099	0.0279	0.0219	62,363
1,000	0.0035	0.0017	0.0027	−0.0156	0.0438	0.0241	73,737

The table displays a subset of the simulation results. The second through seventh columns display the returns of each risk factor in each scenario. The rightmost column displays the simulation results for changes in portfolio value. Horizontal lines mark the VaR scenarios.

5.5 OPTION VEGA RISK

In Chapter 4, we took a first look at the risks of options. But we took into account only an option's exposure to the price of its underlying asset. In reality, most options and option strategies have significant *vega risk*, that is, exposure to changes in implied volatility. Implied volatility thus becomes an additional risk factor to changes in the underlying asset. Implied volatility often has a strong correlation to asset returns, so a portfolio approach is necessary to correctly measure the risk even of a single option position.

We'll discuss vega in two steps. First, we take a look at the behavior of implied volatility and how it differs from what standard models suggest. This is an important subject, not only in the context of option pricing and option risks, but also because of what it tells us about the behavior of the underlying asset returns. We can use implied volatilities to compute the risk-neutral probability distributions we introduced in Chapter 2, using procedures to be outlined in Chapter 10. Later in this section, in the second part of our discussion of vega, we see how we can incorporate vega risk into overall market risk measurement for options.

The differences between the empirical behavior of impled volatility and that predicted by Black-Scholes encompass the behavior of implied volatility *over time*, and the pattern of prices of options with different exercise prices and maturities *at a point in time*. We will discuss each of these, and the challenges they present to option risk measurement, in turn.

5.5.1 Vega Risk and the Black-Scholes Anomalies

We start by defining implied volatility more carefully, a concept that did not exist before the Black-Scholes model became the standard for option modeling, and options and option-like derivatives became widespread during the financial innovation wave of the 1980s (as described in Chapter 1).

Vega risk arises because option prices do not behave precisely as predicted by the Black-Scholes model. The Black-Scholes model posits the same assumption about asset price behavior as the standard asset return forecasting model described in Chapter 2. In the Black-Scholes model, there is a unique, constant, never-changing volatility for each asset. In empirical fact, implied volatilities change widely and abruptly over time. Figure 5.1 shows

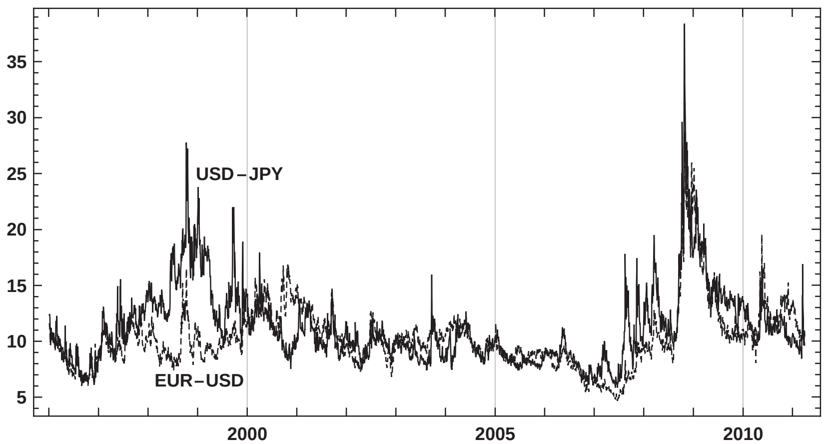

FIGURE 5.1 Time Variation of Implied Volatility
At-the-money (approximately 50δ) one-month implied volatility of the dollar-yen (solid line) and euro-dollar (dashed line) exchange rates, percent p.a., January 6, 1997, to April 1, 2011. Euro vols are for the Deutsche mark prior to January 1, 1999, and for the euro after January 1, 1999.
Source: Bloomberg Financial L.P.

one of myriad examples, the at-the-money forward implied volatilities of one-month foreign exchange options for the EUR-USD and USD-JPY exchange rates.

Let's look at a precise definition of implied volatility. Denote the quoted or market price of a European call option with maturity date T and exercise price X at time t by $c(t, T - t, X)$ and that of the congruent put by $p(t, T - t, X)$. We are using the notation $c(\cdot)$ and $p(\cdot)$ to indicate that we are focusing on observable market prices, as functions of the design parameters strike and maturity, rather than the theoretical Black-Scholes model values of Chapter 4 and Appendix A.3, $v(\cdot)$ and $w(\cdot)$. The implied volatility is the value obtained by solving

$$c(t, \tau, X) = v(S_t, \tau, X, \sigma, r, q) \tag{5.2}$$

for σ. Thus to be precise, we should refer to the *Black-Scholes* implied volatility, since it is defined by that particular model. Since we are now acknowledging the time-variation in implied volatility, we will add a time subscript to the symbol σ_t, which now stands for the value of σ that satisfies the equation in (5.2) of market to theoretical price.

