

## CHAPTER 4

# Time Value of Money

**W**hen you graduate and go to work, either a **defined benefit (DB)** or a **defined contribution (DC)** pension plan will almost certainly be part of your compensation package. Under a DB plan, the company will put funds into its pension fund, which will then invest in stocks, bonds, real estate, and so forth and then use those funds to make the promised payments after you retire. Under a DC plan, the company will put money into your 401(k) plan (which is essentially a mutual fund), you will decide what type of assets to buy, and you will withdraw money after you retire. The analysis required to set up a good retirement program is based on the subject of this chapter, the time value of money (TVM).

How do you suppose a stock market crash like we had in 2008, with the average stock down about 40%, will affect DB and DC retirement plans? If you have a 401(k) plan that holds stocks, as most people do, TVM analysis would show clearly that you will have to work longer than you expected, reduce your post-retirement standard of living, or both.

With a DB plan, a stock market decline reduces the value of the investments set aside for you by the company. If there is also a decline in interest rates, as there was in 2008, TVM analysis shows that the amount of money the company should set aside for you goes up. Thus, the company's pension funding status, which is the difference between the value of the pension plan's investments and the amount the plan should have on hand to cover the future obligations, becomes severely underfunded if the market crashes and interest rates fall. This can even lead to bankruptcy, in which case you might end up with retirement payments from the government instead of from the company, with the government's payments a lot lower than those promised by the company's plan. If you don't believe us, ask someone who recently retired from a bankrupt airline or auto company.<sup>1</sup>

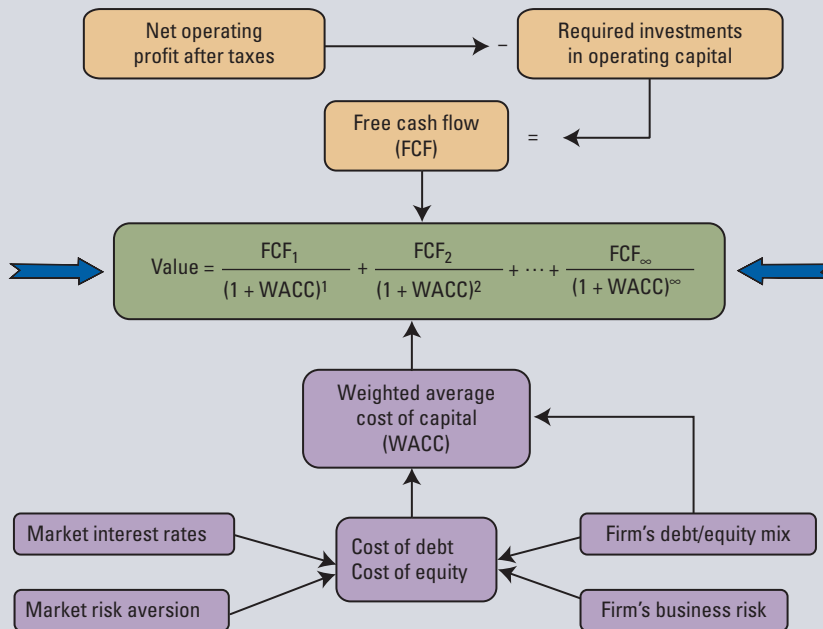
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<sup>1</sup>If you want to see something alarming, apply the procedures set forth in this chapter to the pension plan of almost any municipal government. Politicians find it a lot easier to promise high future benefits than to raise taxes to pay for those benefits. Of course, the federal government is doing the same thing with Social Security, Medicare, and Medicaid, and with federal employees. Politicians need to study this chapter!

## Corporate Valuation and the Time Value of Money

In Chapter 1 we explained (1) that managers should strive to make their firms more valuable and (2) that the value of a firm is determined by the size, timing, and risk of its free cash flows (FCF). Recall from Chapter 2 that free cash flows are the cash flows available for distribution to all of a firm's investors

(stockholders and creditors). We explain how to calculate the weighted average cost of capital (WACC) in Chapter 9, but it is enough for now to think of the WACC as the average rate of return required by all of the firm's investors. The intrinsic value of a company is given by the following diagram.



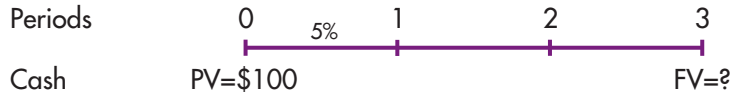
In Chapter 1 we saw that the primary objective of financial management is to maximize the intrinsic value of a firm's stock. We also saw that stock values depend on the timing of the cash flows investors expect from an investment—a dollar expected sooner is worth more than a dollar expected further in the future. Therefore, it is essential for financial managers to understand the time value of money and its impact on stock prices. In this chapter we will explain exactly how the timing of cash flows affects asset values and rates of return.

The principles of time value analysis have many applications, including retirement planning, loan payment schedules, and decisions to invest (or not) in new equipment. *In fact, of all the concepts used in finance, none is more important than the **time value of money (TVM)**, also called **discounted cash flow (DCF) analysis**.* Time value concepts are used throughout the remainder of the book, so it is vital that you understand the material in Chapter 4 and be able to work the chapter's problems before you move on to other topics.<sup>2</sup>

<sup>2</sup>The problems can be worked with either a calculator or an *Excel* spreadsheet. Calculator manuals tend to be long and complicated, partly because they cover a number of topics that aren't used in the basic finance course. Therefore, on this textbook's Web site we provide tutorials for the most commonly used calculators. The tutorials are keyed to this chapter, and they show exactly how to do the calculations used in the chapter. If you don't know how to use your calculator, go to the Web site, get the relevant tutorial, and go through it as you study the chapter. The chapter's *Tool Kit* also explains how to do all of the within-chapter calculations using *Excel*. The *Tool Kit*, along with an *Excel* tutorial designed for this book, is provided on the book's Web site.

## 4.1 TIME LINES

The first step in a time value analysis is to set up a **time line** to help you visualize what's happening in the particular problem. To illustrate, consider the following diagram, where PV represents \$100 that is in a bank account today and FV is the value that will be in the account at some future time (3 years from now in this example):



The intervals from 0 to 1, 1 to 2, and 2 to 3 are time periods such as years or months. Time 0 is today, and it is the beginning of Period 1; Time 1 is one period from today, and it is both the end of Period 1 and the beginning of Period 2; and so on. In our example the periods are years, but they could also be quarters or months or even days. Note again that each tick mark corresponds to both the *end* of one period and the *beginning* of the next one. Thus, if the periods are years, the tick mark at Time 2 represents both the end of Year 2 and the beginning of Year 3.

Cash flows are shown directly below the tick marks, and the relevant interest rate is shown just above the time line. Unknown cash flows, which you are trying to find, are indicated by question marks. Here the interest rate is 5%; a single cash outflow, \$100, is invested at Time 0; and the Time-3 value is unknown and must be found. In this example, cash flows occur only at Times 0 and 3, with no flows at Times 1 or 2. We will, of course, deal with situations where multiple cash flows occur. Note also that in our example the interest rate is constant for all 3 years. The interest rate is generally held constant, but if it varies then in the diagram we show different rates for the different periods.

Time lines are especially important when you are first learning time value concepts, but even experts use them to analyze complex problems. Throughout the book, our procedure is to set up a time line to show what's happening, provide an equation that must be solved to find the answer, and then explain how to solve the equation with a regular calculator, a financial calculator, and a computer spreadsheet.

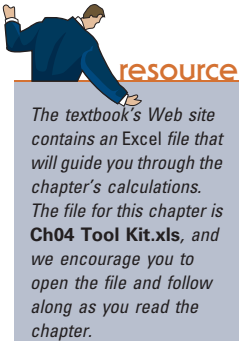
### Self-Test

Do time lines deal only with years, or could other periods be used?

Set up a time line to illustrate the following situation: You currently have \$2,000 in a 3-year certificate of deposit (CD) that pays a guaranteed 4% annually. You want to know the value of the CD after 3 years.

## 4.2 FUTURE VALUES

A dollar in hand today is worth more than a dollar to be received in the future—if you had the dollar now you could invest it, earn interest, and end up with more than one dollar in the future. The process of going forward, from **present values (PVs)** to **future values (FVs)**, is called **compounding**. To illustrate, refer back to our 3-year time line and assume that you have \$100 in a bank account that pays a guaranteed 5% interest each year. How much would you have at the end of Year 3? We first define some terms, after which we set up a time line and show how the future value is calculated.