Implied volatility is clearly itself volatile. What is its impact on position value? One answer is provided by the Black-Scholes option *vega*, the sensitivity of an option to changes in implied volatility. This sensitivity is analogous to the other "Greeks," those with respect to the underlying price and the time to maturity discussed in Chapter 4. The vega of a European call is:

$$\nu_{c,t} \equiv \frac{\partial}{\partial \sigma_t} v(S_t, \tau, X, \sigma_t, r, q)$$

Since options are often quoted in implied volatility units, vega risk can be thought of as the "own" price risk of an option position, as opposed to that of the underlying asset. Like gamma, vega is the same for European puts and calls with the same exercise price and tenor. The notion of a Black-Scholes vega is strange in the same way as the notion of a time-varying Black-Scholes implied volatility: We are using the model to define and measure a phenomenon that is incompatible with the model.

Implied volatility can be computed numerically or algebrically for European options. One needs to be careful with the units of vega. We generally have to divide $\nu_{c,t}$ by 100 before using it in a calculation. This is because $\nu_{c,t}$ is defined as the change in option value for a "unit" change in σ_t, meaning a 1 percent or 1 vol change in the level of volatility. But volatility has to be expressed as a decimal when using it in the Black-Scholes formulas.

For both puts and calls, and for any values of the other parameters, $\nu_{c,t} \geq 0$; higher volatility always adds value to an option. Figure 5.2 shows

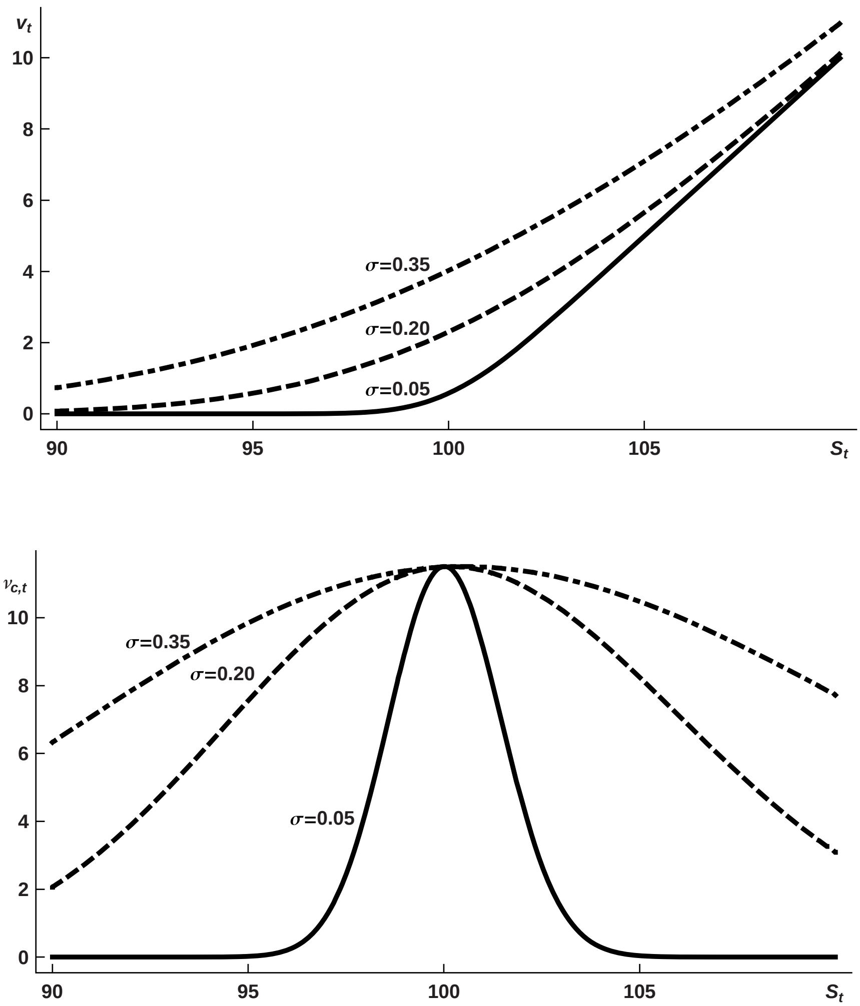

FIGURE 5.2 Option Vega
Upper panel: Black-Scholes value of a European call option as a function of underlying asset price for implied volatilities $\sigma = 0.05, 0.20, 0.35$.
Lower panel: Black-Scholes European call option vega as a function of underlying asset price for implied volatilities $\sigma = 0.05, 0.20, 0.35$.
The time to maturity is one month, and the risk-free rate and dividend rate on the underlying asset are both set to 1 percent.

two ways of looking at the impact of volatility on option value. The upper panel shows the standard hockey-stick plot of a call option's value against the underlying asset price for a low, an intermediate, and a high option price. For all values of the underlying price S_t, the option value is higher for a higher vol.

An option's value can be decomposed into two parts, *intrinsic value* and *time value*. The intrinsic value is the value the option would have if it were exercised right now. An at-the-money or out-of-the-money option has zero intrinsic value, while an in-the-money option has positive intrinsic value. The rest of the option's value is the time value derives from the possibility of ending in-the-money or deeper in-the-money. The time value is driven mainly by volatility and the remaining time to maturity; doubling the volatility, for a given τ, doubles the time value.

The curvature of the plots in the upper panel of Figure 5.2 also changes with volatility. At lower volatilities, the sensitivity of the vega—the "gamma of vega"—changes fast for changes in S_t, that is, there is a lot of curvature in the function. At higher volatilities, there is less curvature, and the sensitivity of the vega to underlying price is smaller.

The lower panel of Figure 5.2 shows the same phenomena in a different way. It plots the vega itself for different levels of implied volatility. When the option is at-the-money, the vega is the same regardless of the *level* of implied volatility. The peak vega occurs at the underlying price at which the option's *forward call delta* is 0.50, in other words, at the underlying price at which the option is "50 delta forward." The forward call delta is given by

$$\delta_{c,t}^{\text{fwd}} \equiv e^{r\tau} \frac{\partial}{\partial S_t} v(S_t, \tau, X, \sigma, r, q) = e^{r\tau} \delta_{c,t}$$

which is always on $(0, 1)$. The forward put delta is $1 - \delta_{c,t}^{\text{fwd}}$, so the peak vega for a European put occurs at the same underlying price as for a call. This price is close but not exactly equal to the current and forward prices. It is also equal to the 50th percentile of the future underlying price under the risk-neutral probability distribution, discussed in Chapter 10 and Appendix A.3.

From a risk monitoring standpoint, the implication of these properties is that at-the-money options, whether puts or calls, have the highest vega risk, and that vega risk increases with both implied volatility and time to maturity; vega risk is a function, not of σ_t alone, but of $\sigma_t\sqrt{\tau}$. In a low-volatility environment, such as that which prevailed in the early 2000s, the vega of a portfolio was likely to drop off rapidly if the market moved away from the strikes of the options. In a high-volatility environment such as that prevailing in the subprime crisis, high vega risk is more persistent.

Example 5.4 (Option Vega) Consider a one-month at-the-money forward European call options, exercising into one share of a stock trading at \$100. The implied volatility is 20 percent, and the risk-free rate and dividend rate on the underlying asset are both set to 1 percent.

The Black-Scholes model value of the option is \$2.3011. The delta of the option is 0.511, and its vega is 11.50. If the implied volatility were to increase to 21 percent, the model value of the option would increase to \$2.4161 = \$2.3011 + 0.1150.

5.5.2 The Option Implied Volatility Surface

Apart from time variation, implied volatilities display other important departures from the Black-Scholes model's predictions. The implied volatility "biases," as these systematic disagreements between real-world behavior and the model are sometimes called, can only be defined and discussed in the context of the Black-Scholes model. Implied volatility is a Black-Scholes concept; without a model, there are only option *prices*.

The key Black-Scholes biases are:

- Options with the same exercise price but different maturities generally have different implied volatilities, giving rise to a *term structure of implied volatility*. A rising term structure indicates that market participants expect short-term implied volatility to rise or are willing to pay more to protect against longer-term return volatility.
- Out-of-the-money call options often have implied volatilities that differ from those of equally out-of-the-money puts, a pattern called the *option skew*. As we see in Chapter 10, it indicates that the market perceives the return distribution to be skewed or is willing to pay more for protection against sharp asset price moves in one direction than in the other.
- Out-of-the money options generally have higher average implied volatilities than at-the-money options, a pattern called the *volatility smile*. It indicates that the market perceives returns to be leptokurtotic or is assuming a defensive posture on large asset price moves.

The latter two phenomena are sometimes referred to collectively as the volatility smile. Figure 5.3 shows a typical example, for the S&P 500. The implied volatility of each put or call is plotted against the forward call delta corresponding to the option's exercise price. We can see that options with high call deltas, that is, low exercise prices compared to the current index level, have much higher implied volatilities than options with high strikes and low call deltas.