$PV$  = Present value, or beginning amount. In our example,  $PV = \$100$ .

$FV_N$  = Future value, or ending amount, in the account after  $N$  periods. Whereas  $PV$  is the value *now*, or the *present value*,  $FV_N$  is the value  $N$  periods into the *future*, after interest earned has been added to the account.

$CF_t$  = Cash flow. Cash flows can be positive or negative. For a borrower, the first cash flow is positive and the subsequent cash flows are negative, and the reverse holds for a lender. The cash flow for a particular period is often given a subscript,  $CF_t$ , where  $t$  is the period. Thus,  $CF_0 = PV$  = the cash flow at Time 0, whereas  $CF_3$  would be the cash flow at the end of Period 3. In this example the cash flows occur *at the ends* of the periods, but in some problems they occur at the beginning.

$I$  = Interest rate earned per year. (Sometimes a lowercase  $i$  is used.) Interest earned is based on the balance at the beginning of each year, and we assume that interest is paid at the end of the year. Here  $I = 5\%$  or, expressed as a decimal,  $0.05$ . Throughout this chapter, we designate the interest rate as  $I$  (or  $I/YR$ , for interest rate per year) because that symbol is used on most financial calculators. Note, though, that in later chapters we use the symbol “ $r$ ” to denote the rate because  $r$  (for *rate* of return) is used more often in the finance literature. Also, in this chapter we generally assume that interest payments are guaranteed by the U.S. government and hence are riskless (i.e., certain). In later chapters we will deal with risky investments, where the rate actually earned might be different from its expected level.

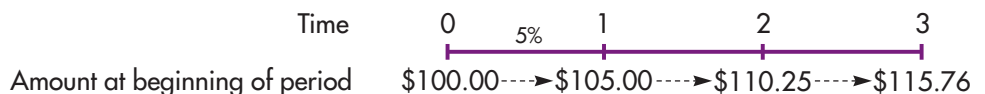
$INT$  = Dollars of interest earned during the year = (Beginning amount)  $\times I$ . In our example,  $INT = \$100(0.05) = \$5$  for Year 1, but it rises in subsequent years as the amount at the beginning of each year increases.

$N$  = Number of periods involved in the analysis. In our example,  $N = 3$ . Sometimes the number of periods is designated with a lowercase  $n$ , so both  $N$  and  $n$  indicate number of periods.

We can use four different procedures to solve time value problems.<sup>3</sup> These methods are described next.

## Step-by-Step Approach

The time line itself can be modified and used to find the  $FV$  of  $\$100$  compounded for 3 years at  $5\%$ , as shown below:



<sup>3</sup>A fifth procedure is called the *tabular approach*, which uses tables that provide “interest factors;” this procedure was used before financial calculators and computers became available. Now, though, calculators and spreadsheets such as *Excel* are programmed to calculate the specific factor needed for a given problem, which is then used to find the  $FV$ . This is much more efficient than using the tables. Also, calculators and spreadsheets can handle fractional periods and fractional interest rates. For these reasons, tables are not used in business today; hence we do not discuss them in the text. However, because some professors cover the tables for pedagogic purposes, we discuss them in *Web Extension 4A*, on the textbook’s Web site.

We start with \$100 in the account, which is shown at  $t = 0$ . We then multiply the initial amount, and each succeeding beginning-of-year amount, by  $(1 + I) = (1.05)$ .

- You earn  $\$100(0.05) = \$5$  of interest during the first year, so the amount at the end of Year 1 (or at  $t = 1$ ) is

$$\begin{aligned} FV_1 &= PV + INT \\ &= PV + PV(I) \\ &= PV(1 + I) \\ &= \$100(1 + 0.05) = \$100(1.05) = \$105 \end{aligned}$$

- We begin the second year with \$105, earn  $0.05(\$105) = \$5.25$  on the now larger beginning-of-period amount, and end the year with \$110.25. Interest during Year 2 is \$5.25, and it is higher than the first year's interest, \$5, because we earned  $\$5(0.05) = \$0.25$  interest on the first year's interest. This is called "compounding," and interest earned on interest is called "compound interest."
- This process continues, and because the beginning balance is higher in each successive year, the interest earned each year increases.
- The total interest earned, \$15.76, is reflected in the final balance, \$115.76.

The step-by-step approach is useful because it shows exactly what is happening. However, this approach is time-consuming, especially if the number of years is large and you are using a calculator rather than *Excel*, so streamlined procedures have been developed.

## Formula Approach

In the step-by-step approach, we multiplied the amount at the beginning of each period by  $(1 + I) = (1.05)$ . Notice that the value at the end of Year 2 is

$$\begin{aligned} FV_2 &= FV_1(1 + I) \\ &= PV(1 + I)(1 + I) \\ &= PV(1 + I)^2 \\ &= 100(1.05)^2 = \$110.25 \end{aligned}$$

If  $N = 3$ , then we multiply  $PV$  by  $(1 + I)$  three different times, which is the same as multiplying the beginning amount by  $(1 + I)^3$ . This concept can be extended, and the result is this key equation:

$$FV_N = PV(1 + I)^N \quad (4-1)$$

We can apply Equation 4-1 to find the  $FV$  in our example:

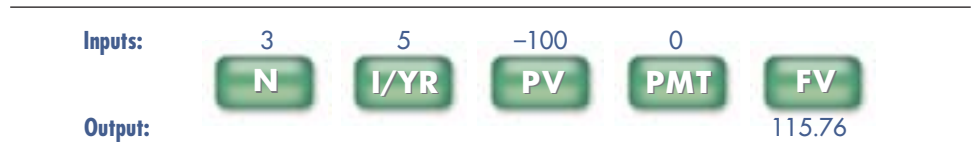
$$FV_3 = \$100(1.05)^3 = \$115.76$$

Equation 4-1 can be used with any calculator, even a nonfinancial calculator that has an exponential function, making it easy to find  $FVs$  no matter how many years are involved.

## Financial Calculators

Financial calculators were designed specifically to solve time value problems. First, note that financial calculators have five keys that correspond to the five variables in the basic time value equations. Equation 4-1 has only four variables, but we will shortly deal with situations where a fifth variable (a set of periodic additional

payments) is involved. We show the inputs for our example above their keys in the following diagram, and the output, which is the FV, below its key. Since in this example there are no periodic payments, we enter 0 for PMT. We describe the keys in more detail below the diagram.



N = Number of periods = 3. Some calculators use  $n$  rather than N.

I/YR = Interest rate per period = 5. Some calculators use  $i$  or I rather than I/YR. Calculators are programmed to automatically convert the 5 to the decimal 0.05 before doing the arithmetic.

PV = Present value = 100. In our example we begin by making a deposit, which is an outflow of 100, so the PV is entered with a negative sign. On most calculators you must enter the 100, then press the  $\pm$  key to switch from +100 to -100. If you enter -100 directly, this will subtract 100 from the last number in the calculator, which will give you an incorrect answer unless the last number was zero.

PMT = Payment. This key is used if we have a series of equal, or constant, payments. Since there are no such payments in our current problem, we enter PMT = 0. We will use the PMT key later in this chapter.

FV = Future value. In our example, the calculator automatically shows the FV as a positive number because we entered the PV as a negative number. If we had entered the 100 as a positive number, then the FV would have been negative. Calculators automatically assume that either the PV or the FV must be negative.

As noted in our example, you first enter the four known values (N, I/YR, PMT, and PV) and then press the FV key to get the answer, FV = 115.76.

## Spreadsheets

Spreadsheets are ideally suited for solving many financial problems, including those dealing with the time value of money.<sup>4</sup> Spreadsheets are obviously useful for calculations, but they can also be used like a word processor to create exhibits like our Figure 4-1, which includes text, drawings, and calculations. We use this figure to show that four methods can be used to find the FV of \$100 after 3 years at an interest rate of 5%. The time line on Rows 43 to 45 is useful for visualizing the problem, after which the spreadsheet calculates the required answer. Note that the letters across the top designate columns, the numbers down the left column designate rows, and the rows and columns jointly designate cells. Thus, cell C39 shows the amount of the investment, \$100, and it is given a minus sign because it is an outflow.