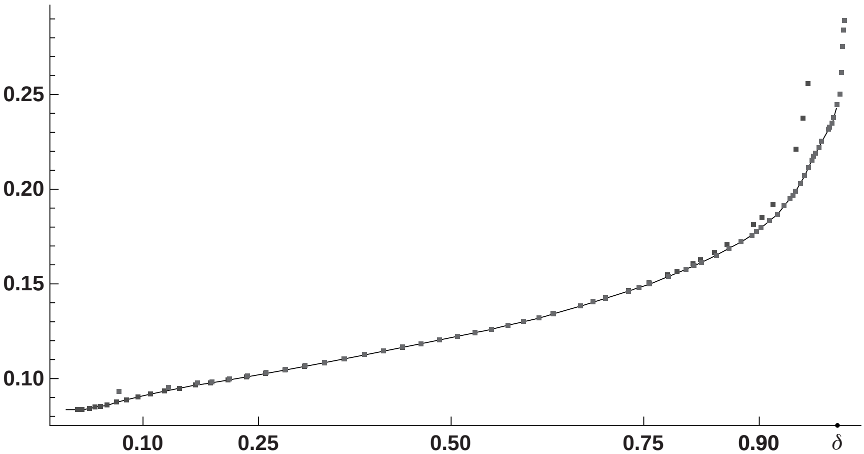

FIGURE 5.3 S&P 500 Implied Volatility Smile
June options, prices as of March 9, 2007. Exercise prices expressed as forward call deltas, annual implied volatilities expressed as decimals.

The volatility term structure and volatility smile together form the *implied volatility surface*. Figure 5.4 illustrates with the EUR-USD volatility surface for March 19, 2007. Apart from the skew in implied volatilities toward high-strike options that pay off if there is a sharp dollar depreciation, the surface also displays an upward-sloping term structure.

We can describe these observed volatility phenomena as a *volatility function* $\sigma_t(X, \tau)$ that varies both with exercise price and term to maturity, and also varies over time. Such a function is a far more realistic description of option prices than a constant, fixed volatility σ for each asset. The variation of at-the-money implied volatility over time is called the *volatility of volatility* or "vol of vol" (though the term is also sometimes used to describe the variability of historical volatility). Vol of vol is, for most portfolios containing options, the main driver of vega risk. However, the term structure and option skew also change over time, and are important additional sources of vega risk.

The exchange-traded and OTC markets have different conventions for trading options. The volatility smile has encouraged the OTC option markets to adapt the way in which they quote prices. Most of the time, most of the fluctuations in option prices are attributable to fluctuations in the underlying price. Changes in implied volatility tend to be less frequent. To make it easier to quote options, dealers in OTC option markets often quote the Black-Scholes implied volatility of an option rather than the price in

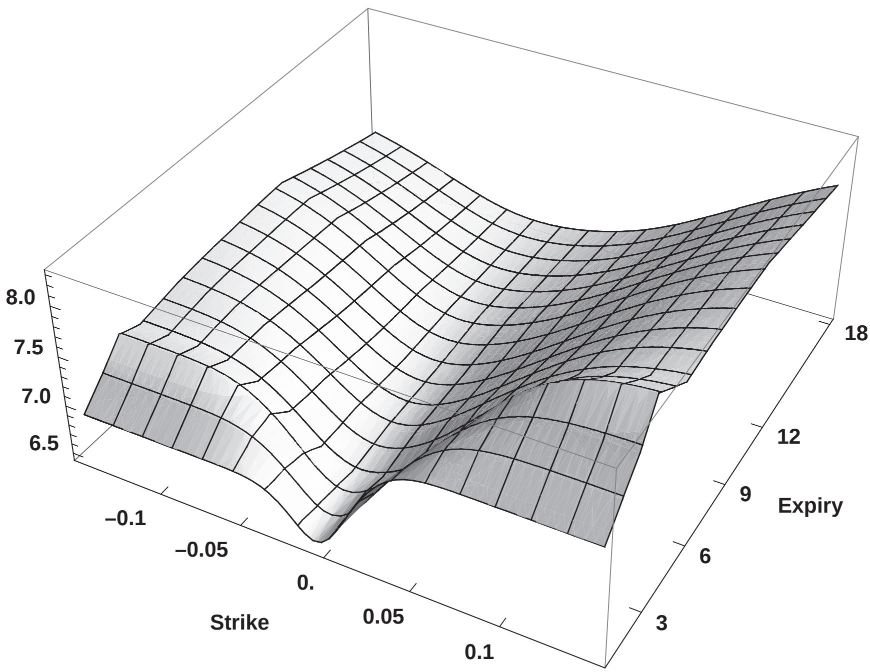

FIGURE 5.4 EUR-USD Volatility Surface
Strike expressed as decimal deviation from ATMF (at-the-money forward) strike. Data as of March 19, 2007. Spot FX rate \$1.33.

currency units. Traders use the Black-Scholes model even though they do not believe it holds exactly, because the implied volatilities of an option with a given maturity and strike will typically be steadier than the price in currency units. When a trade is agreed on based on implied volatility, it is easy to ascertain the current levels of the other market inputs to the price, the underlying price and financing rate that enter into the Black-Scholes value of the option, and settle the trade. This enables them to revise their price schedule only in response to less-frequent changes in $\sigma_t(X, \tau)$, rather than to more-frequent changes in the option price in currency units.