<sup>4</sup>The file *Ch04 Tool Kit.xls* on the book's Web site does the calculations in the chapter using *Excel*. We highly recommend that you go through this *Tool Kit*. This will give you practice with *Excel*, and that will help tremendously in later courses, in the job market, and in the workplace. Also, going through the models will improve your understanding of financial concepts.

## Hints on Using Financial Calculators

When using a financial calculator, make sure your machine is set up as indicated below. Refer to your calculator manual or to our calculator tutorial on the text's Web site for information on setting up your calculator.

- ◆ **One payment per period.** Many calculators “come out of the box” assuming that 12 payments are made per year; that is, they assume monthly payments. However, in this book we generally deal with problems in which only one payment is made each year. *Therefore, you should set your calculator at one payment per year and leave it there. See our tutorial or your calculator manual if you need assistance. We will show you how to solve problems with more than 1 payment per year in Section 4.15.*
- ◆ **End mode.** With most contracts, payments are made at the *end* of each period. However, some contracts call for payments at the *beginning* of each period. You can switch between “End Mode” and “Begin Mode” depending on the problem you are solving. *Because most of the problems in this book call for end-of-period payments, you should return your calculator to End Mode after you work a problem in which payments are made at the beginning of periods.*
- ◆ **Negative sign for outflows.** When first learning how to use financial calculators, students often forget that one cash flow must be negative. Mathematically, financial calculators solve a version of this equation:

$$PV(1 + I)^N + FV_N = 0$$

(4-2)

Notice that for reasonable values of  $I$ , either  $PV$  or  $FV_N$  must be negative, and the other one must be positive to make the equation equal 0. This is reasonable because, in all realistic situations, one cash flow is an outflow (which should have a negative sign) and one is an inflow (which should have a positive sign). For example, if you make a deposit (which is an outflow, and hence should have a negative sign) then you will expect to make a later withdrawal (which is an inflow with a positive sign). *The bottom line is that one of your inputs for a cash flow must be negative and one must be positive. This generally means typing the outflow as a positive number and then pressing the +/- key to convert from + to - before hitting the enter key.*

- ◆ **Decimal places.** When doing arithmetic, calculators use a great many decimal places. However, they allow you to show from 0 to 11 decimal places on the display. When working with dollars, we generally specify two decimal places. When dealing with interest rates, we generally specify two places if the rate is expressed as a percentage, like 5.25%, but we specify four places if the rate is expressed as a decimal, like 0.0525.
- ◆ **Interest rates.** *For arithmetic operations with a nonfinancial calculator, the rate 5.25% must be stated as a decimal, .0525. However, with a financial calculator you must enter 5.25, not .0525, because financial calculators are programmed to assume that rates are stated as percentages.*

It is useful to put all of the problem's inputs in a section of the spreadsheet designated “Inputs.” In Figure 4-1 we put the inputs in the range A38:C41, with C39 being the cell where we specify the investment, C40 the interest rate, and C41 the number of periods. We can use these three cell references, rather than the fixed numbers themselves, in the formulas in the remainder of the model. This makes it easy to modify the problem by changing the inputs and then having the new data automatically used in the calculations.

Time lines are important for solving finance problems because they help us visualize what's happening. When we work a problem by hand we usually draw a time line, and when we work a problem with *Excel*, we actually set the model up as a time line. For example, in Figure 4-1 Rows 43 to 45 are indeed a time line. It's easy



FIGURE 4-1 Alternative Procedures for Calculating Future Values

	A	B	C	D	E	F	G
38	INPUTS:						
39	Investment	= CF <sub>0</sub> = PV =	-\$100.00				
40	Interest rate	= I =	5.00%				
41	No. of periods	= N =	3				
42							
43	Setup of the problem as a	Periods:	0	5%	1	2	3
44	Time Line		—				
45		Cash Flow:	-\$100		0	0	FV = ?
46							
47	1. Step-by-Step: Multiply \$100 by (1 + I)		\$100	→	\$105.00	→	\$110.25 → \$115.76
48							
49	2. Formula: $FV_N = PV(1 + I)^N$				$FV_3 = \$100(1.05)^3$	=	\$115.76
50							
51		3	5	-\$100.00	\$0		
52	3. Financial Calculator:	N	I/YR	PV	PMT		FV
53							\$115.76
54							
55	4. Excel Spreadsheet:	FV Function:	$FV_N =$	$= FV(I,N,0,PV)$			
56		Fixed inputs:	$FV_N =$	$= FV(0.05,3,0,-100)$			\$115.76
57		Cell references:	$FV_N =$	$= FV(C40,C41,0,C39)$			\$115.76
58	In the Excel formula, the terms are entered in the sequence: interest, periods, 0 to indicate no periodic cash flows, and then the PV. The data can be entered as fixed numbers or, better yet, as cell references.						

to construct time lines with *Excel*, with each column designating a different period on the time line.

On Row 47 we use *Excel* to go through the step-by-step calculations, multiplying the beginning-of-year values by (1 + I) to find the compounded value at the end of each period. Cell G47 shows the final result of the step-by-step approach.

We illustrate the formula approach in Row 49, using *Excel* to solve Equation 4-1 to find the FV. Cell G49 shows the formula result, \$115.76. As it must, it equals the step-by-step result.

Rows 51 to 53 illustrate the financial calculator approach, which again produces the same answer, \$115.76.

The last section, in Rows 55 to 58, illustrates *Excel's* future value (FV) function. You can access the function wizard by clicking the  $f_x$  symbol in *Excel's* formula bar. Then select the category for Financial functions, and then the FV function, which is  $=FV(I,N,0,PV)$ , as shown in Cell E55.<sup>5</sup> Cell E56 shows how the formula would look with numbers as inputs; the actual function itself is entered in Cell G56, but it shows up in the table as the answer, \$115.76. If you access the model and put the pointer on Cell G56, you will see the full formula. Finally, Cell E57 shows how the formula would look with cell references rather than fixed values as inputs, with

<sup>5</sup>All functions begin with an equal sign. The third entry is zero in this example, which indicates that there are no periodic payments. Later in this chapter we will use the FV function in situations where we have nonzero periodic payments. Also, for inputs we use our own notation, which is similar but not identical to *Excel's* notation.



the actual function again in Cell G57. We generally use cell references as function inputs because this makes it easy to change inputs and see how those changes affect the output. This is called “sensitivity analysis.” Many real-world financial applications use sensitivity analysis, so it is useful to get in the habit of setting up an input data section and then using cell references rather than fixed numbers in the functions.

When entering interest rates in *Excel*, you can use either actual numbers or percentages, depending on how the cell is formatted. For example, in cell C40, we first formatted to Percentage, and then typed in 5, which showed up as 5%. However, *Excel* uses 0.05 for the arithmetic. Alternatively, we could have formatted C40 as a Number, in which case we would have typed “0.05.” If C40 is formatted to Number and you enter 5, then *Excel* would think you meant 500%. Thus, *Excel*’s procedure is quite different from the convention used in financial calculators.

## Comparing the Procedures

The first step in solving any time value problem is to understand what is happening and then to diagram it on a time line. Woody Allen said that 90% of success is just showing up. With time value problems, 90% of success is correctly setting up the time line.

After you diagram the problem on a time line, your next step is to pick one of the four approaches shown in Figure 4-1 to solve the problem. Any may be used, but your choice of method will depend on the particular situation.

All business students should know Equation 4-1 by heart and should also know how to use a financial calculator. So, for simple problems such as finding the future value of a single payment, it is generally easiest and quickest to use either the formula approach or a financial calculator. However, for problems that involve several cash flows, the formula approach usually is time-consuming, so either the calculator or spreadsheet approach would generally be used. Calculators are portable and quick to set up, but if many calculations of the same type must be done, or if you want to see how changes in an input such as the interest rate affect the future value, then the spreadsheet approach is generally more efficient. If the problem has many irregular cash flows, or if you want to analyze alternative scenarios using different cash flows or interest rates, then the spreadsheet approach definitely is the most efficient procedure.

Spreadsheets have two additional advantages over calculators. First, it is easier to check the inputs with a spreadsheet—they are visible, whereas with a calculator they are buried somewhere in the machine. Thus, you are less likely to make a mistake in a complex problem when you use the spreadsheet approach. Second, with a spreadsheet, you can make your analysis much more transparent than you can when using a calculator. This is not necessarily important when all you want is the answer, but if you need to present your calculations to others, like your boss, it helps to be able to show intermediate steps, which enables someone to go through your exhibit and see exactly what you did. Transparency is also important when you must go back, some-time later, and reconstruct what you did.