In exchange-traded markets, option prices are quoted in currency units, since these markets are set up to keep frequently fluctuating prices posted and are not well set up to have a detailed conversation between counterparties each time a trade is made. The OTC option trader first fixes a particular value of the implied volatility for the particular exercise price and tenor of the option on the "price schedule" $\sigma_t(X, \tau)$. That price schedule will vary, of course, during the trading day or by date t. The trader then substitutes that implied volatility, together with the design parameters of the option and the observable market prices into the Black-Scholes pricing function to get the option price in currency units:

$$c(t, T - t, X) = v(S_t, \tau, X, \sigma_t(X, \tau), r, q)$$

The trader's pricing process reverses the logic of Equation (5.2), which solves for an implied volatility from an option price. Information about the volatility smile is also expressed in the pricing of option combinations and spreads, such as the strangle and the risk reversal we encountered in Chapter 4. Dealers usually quote strangle prices by stating the implied volatility—the "strangle volatility"—at which they buy or sell both options. For example, the dealer might quote his selling price as 14.6 vols, meaning that he sells a 25-delta call and a 25-delta put at an implied volatility of 14.6 vols each. Dealers generally record strangle prices as the spread of the strangle volatility over the at-the-money forward volatility. If market participants were convinced that exchange rates move lognormally, the out-of-the-money options would have the same implied volatility as at-the-money options, and strangle spreads would be centered at zero. Strangles, therefore, indicate the degree of curvature of the volatility smile.

In a risk reversal, the dealer quotes the implied volatility differential at which he is prepared to exchange a 25-delta call for a 25-delta put. For example, if the dollar-yen exchange rate is strongly expected to fall (dollar depreciation), an options dealer might quote dollar-yen risk reversals as follows: "One-month 25-delta risk reversals are 0.8 at 1.2 yen calls over." This means he stands ready to pay a net premium of 0.8 vols to buy a 25-delta yen call and sell a 25-delta yen put against the dollar, and charges a net premium of 1.2 vols to sell a 25-delta yen call and buy a 25-delta yen put. The pricing thus expresses both a "yen premium" and a bid-ask spread.

5.5.3 Measuring Vega Risk

If there were time-variation in volatility, but neither a smile nor a term structure of volatility, the vol surface would be a plane parallel to the maturity and strike axes. Vega risk could then be handled precisely in the quadratic approximation framework. For now, let's assume a flat, but time-varying vol surface. If we include implied volatility as a risk factor, the approximate P&L of a call is:

$$\Delta v(S_t, \tau, X, \sigma_t, r, q) \approx \theta_{c,t}\Delta\tau + \delta_{c,t}\Delta S + \frac{1}{2}\gamma_t\Delta S^2 + \nu_t\Delta\sigma$$

We now have two risk factors, the underlying asset and the implied volatility. This introduces new issues and new parameters:

1. Returns on the implied volatility may behave very differently over time from those on "conventional" risk factors. If so, then simply estimating the volatility of logarithmic changes in implied volatility and treating it as an ordinary risk factor may lead to even more inaccurate VaR estimates than is the case for cash assets. As we see in Chapter 10, however, there is some evidence that the assumption of normally-distributed log returns may not be that much more inaccurate for implied volatility than for many cash assets.
2. We now also need to estimate the correlation between returns on the underlying asset and log changes in implied vol.

How significant is vega risk relative to the "delta-gamma" risk arising from fluctuations in the underlying price? Figure 5.5 provides an illustration. It incorporates a typical-size shock to the underlying price and to implied volatility for a plain-vanilla naked call. In the illustration, implied volatility causes a P&L event about one-quarter the size of that induced by the spot price. For other exercise prices or other option portfolios, however, vega risk can be more dominant, even with typical-size asset return and implied volatility shocks. For example, straddles have *de minimis* exposure to the underlying, but a great deal of exposure to vega.