You should understand the various approaches well enough to make a rational choice, given the nature of the problem and the equipment you have available. In any event, you must understand the concepts behind the calculations, and you must also know how to set up time lines in order to work complex problems. This is true for stock and bond valuation, capital budgeting, lease analysis, and many other important financial problems.

## The Power of Compound Interest

Assume that you are 26 and just received your MBA. After reading the introduction to this chapter, you decide to start investing in the stock market for your retirement. Your goal is to have \$1 million when you retire at age 65. Assuming you earn 10% annually on your stock investments, how much must you invest at the end of each year in order to reach your goal?

The answer is \$2,491, but this amount depends critically on the return earned on your investments. If your return drops to 8%, the required annual contribution would rise to \$4,185. On the other hand, if the return rises to 12%, you would need to put away only \$1,462 per year.

What if you are like most 26-year-olds and wait until later to worry about retirement? If you wait until age 40,

you will need to save \$10,168 per year to reach your \$1 million goal, assuming you can earn 10%, but \$13,679 per year if you earn only 8%. If you wait until age 50 and then earn 8%, the required amount will be \$36,830 per year!

Although \$1 million may seem like a lot of money, it won't be when you get ready to retire. If inflation averages 5% a year over the next 39 years, then your \$1 million nest egg would be worth only \$149,148 in today's dollars. If you live for 20 years after retirement and earn a real 3% rate of return, your annual retirement income in today's dollars would be only \$9,733 before taxes. So, after celebrating your graduation and new job, start saving!

## Graphic View of the Compounding Process

Figure 4-2 shows how a \$100 investment grows (or declines) over time at different interest rates. Interest rates are normally positive, but the “growth” concept is broad enough to include negative rates. We developed the curves by solving Equation 4-1 with different values for  $N$  and  $I$ . The interest rate is a growth rate: If money is deposited and earns 5% per year, then your funds will grow by 5% per year. Note also that time value concepts can be applied to anything that grows—sales, population, earnings per share, or your future salary. Also, as noted before, the “growth rate” can be negative, as was sales growth for a number of auto companies in recent years.



## Simple Interest versus Compound Interest

As explained earlier, when interest is earned on the interest earned in prior periods, we call it **compound interest**. If interest is earned only on the principal, we call it **simple interest**. The total interest earned with simple interest is equal to the principal multiplied by the interest rate times the number of periods:  $PV(I)(N)$ . The future value is equal to the principal plus the interest:  $FV = PV + PV(I)(N)$ . For example, suppose you deposit \$100 for 3 years and earn simple interest at an annual rate of 5%. Your balance at the end of 3 years would be:

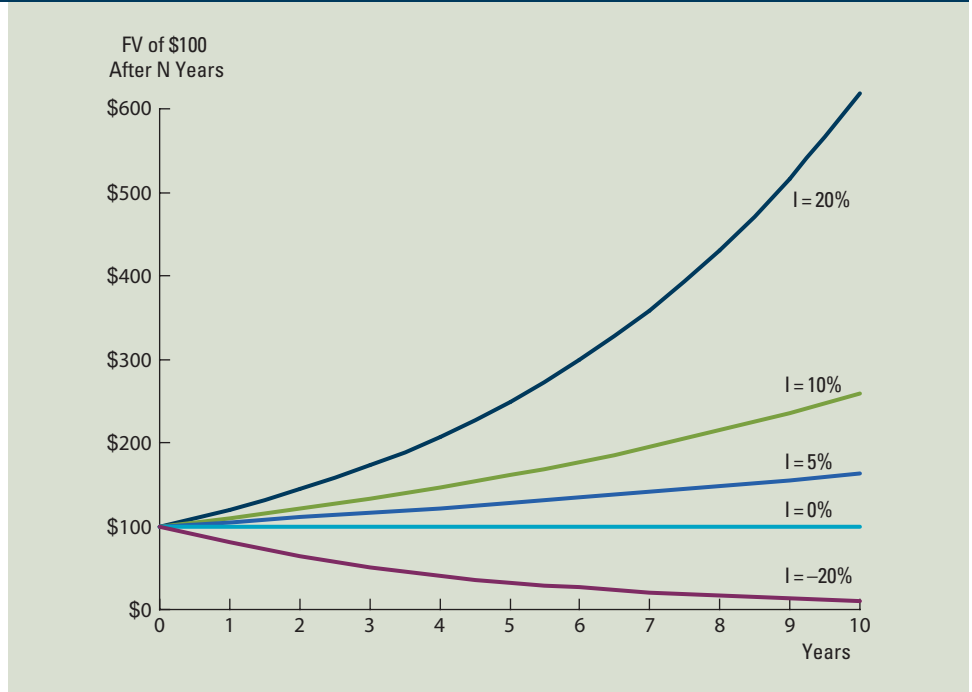
$$\begin{aligned} FV &= PV + PV(I)(N) \\ &= \$100 + \$100(5\%)(3) \\ &= \$100 + \$15 = \$115 \end{aligned}$$

Notice that this is less than the \$115.76 we calculated earlier using compound interest. Most applications in finance are based on compound interest, but you should be aware that simple interest is still specified in some legal documents.

### Self-Test

Explain why this statement is true: “A dollar in hand today is worth more than a dollar to be received next year, assuming interest rates are positive.”

What is compounding? What would the future value of \$100 be after 5 years at 10% *compound* interest? (**\$161.05**)

**FIGURE 4-2** Growth of \$100 at Various Interest Rates and Time Periods

Suppose you currently have \$2,000 and plan to purchase a 3-year certificate of deposit (CD) that pays 4% interest, compounded annually. How much will you have when the CD matures? (**\$2,249.73**) How would your answer change if the interest rate were 5%, or 6%, or 20%? (*Hint:* With a calculator, enter  $N = 3$ ,  $I/YR = 4$ ,  $PV = -2000$ , and  $PMT = 0$ ; then press FV to get 2,249.73. Then, enter  $I/YR = 5$  to override the 4% and press FV again to get the second answer. In general, you can change one input at a time to see how the output changes.) (**\$2,315.25; \$2,382.03; \$3,456.00**)

A company's sales in 2009 were \$100 million. If sales grow by 8% annually, what will they be 10 years later? (**\$215.89 million**) What would they be if they decline by 8% per year for 10 years? (**\$43.44 million**)

How much would \$1, growing at 5% per year, be worth after 100 years? (**\$131.50**)

What would FV be if the growth rate were 10%? (**\$13,780.61**)

## 4.3 PRESENT VALUES

Suppose you have some extra money and want to make an investment. A broker offers to sell you a bond that will pay a guaranteed \$115.76 in 3 years. Banks are currently offering a guaranteed 5% interest on 3-year certificates of deposit (CDs), and if you don't buy the bond you will buy a CD. The 5% rate paid on the CD is defined as your **opportunity cost**, or the rate of return you would earn on an alternative investment of similar risk if you don't invest in the security under consideration. Given these conditions, what's the most you should pay for the bond?

First, recall from the future value example in the last section that if you invested \$100 at 5% in a CD, it would grow to \$115.76 in 3 years. You would also have \$115.76 after 3 years if you bought the bond. Therefore, the most you should pay for the bond is \$100—this is its “fair price,” which is also its intrinsic, or fundamental, value. If you

could buy the bond for *less than* \$100, then you should buy it rather than invest in the CD. Conversely, if its price were *more than* \$100, you should buy the CD. If the bond's price were exactly \$100, you should be indifferent between the bond and the CD.

The \$100 is defined as the present value, or PV, of \$115.76 due in 3 years when the appropriate interest rate is 5%. In general, *the present value of a cash flow due N years in the future is the amount which, if it were on hand today, would grow to equal the given future amount.* Since \$100 would grow to \$115.76 in 3 years at a 5% interest rate, \$100 is the present value of \$115.76 due in 3 years at a 5% rate.

Finding present values is called **discounting**, and as previously noted, it is the reverse of compounding: If you know the PV, you can compound to find the FV; or if you know the FV, you can discount to find the PV. Indeed, we simply solve Equation 4-1, the formula for the future value, for the PV to produce the present value equation as follows.

$$\text{Compounding to find future values : Future value} = FV_N = PV(1 + I)^N \quad (4-1)$$

$$\text{Discounting to find present values : Present value} = PV = \frac{FV_N}{(1 + I)^N} \quad (4-3)$$



The top section of Figure 4-3 shows inputs and a time line for finding the present value of \$115.76 discounted back for 3 years. We first calculate the PV using the step-by-step approach. When we found the FV in the previous section, we worked from left to right, *multiplying* the initial amount and each subsequent amount by (1 + I). To find

**FIGURE 4-3** Alternative Procedures for Calculating Present Values

	A	B	C	D	E	F	G	
109	INPUTS:							
110	Future payment = CF <sub>N</sub> = FV =		\$115.76					
111	Interest rate = I =		5.00%					
112	No. of periods = N =		3					
113								
114	Problem as a Time Line		Periods:	0	1	2	3	
115								
116	Cash Flow Time Line:			PV = ?			\$115.76	
117								
118	1. Step-by-Step:			\$100.00 ←	\$105.00 ←	\$110.25 ←	\$115.76	
119								
120	2. Formula: $FV_N = PV/(1+I)^N$			PV = \$115.76(1.05) <sup>3</sup> =			\$100.00	
121								
122		3	5		\$0		\$115.76	
123	3. Financial Calculator:		N	I/YR	PV	PMT	FV	
124					-\$100.00			
125								
126		PV Function:	PV =	= PV(I,N,0,FV)				
127	4. Excel Spreadsheet:		Fixed inputs:	PV =	= PV(0.05,3,0,115.76) =			-\$100.00
128		Cell references:	PV =	= PV(C111,C112,0,C110) =			-\$100.00	
129	In the Excel formula, the terms are entered in the sequence: interest, periods, 0 to indicate no periodic cash flows, and then the FV. The data can be entered as fixed numbers or, better yet, as cell references.							

present values, we work backwards, or from right to left, *dividing* the future value and each subsequent amount by  $(1 + I)$ , with the present value of \$100 shown in Cell D118. The step-by-step procedure shows exactly what's happening, and that can be quite useful when you are working complex problems or trying to explain a model to others. However, it's inefficient, especially if you are dealing with more than a year or two.

A more efficient procedure is to use the formula approach in Equation 4-3, simply dividing the future value by  $(1 + I)^N$ . This gives the same result, as we see in Figure 4-3, Cell G120.

Equation 4-2 is actually programmed into financial calculators. As shown in Figure 4-3, Rows 122 to 124, we can find the PV by entering values for  $N=3$ ,  $I/YR=5$ ,  $PMT=0$ , and  $FV=115.76$ , and then pressing the PV key to get  $-100$ .

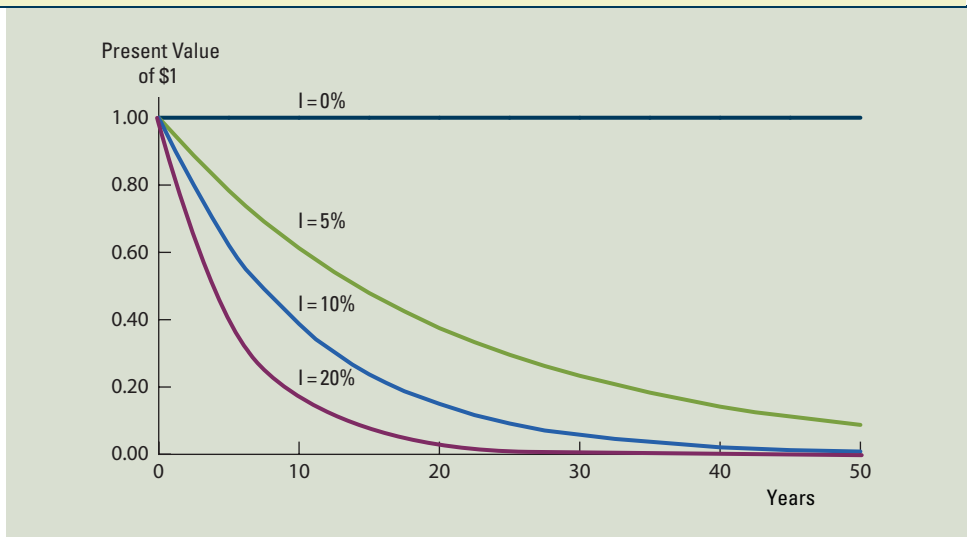
*Excel* also has a function that solves Equation 4-3—this is the PV function, and it is written as  $=PV(I,N,0,FV)$ .<sup>6</sup> Cell E126 shows the inputs to this function. Next, Cell E127 shows the *Excel* function with fixed numbers as inputs, with the actual function and the resulting  $-\$100$  in Cell G127. Cell E128 shows the *Excel* function using cell references, with the actual function and the resulting  $-\$100$  in Cell G128.

The fundamental goal of financial management is to maximize the firm's intrinsic value, and the intrinsic value of a business (or any asset, including stocks and bonds) is the *present value* of its expected future cash flows. Because present value lies at the heart of the valuation process, we will have much more to say about it in the remainder of this chapter and throughout the book.

## Graphic View of the Discounting Process

Figure 4-4 shows that the present value of a sum to be received in the future decreases and approaches zero as the payment date is extended further and further into the future; it also shows that, the higher the interest rate, the faster the present value falls. At relatively high rates, funds due in the future are worth very little today, and even at relatively

**FIGURE 4-4** Present Value of \$1 at Various Interest Rates and Time Periods



<sup>6</sup>The third entry in the PV function is zero to indicate that there are no intermediate payments in this particular example.

## Self-Test

low rates present values of sums due in the very distant future are quite small. For example, at a 20% discount rate, \$1 million due in 100 years would be worth just over 1 cent today. (However, 1 cent would grow to almost \$1 million in 100 years at 20%.)

What is “discounting,” and how is it related to compounding? How is the future value equation (4-1) related to the present value equation (4-3)?

How does the present value of a future payment change as the time to receipt is lengthened? As the interest rate increases?

Suppose a risk-free bond promises to pay \$2,249.73 in 3 years. If the going risk-free interest rate is 4%, how much is the bond worth today? (**\$2,000**) How would your answer change if the bond matured in 5 rather than 3 years? (**\$1,849.11**) If the risk-free interest rate is 6% rather than 4%, how much is the 5-year bond worth today? (**\$1,681.13**)

How much would \$1 million due in 100 years be worth today if the discount rate were 5%? (**\$7,604.49**) What if the discount rate were 20%? (**\$0.0121**)

## 4.4 FINDING THE INTEREST RATE, I

Thus far we have used Equations 4-1, 4-2, and 4-3 to find future and present values. Those equations have four variables, and if we know three of them, then we (or our calculator or *Excel*) can solve for the fourth. Thus, if we know PV, I, and N, we can solve Equation 4-1 for FV, or if we know FV, I, and N, we can solve Equation 4-3 to find PV. That’s what we did in the preceding two sections.

Now suppose we know PV, FV, and N, and we want to find I. For example, suppose we know that a given security has a cost of \$100 and that it will return \$150 after 10 years. Thus, we know PV, FV, and N, and we want to find the rate of return we will earn if we buy the security. Here’s the solution using Equation 4-1:

$$\begin{aligned} FV &= PV(1 + I)^N \\ \$150 &= \$100(1 + I)^{10} \\ \$150/\$100 &= (1 + I)^{10} \\ (1 + I)^{10} &= 1.5 \\ (1 + I) &= 1.5^{(1/10)} \\ 1 + I &= 1.0414 \\ I &= 0.0414 = 4.14\%. \end{aligned}$$

Finding the interest rate by solving the formula takes a little time and thought, but financial calculators and spreadsheets find the answer almost instantly. Here’s the calculator setup:

Inputs:	10	-100	0	150
	<b>N</b>	<b>PV</b>	<b>PMT</b>	<b>FV</b>
Output:		4.14		

Enter N=10, PV=-100, PMT=0 (because there are no payments until the security matures), and FV=150. Then, when you press the I/YR key, the calculator gives the answer, 4.14%. Notice that the PV is a negative value because it is a cash outflow (an investment) and the FV is positive because it is a cash inflow (a return of the investment). If you enter both PV and FV as positive numbers (or both as negative numbers), you will get an error message rather than the answer.