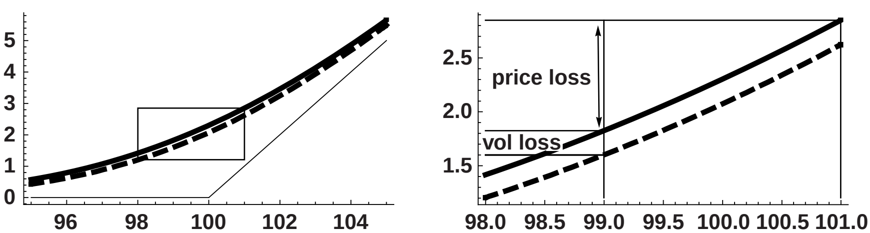

FIGURE 5.5 Impact of Vega Risk
The market value of a long position in a one-month at-the-money forward call on an asset with initial underlying price of 101.0. The time to maturity is one month, and the risk-free rate and dividend rate on the underlying asset are both set to 1 percent. The right panel is a blowup of the area in the box in the left panel. The left panel shows the option's current and intrinsic value as the current and terminal underlying asset price change. The right panel shows how much the plot of current option price against the underlying price shifts down with a decrease in implied volatility.

We'll use the delta-normal method of computing the VaR, taking vega risk into account. While other approaches to option risk measurement can be and are used in practice, the delta-normal model illustrates the issues specific to options. A single option position is treated as a portfolio containing the two risk factors, the underlying asset price and implied volatility. In this approach, we take account of the time-variation of implied volatility, but not changes in the shape of the volatility surface.

In the delta-normal approach, we have to define the exposure amounts, and measure two return volatilities and a return correlation. To avoid confusion with the implied volatility, we'll denote the underlying price volatility σ^{price} and the vol of vol σ^{vol}. As in any delta-normal approach, we have to make sure the delta equivalents are defined appropriately for the risk factors and for the way we are defining and measuring their volatilities and correlations. The delta equivalent has already been defined as $xS_t\delta_{c,t}$, where x is the number of options. Similarly, the vega exposure or "vega equivalent" is $x\sigma_t\nu_{c,t}$.

If we are dealing with just one option position, or a portfolio of options, all on the same underlying asset and with the same option maturity, there are two risk factors, the underlying asset return and the implied volatility. The vector of delta equivalents is

$$\mathbf{d}_t = x\begin{pmatrix} S_t\delta_{c,t} \\ \sigma_t\nu_{c,t} \end{pmatrix}$$

where x is the number of units of the underlying asset the option is written on. The covariance matrix of logarithmic underlying asset and volatility returns is

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^{\text{price}} & 0 \\ 0 & \sigma^{\text{vol}} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} \sigma^{\text{price}} & 0 \\ 0 & \sigma^{\text{vol}} \end{pmatrix}$$

where ρ is the correlation between logarithmic changes in the underlying price and implied volatility.

The portfolio may contain long or short hedge positions in the underlying asset. The number of risk factors may be greater than two if the options have different exercise prices or tenors.

Example 5.5 Consider a one-month at-the-money forward European call option on \$1,000,000 worth of euros. The option prices are denominated in U.S. dollars. We'll compute the one-day VaR at a 99-percent confidence level as of June 4, 2010. The spot and forward foreign exchange rates were

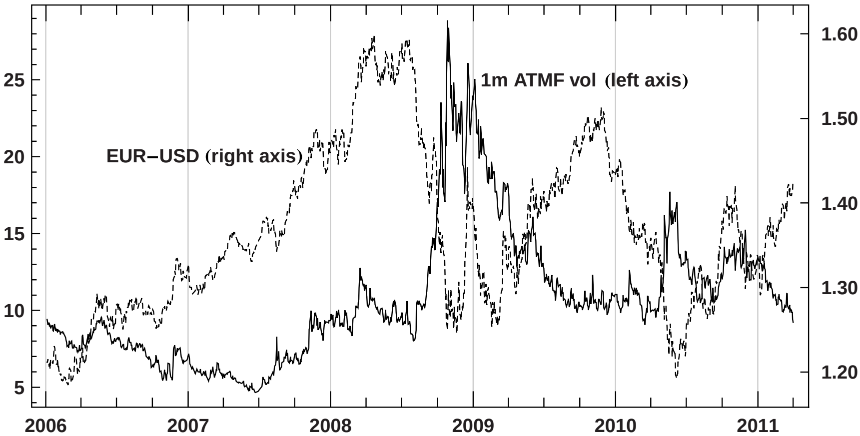

FIGURE 5.6 Euro Foreign Exchange Implied Volatilities
Spot exchange rate (dashed line) and one-month at-the-money forward implied volatility (solid line) of the euro-dollar exchange rate, January 3, 2006, to April 1, 2010.
Source: Bloomberg Financial L.P.

1.1967 and 1.1970, the implied volatility 16.595 percent, and the U.S. and euro money market rates 35 and 43 basis points.