**Self-Test**

In *Excel*, the **RATE** function can be used to find the interest rate:  $\text{=RATE}(N, \text{PMT}, \text{PV}, \text{FV})$ . For this example, the interest rate is found as  $\text{=RATE}(10, 0, -100, 150) = 0.0414 = 4.14\%$ . See the file *Ch04 Tool Kit.xls* on the textbook's Web site for an example.

Suppose you can buy a U.S. Treasury bond that makes no payments until the bond matures 10 years from now, at which time it will pay you \$1,000.<sup>7</sup> What interest rate would you earn if you bought this bond for \$585.43? (**5.5%**) What rate would you earn if you could buy the bond for \$550? (**6.16%**) For \$600? (**5.24%**) Microsoft earned \$0.33 per share in 1997. Ten years later, in 2007, it earned \$1.42. What was the growth rate in Microsoft's earnings per share (EPS) over the 10-year period? (**15.71%**) If EPS in 2007 had been \$1.00 rather than \$1.42, what would the growth rate have been? (**11.72%**)

## 4.5 FINDING THE NUMBER OF YEARS, N

We sometimes need to know how long it will take to accumulate a specific sum of money, given our beginning funds and the rate we will earn. For example, suppose we now have \$500,000 and the interest rate is 4.5%. How long will it be before we have \$1 million?

Here's Equation 4-1, showing all the known variables.

$$\$1,000,000 = \$500,000(1 + 0.045)^N \quad (4-1)$$

We need to solve for N, and we can use three procedures: a financial calculator, *Excel* (or some other spreadsheet), or by working with natural logs. As you might expect, the calculator and spreadsheet approaches are easier.<sup>8</sup> Here's the calculator setup:

<b>Inputs:</b>	4.5	-500000	0	1000000
	<b>N</b>	<b>I/YR</b>	<b>PV</b>	<b>PMT</b>
<b>Output:</b>	15.7473			

**Self-Test**

Enter I/YR = 4.5, PV = -500000, PMT = 0, and FV = 1000000. We press the N key to get the answer, 15.7473 years. In *Excel*, we would use the **NPER** function:  $\text{=NPER}(I, \text{PMT}, \text{PV}, \text{FV})$ . Inserting data, we have  $\text{=NPER}(0.045, 0, -500000, 1000000) = 15.7473$ . The chapter's tool kit, *Ch04 Tool Kit.xls*, shows this example.

How long would it take \$1,000 to double if it were invested in a bank that pays 6% per year? (**11.9 years**) How long would it take if the rate were 10%? (**7.27 years**) Microsoft's 2007 earnings per share were \$1.42, and its growth rate during the prior 10 years was 15.71% per year. If that growth rate were maintained, how long would it take for Microsoft's EPS to double? (**4.75 years**)

<sup>7</sup>This is a STRIP bond, which we explain in Chapter 5.

<sup>8</sup>Here's the setup for the log solution. First, transform Equation 4-1 as indicated, then find the natural logs using a financial calculator, and then solve for N:

$$\begin{aligned} \$1,000,000 &= \$500,000(1 + 0.045)^N \\ 2 &= (1 + 0.045)^N \\ \ln(2) &= N[\ln(1.045)] \\ N &= 0.6931/0.0440 = 15.7473 \text{ years} \end{aligned}$$



## 4.6 ANNUITIES

Thus far we have dealt with single payments, or “lump sums.” However, assets such as bonds provide a series of cash inflows over time, and obligations such as auto loans, student loans, and mortgages call for a series of payments. If the payments are equal and are made at fixed intervals, then we have an **annuity**. For example, \$100 paid at the end of each of the next 3 years is a 3-year annuity.

If payments occur at the *end* of each period, then we have an **ordinary** (or **deferred**) **annuity**. Payments on mortgages, car loans, and student loans are generally made at the ends of the periods and thus are ordinary annuities. If the payments are made at the *beginning* of each period, then we have an **annuity due**. Rental lease payments, life insurance premiums, and lottery payoffs (if you are lucky enough to win one!) are examples of annuities due. Ordinary annuities are more common in finance, so when we use the term “annuity” in this book, you may assume that the payments occur at the ends of the periods unless we state otherwise.

Next we show the time lines for a \$100, 3-year, 5%, ordinary annuity and for the same annuity on an annuity due basis. With the annuity due, each payment is shifted back (to the left) by 1 year. In our example, we assume that a \$100 payment will be made each year, so we show the payments with minus signs.

### Ordinary Annuity:



### Annuity Due:



As we demonstrate in the following sections, we can find an annuity’s future value, present value, the interest rate built into the contracts, how long it takes to reach a financial goal using the annuity, and, if we know all of those values, the size of the annuity payment. Keep in mind that annuities must have *constant payments* and a *fixed number of periods*. If these conditions don’t hold, then the series is not an annuity.

#### Self-Test

What’s the difference between an ordinary annuity and an annuity due?

Why should you prefer to receive an *annuity due* with payments of \$10,000 per year for 10 years than an otherwise similar *ordinary annuity*?

## 4.7 FUTURE VALUE OF AN ORDINARY ANNUITY

Consider the ordinary annuity whose time line was shown previously, where you deposit \$100 at the *end* of each year for 3 years and earn 5% per year. Figure 4-5 shows how to calculate the **future value of the annuity**,  $FVA_N$ , using the same approaches we used for single cash flows.

As shown in the step-by-step section of Figure 4-5, we compound each payment out to Time 3, then sum those compounded values in Cell F226 to find the annuity’s FV,  $FVA_3 = \$315.25$ . The first payment earns interest for two periods, the second for

FIGURE 4-5 Summary: Future Value of an Ordinary Annuity

	A	B	C	D	E	F	G
214	INPUTS:						
215	Payment amount	= PMT =	-\$100.00				
216	Interest rate	= I =	5.00%				
217	No. of periods	= N =	3				
218							
219	1. Step-by-Step:						
220			0	1	2	3	
221				-\$100	-\$100	-\$100	
222						\$100.00	
223						\$105.00	
224						\$110.25	
225						\$315.25	
226							
227							
228	2. Formula:						
229							
230							
231							
232							
233	3. Financial Calculator:						
234			3	5	\$0	-\$100.00	
235			N	I/YR	PV	PMT	FV
236							\$315.25
237	4. Excel Spreadsheet:						
238		FV Function:	FVA <sub>N</sub> =	= FV(I,N,PMT,PV)			
239		Fixed inputs:	FVA <sub>N</sub> =	= FV(0.05,3,-100,0)			\$315.25
240		Cell references:	FVA <sub>N</sub> =	= FV(C216,C217,C215,0)			\$315.25

one period, and the third earns no interest because it is made at the end of the annuity's life. This approach is straightforward, but if the annuity extends out for many years, it is cumbersome and time-consuming.

As you can see from the time line diagram, with the step-by-step approach we apply the following equation with  $N = 3$  and  $I = 5\%$ :

$$\begin{aligned} FVA_N &= PMT(1 + I)^{N-1} + PMT(1 + I)^{N-2} + PMT(1 + I)^{N-3} \\ &= \$100(1.05)^2 + \$100(1.05)^1 + \$100(1.05)^0 \\ &= \$315.25 \end{aligned}$$

For the general case, the future value of an annuity is

$$FVA_N = PMT(1 + I)^{N-1} + PMT(1 + I)^{N-2} + PMT(1 + I)^{N-3} + \dots + PMT(1 + I)^0$$

As shown in **Web Extension 4B** on the textbook's Web site, the future value of an annuity can be written as follows:<sup>9</sup>

<sup>9</sup>Section 4.11 shows that the present value of an infinitely long annuity, called a perpetuity, is equal to  $PMT/I$ . The cash flows of an ordinary annuity of  $N$  periods are equal to the cash flows of a perpetuity minus the cash flows of a perpetuity that begins at year  $N+1$ . Therefore, the future value of an  $N$ -period annuity is equal to the future value (as of year  $N$ ) of a perpetuity minus the value (as of year  $N$ ) of a perpetuity that begins at year  $N+1$ . See **Web Extension 4B** on the textbook's Web site for details regarding derivations of Equation 4-4.

$$FVA_N = PMT \left[ \frac{(1+I)^N}{I} - \frac{1}{I} \right] \quad (4-4)$$

Using Equation 4-4, the future value of the annuity is found to be \$315.25:

$$FVA_3 = \$100 \left[ \frac{(1+0.05)^3}{0.05} - \frac{1}{0.05} \right] = \$315.25$$

As you might expect, annuity problems can be solved easily using a financial calculator or a spreadsheet, most of which have the following formula built into them:

$$PV(1+I)^N + PMT \left[ \frac{(1+I)^N}{I} - \frac{1}{I} \right] + FV = 0 \quad (4-5)$$

The procedure when dealing with annuities is similar to what we have done thus far for single payments, but the presence of recurring payments means that we must use the PMT key. Here's the calculator setup for our illustrative annuity:

<b>Inputs:</b>	3	5	0	-100	End Mode
	<b>N</b>	<b>I/YR</b>	<b>PV</b>	<b>PMT</b>	<b>FV</b>
<b>Output:</b>					315.25

We enter  $PV = 0$  because we start off with nothing, and we enter  $PMT = -100$  because we will deposit this amount in the account at the end of each of the 3 years. The interest rate is 5%, and when we press the FV key we get the answer,  $FVA_3 = 315.25$ .

Since this is an ordinary annuity, with payments coming at the *end* of each year, we must set the calculator appropriately. As noted earlier, most calculators “come out of the box” set to assume that payments occur at the end of each period—that is, to deal with ordinary annuities. However, there is a key that enables us to switch between ordinary annuities and annuities due. For ordinary annuities, the designation “End Mode” or something similar is used, while for annuities due the designator is “Begin,” “Begin Mode,” “Due,” or something similar. If you make a mistake and set your calculator on Begin Mode when working with an ordinary annuity, then each payment will earn interest for one extra year, which will cause the compounded amounts, and thus the  $FVA$ , to be too large.

The spreadsheet approach uses *Excel's* FV function,  $=FV(I,N,PMT,PV)$ . In our example, we have  $=FV(0.05,3,-100,0)$ , and the result is again \$315.25.



#### Self-Test

For an ordinary annuity with 5 annual payments of \$100 and a 10% interest rate, for how many years will the *first payment* earn interest, and what is the compounded value of this payment at the end? (**4 years, \$146.41**) Answer this same question for the fifth payment. (**0 years, \$100**)

Assume that you plan to buy a condo 5 years from now, and you estimate that you can save \$2,500 per year toward a down payment. You plan to deposit the money in a bank that pays 4% interest, and you will make the first deposit at the end of this year. How much will you have after 5 years? (**\$13,540.81**) How would your answer change if the bank's interest rate were increased to 6%, or decreased to 3%? (**\$14,092.73; \$13,272.84**)

## 4.8 FUTURE VALUE OF AN ANNUITY DUE

Because each payment occurs one period earlier with an annuity due, the payments will all earn interest for one additional period. Therefore, the FV of an annuity due will be greater than that of a similar ordinary annuity.

If you went through the step-by-step procedure, you would see that our illustrative annuity due has a FV of \$331.01 versus \$315.25 for the ordinary annuity. See *Ch04 Tool Kit.xls* on the textbook's Web site for a summary of future value calculations.

With the formula approach, we first use Equation 4-4, but since each payment occurs one period earlier, we multiply the Equation 4-4 result by  $(1 + I)$ :

$$FVA_{\text{due}} = FVA_{\text{ordinary}}(1 + I) \quad (4-6)$$

Thus, for the annuity due,  $FVA_{\text{due}} = \$315.25(1.05) = \$331.01$ , which is the same result as found with the step-by-step approach.

With a calculator we input the variables just as we did with the ordinary annuity, but we now set the calculator to Begin Mode to get the answer, \$331.01.

<b>Inputs:</b>	3	5	0	-100	Begin Mode
	<b>N</b>	<b>I/YR</b>	<b>PV</b>	<b>PMT</b>	<b>FV</b>
<b>Output:</b>					331.01



In *Excel*, we still use the FV function, but we must indicate that we have an annuity due. The function is  $=FV(I,N,PMT,PV,Type)$ , where “Type” indicates the type of annuity. If Type is omitted then *Excel* assumes that it is 0, which indicates an ordinary annuity. For an annuity due, Type = 1. As shown in *Ch04 Tool Kit.xls*, the function is  $=FV(0.05,3,-100,0,1) = \$331.01$ .

### Self-Test

Why does an annuity due always have a higher future value than an ordinary annuity?

If you know the value of an ordinary annuity, explain why you could find the value of the corresponding annuity due by multiplying by  $(1+i)$ .

Assume that you plan to buy a condo 5 years from now and that you need to save for a down payment. You plan to save \$2,500 per year, with the first payment being made *immediately* and deposited in a bank that pays 4%. How much will you have after 5 years? (**\$14,082.44**) How much would you have if you made the deposits at the *end* of each year? (**\$13,540.81**)

## 4.9 PRESENT VALUE OF ORDINARY ANNUITIES AND ANNUITIES DUE

The present value of any annuity,  $PVA_N$ , can be found using the step-by-step, formula, calculator, or spreadsheet methods. We begin with ordinary annuities.

### Present Value of an Ordinary Annuity

See Figure 4-6 for a summary of the different approaches for calculating the present value of an ordinary annuity.

FIGURE 4-6 Summary: Present Value of an Ordinary Annuity

	A	B	C	D	E	F	G
283	INPUTS:						
284	Payment amount	= PMT =	-\$100.00				
285	Interest rate	= I =	5.00%				
286	No. of periods	= N =	3				
287							
288	Cash Flow Time Line:						
289	Periods:	0	1	2	3		
290			-\$100	-\$100	-\$100		
291			\$95.24				
292	1. Step-by-Step:		\$90.70				
293	Divide each payment by		\$86.38				
294	(1+I) <sup>t</sup> and sum these PVs to						
295	find PVA <sub>N</sub> :		\$272.32				
296							
297	2. Formula:						
298							
299							\$272.32
300							
301							
302							
303	3. Financial Calculator:						
304		3	5		-100.00		0
305		N	I	PV	PMT	FV	
306				272.32			
307	4. Excel Spreadsheet:						
308		FV Function:	PVA <sub>N</sub> =	= PV(I,N,PMT,FV)			
309		Fixed inputs:	PVA <sub>N</sub> =	= PV(0.05,3,-100,0)	=	\$272.32	
		Cell references:	PVA <sub>N</sub> =	= PV(C285,C286,C284,0)	=	\$272.32	



As shown in the step-by-step section of Figure 4-6, we discount each payment back to Time 0, then sum those discounted values to find the annuity's PV,  $PVA_3 = \$272.32$ . This approach is straightforward, but if the annuity extends out for many years, it is cumbersome and time-consuming.

The time line diagram shows that with the step-by-step approach we apply the following equation with  $N = 3$  and  $I = 5\%$ :

$$PVA_N = PMT/(1+I)^1 + PMT/(1+I)^2 + \dots + PMT/(1+I)^N$$

The present value of an annuity can be written as<sup>10</sup>

$$PVA_N = PMT \left[ \frac{1}{I} - \frac{1}{I(1+I)^N} \right] \quad (4-7)$$

For our illustrative annuity, the present value is

$$PVA_3 = PMT \left[ \frac{1}{0.05} - \frac{1}{0.05(1+0.05)^3} \right] = \$272.32$$

<sup>10</sup>See *Web Extension 4B* on the textbook's Web site for details of this derivation.

Financial calculators are programmed to solve Equation 4-7, so we merely input the variables and press the PV key, *first making sure the calculator is set to End Mode*. The calculator setup is shown below:

Inputs:	3	5	-100	0	End Mode (Ordinary Annuity)
	<b>N</b>	<b>I/YR</b>	<b>PV</b>	<b>PMT</b>	<b>FV</b>
Output:			272.32		



Section 4 of Figure 4-6 shows the spreadsheet solution using *Excel's* built-in PV function:  $=PV(I,N,PMT,FV)$ . In our example, we have  $=PV(0.05,3,-100,0)$  with a resulting value of \$272.32.

## Present Value of Annuities Due

Because each payment for an annuity due occurs one period earlier, the payments will all be discounted for one less period. Therefore, the PV of an annuity due must be greater than that of a similar ordinary annuity.

If you went through the step-by-step procedure, you would see that our illustrative annuity due has a PV of \$285.94 versus \$272.32 for the ordinary annuity. See *Ch04 Tool Kit.xls* for this and the other calculations.