As seen in Figure 5.6, the exchange rate and implied volatility have both been quite volatile during the subprime crisis and the correlation between their returns has also been subject to wide swings. During the first half of 2010, the correlation between underlying price and implied volatility returns is negative: Implied vol goes up as the euro depreciates against the dollar. The vols and correlation are

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^{\text{price}} & 0 \\ 0 & \sigma^{\text{vol}} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} \sigma^{\text{price}} & 0 \\ 0 & \sigma^{\text{vol}} \end{pmatrix}$$

$$= \begin{pmatrix} 0.1619 & 0 \\ 0 & 0.8785 \end{pmatrix} \begin{pmatrix} 1 & -0.3866 \\ -0.3866 & 1 \end{pmatrix} \begin{pmatrix} 0.1619 & 0 \\ 0 & 0.8785 \end{pmatrix}$$

$$= \begin{pmatrix} 0.0262 & -0.0550 \\ -0.0550 & 0.7717 \end{pmatrix}$$

The number of units of the option is $x = € \, 835{,}415 = \$1{,}000{,}000 \times 1.1967^{-1}$. The vector of delta equivalents in U.S. dollars is

$$\mathbf{d}_t = x \begin{pmatrix} S_t \delta_{c,t} \\ \sigma_t \nu_{c,t} \end{pmatrix} = 835{,}415 \begin{pmatrix} 0.6099 \\ 0.0229 \end{pmatrix} = \begin{pmatrix} 509{,}553 \\ 19{,}106 \end{pmatrix}$$

We can thus quantify the exposure to vol as just under 4 percent of the total exposure to market prices via the option. The VaR is

$$\text{VaR}_t(\alpha, \tau)(x) = -z_* \sqrt{\tau} \sqrt{\mathbf{d}_t' \boldsymbol{\Sigma} \mathbf{d}_t} = \frac{2.33}{\sqrt{252}} 74{,}570.4$$

or about \$11,366. The VaR of a long cash position in \$1,000,000 worth of euros would be slightly higher at \$12,088. That is, expressing the exposure through a long option position is slightly less risky "in the small," as there is a negative correlation between volatility and the value of the euro: As the euro declines, implied volatility tends to rise, damping the decline in the option's value. If the correlation between implied vol and the value of the euro turns positive, as during the early phase of the subprime crisis, the risk of a long euro position would be higher if expressed through options, since a decline in the value of the euro would tend to be accompanied by a decline in vol.

The delta-normal VaR estimate takes into account variability in the level of implied volatility over time, but not the variation in implied volatility along the volatility surface, and as the shape of the volatility surface changes. VaR can be made more accurate by taking account of the term structure and smile. Furthermore, the level of implied volatility is correlated with both the volatility of underlying returns and with the option skew.

To better understand these additional sources of risk, let's start with the simplest case, in which the shape of the volatility surface does not change, regardless of changes in the asset price and in the level of implied volatility. The shape of the volatility surface, however, has an impact on risk even if it doesn't change. The scenario is illustrated in Figure 5.7. The initial underlying price and implied volatility are indicated as a point on the initial smile (solid curve). Suppose implied volatilities increase across the board, so the volatility smile shifts up by 0.01 (one vol), to the dashed curve. If the asset price does not change, the initial underlying price and new implied volatility are represented by point *a*. If the cash price also experiences fluctuations, the new state might be at point *b* or *c*.[2]

The asset price has changed, while the exercise price of the option hasn't, so the option delta has changed. The increase in implied volatility may then be larger or smaller than the one-vol parallel shift in the smile, as

[2]The volatility smiles displayed in Figure 5.7 are interpolated by fitting a polynomial to five observed implied volatilities for call deltas equal to (0.10, 0.25, 0.50, 0.75, 0.90) using the *Mathematica* function `InterpolatingPolynomial`.

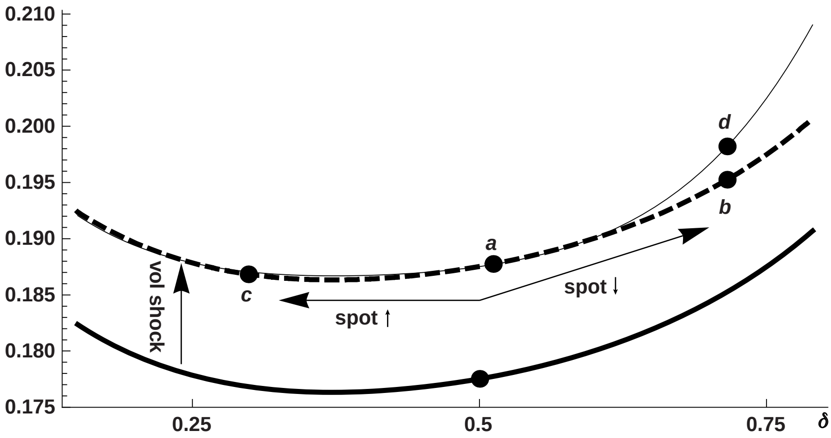

FIGURE 5.7 Vega and the Smile
The figure illustrates the sticky delta approach. The initial delta of the call option is 0.5. The initial volatility smile is that for one-month USD-EUR options, October 17, 2008. The shocked volatility smile represented by the dashed curve is one vol higher for each delta. At point *a*, the spot rate is unchanged. Point *b* corresponds to a lower spot rate and point *c* to a higher one. The thin solid curve represents a shocked curve in which the negative put skew has increased in addition to an increase in the level of volatility.

the market will price the option using an implied volatility appropriate to the new delta. This is known as a *sticky delta* approach to modeling, since implied volatilities remain "attached" to the delta of the option as underlying prices change. It contrasts with the *sticky strike* approach, in which implied volatilities do not adjust to the changed moneyness of the option, but rather remain "attached" to the exercise price. The sticky delta approach is more realistic, especially in periods of higher volatility.

In the example illustrated in Figure 5.7, if the asset price declines, the call delta increases, and the new implied volatility is given by point *b* on the shocked volatility smile, resulting in an additional increase in the implied volatility along the smile. If the asset price increases, so that the call delta decreases, the new implied volatility is given by point *c*, resulting in almost no change in implied volatility along the smile. Thus, even if the shape of the smile does not change, fluctuations in the underlying asset price can induce significant changes in implied volatility—and thus P&L—along the volatility smile.[3]

[3]We are ignoring the shortening of the option's time to maturity and any changes in money market rates in this analysis.

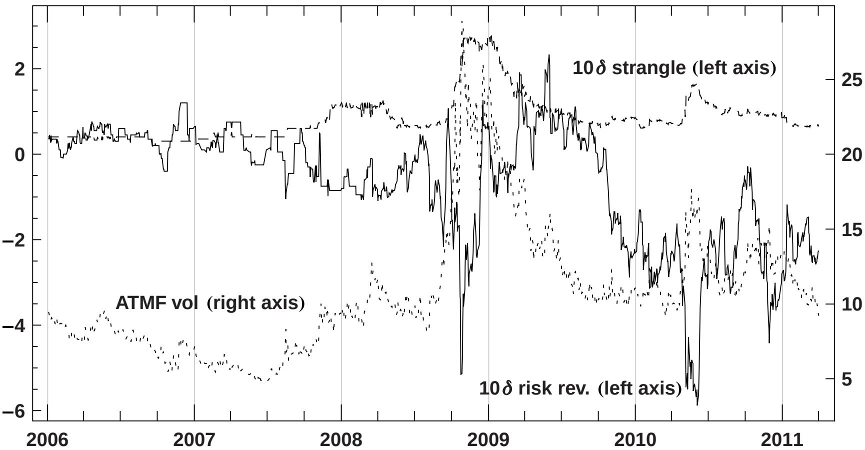

FIGURE 5.8 Euro Implied Volatilities, Risk Reversals, and Strangle Prices
One-month at-the-money forward implied volatility (dotted line) and prices in vols of 10-delta risk reversals (solid line) and strangle prices (dashed line) for the euro-dollar exchange rate, January 3, 2006, to April 1, 2011.
Source: Bloomberg Financial L.P.

The preceding paragraphs examined the case of a parallel shift in the volatility surface. Generally, however, the shape of the volatility surface will change when there are shocks to implied volatility, especially if those shocks are large. The correlation between the option skew and the level of vol can change rapidly, as seen for EUR-USD in Figure 5.8. During the subprime crisis, for example, rising implied volatility was typically accompanied by a more pronounced skew toward euro depreciation. This reflected an investor bias in favor of the U.S. dollar and was part of the "flight to safety" markets exhibited during the financial crisis. But at times, such as during the first half of 2009, this correlation was positive.

Such changes in the shape of the volatility surface can have an important impact on option P&L. Imagine a short position in an out-of-the-money EUR-USD put option in a high-volatility environment. The option may have been sold to express a view that the underlying price will rise, or at least not fall. In an adverse case, a sharp move up in implied volatility and lower in the underlying price might be accompanied by an increase in the negative skew. This is illustrated by the thin solid curve in Figure 5.7. The new state might be a point such as *d*, adding to the losses to the short put option position caused by the underlying price decline and the increase in vol caused by the change in delta.

FURTHER READING

Factor models are discussed in Fama and French (1992, 1993) and Zangari (2003).

See Malz (2000, 2001b) for a more detailed discussion of the material on vega risk. Dumas, Fleming, and (1997), Gatheral (2006), Daglish, Hull, and (2007) are good starting points on implied volatility behavior. Cont and da Fonseca (2002) discusses factor model approaches to the behavior of implied volatility surfaces over time. “Sticky” strikes and deltas are discussed in Derman (1999). See also the Further Reading sections at the end of Chapters 10 and 14.