With the formula approach, we first use Equation 4-7 to find the value of the ordinary annuity and then, since each payment now occurs one period earlier, we multiply the Equation 4-7 result by  $(1 + I)$ :

$$PVA_{\text{due}} = PVA_{\text{ordinary}}(1 + I) \quad (4-8)$$

$$PVA_{\text{due}} = \$272.32(1.05) = \$285.94$$

With a financial calculator, the inputs are the same as for an ordinary annuity, except you must set the calculator to Begin Mode:

Inputs:	3	5	-100	0	Begin Mode (Annuity Due)
	<b>N</b>	<b>I/YR</b>	<b>PV</b>	<b>PMT</b>	<b>FV</b>
Output:			285.94		



In *Excel*, we again use the PV function, but now we must indicate that we have an annuity due. The function is now  $=PV(I,N,PMT,FV,Type)$ , where “Type” is the type of annuity. If Type is omitted then *Excel* assumes that it is 0, which indicates an ordinary annuity; for an annuity due, Type = 1. As shown in *Ch04 Tool Kit.xls*, the function for this example is  $=PV(0.05,3,-100,0,1) = \$285.94$ .

### Self-Test

Why does an annuity due have a higher present value than an ordinary annuity? If you know the present value of an ordinary annuity, what's an easy way to find the PV of the corresponding annuity due?

What is the PVA of an ordinary annuity with 10 payments of \$100 if the appropriate interest rate is 10%? (**\$614.46**) What would the PVA be if the interest rate were 4%? (**\$811.09**) What if the interest rate were 0%? (**\$1,000.00**) What would the PVAs be if we were dealing with annuities due? (**\$675.90, \$843.53, and \$1,000.00**)

## Variable Annuities: Good or Bad?

Retirees appreciate stable, predictable income, so they often buy annuities. Insurance companies have been the traditional suppliers, using the payments they receive to buy high-grade bonds, whose interest is then used to make the promised payments. Such annuities were quite safe and stable and provided returns of around 7.5%. However, returns on stocks (dividends plus capital gains) have historically exceeded bonds' returns (interest). Therefore, some insurance companies in the 1990s began to offer *variable annuities*, which were backed by stocks instead of bonds. If stocks earned in the future as much as they had in the past, then variable annuities could offer returns of about 9%, better than the return on a fixed rate annuities. If stock returns turned out to be lower in the future than they had been in the past (or even had negative returns), then the variable annuities promised a *guaranteed minimum payment* of about 6.5%. Variable annuities appealed to many retirees, so companies that offered them had a significant competitive advantage.

The insurance company that pioneered variable annuities, The Hartford, tried to hedge its position with derivatives that paid off if stocks went down. But like so many other derivatives-based risk management programs, this one went awry in 2008 because stock losses exceeded the assumed worst-case scenario. The Hartford, which was founded in 1810 and was one of the oldest and largest U.S. insurance companies at the beginning of 2008, saw its stock fall from \$85.54 to \$4.16. Because of the general stock market crash, investors feared that The Hartford would be unable to make good on its variable annuity promises, which would lead to bankruptcy. The company was bailed out by the economic stimulus package, but this 199-year-old firm will never be the same again.

*Source:* Leslie Scism and Liam Plevin, "Hartford Aims to Take Risk Out of Annuities," *Online Wall Street Journal*, January 13, 2009.

Assume that you are offered an annuity that pays \$100 at the end of each year for 10 years. You could earn 8% on your money in other equally risky investments. What is the most you should pay for the annuity? **(\$671.01)** If the payments began immediately, then how much would the annuity be worth? **(\$724.69)**

## 4.10 FINDING ANNUITY PAYMENTS, PERIODS, AND INTEREST RATES

In the three preceding sections we discussed how to find the FV and PV of ordinary annuities and annuities due, using these four methods: step-by-step, formula, financial calculator, and *Excel*. Five variables are involved—N, I, PMT, FV, and PV—and if you know any four, you can find the fifth by solving either 4-4 (4-6 for annuities due) or 4-7 (4-8 for annuities due). However, a trial-and-error procedure is generally required to find N or I, and that can be quite tedious. Therefore, we discuss only the financial calculator and spreadsheet approaches for finding N and I.

### Finding Annuity Payments, PMT

We need to accumulate \$10,000 and have it available 5 years from now. We can earn 6% on our money. Thus, we know that  $FV = 10,000$ ,  $PV = 0$ ,  $N = 5$ , and  $I/YR = 6$ . We can enter these values in a financial calculator and then press the PMT key to find our required deposits. However, the answer depends on whether we make deposits at the end of each year (ordinary annuity) or at the beginning (annuity due), so the mode must be set properly. Here are the results for each type of annuity:



Inputs:	5	6	0	10000	End Mode (Ordinary Annuity)
	<b>N</b>	<b>I/YR</b>	<b>PV</b>	<b>PMT</b>	<b>FV</b>
Output:				-1773.96	

Inputs:	5	6	0	10000	Begin Mode (Annuity Due)
	<b>N</b>	<b>I/YR</b>	<b>PV</b>	<b>PMT</b>	<b>FV</b>
Output:				-1673.55	

Thus, you must put away \$1,773.96 per year if you make payments at the *end* of each year, but only \$1,673.55 if the payments begin *immediately*. Finally, note that the required payment for the annuity due is the ordinary annuity payment divided by  $(1 + I)$ :  $\$1,773.96/1.06 = \$1,673.55$ .

*Excel* can also be used to find annuity payments, as shown below for the two types of annuities. For end-of-year (ordinary) annuities, “Type” can be left blank or a 0 can be inserted. For beginning-of-year annuities (annuities due), the same function is used but now Type is designated as 1. Here is the setup for the two types of annuities.

**Function :** = PMT(I, N, PV, FV, Type)

**Ordinary annuity :** = PMT(0.06, 5, 0, 10000) = \$1,773.96

**Annuity due** = PMT(0.06, 5, 0, 10000, 1) = \$1,673.55



## Finding the Number of Periods, N

Suppose you decide to make end-of-year deposits, but you can save only \$1,200 per year. Again assuming that you would earn 6%, how long would it take you to reach your \$10,000 goal? Here is the calculator setup:

Inputs:		6	0	-1200	10000	End Mode
	<b>N</b>	<b>I/YR</b>	<b>PV</b>	<b>PMT</b>	<b>FV</b>	
Output:	6.96					



With these smaller deposits, it would take 6.96 years, not 5 years, to reach the \$10,000 target. If you began the deposits immediately, then you would have an annuity due and N would be slightly less, 6.63 years.

With *Excel*, you can use the NPER function: =NPER(I,PMT,PV,FV, Type). For our ordinary annuity example, Type is left blank (or 0 is inserted) and the function is =NPER(0.06,-1200,0,10000) = 6.96. If we put in 1 for type, we would find N = 6.63.

## Finding the Interest Rate, I

Now suppose you can save only \$1,200 annually, but you still need to have the \$10,000 in 5 years. What rate of return would you have to earn to reach your goal? Here is the calculator setup:

<b>Inputs:</b>	5	0	-1200	10000	End Mode
	<b>N</b>	<b>I/YR</b>	<b>PV</b>	<b>PMT</b>	<b>FV</b>
<b>Output:</b>	25.78				

Thus, you would need to earn a whopping 25.78%! About the only way to earn such a high return would be either to invest in speculative stocks or head to a Las Vegas casino. Of course, speculative stocks and gambling aren't like making deposits in a bank with a guaranteed rate of return, so there would be a high probability that you'd end up with nothing. So, you should probably save more, lower your \$10,000 target, or extend your time horizon. It might be appropriate to seek a somewhat higher return, but trying to earn 25.78% in a 6% market would involve speculation, not investing.

In *Excel*, you can use the RATE function: **=RATE(N,PMT,PV,FV,Type)**. For our example, the function is **=RATE(5,-1200,0,10000)** = 0.2578 = 25.78%. If you decide to make the payments beginning immediately then the required rate of return would decline sharply, to 17.54%.

#### Self-Test

Suppose you inherited \$100,000 and invested it at 7% per year. How large of a withdrawal could you make at the *end* of each of the next 10 years and end up with zero? (**\$14,237.75**) How would your answer change if you made withdrawals at the *beginning* of each year? (**\$13,306.31**)

If you had \$100,000 that was invested at 7% and you wanted to withdraw \$10,000 at the end of each year, how long would your funds last? (**17.8 years**) How long would they last if you earned 0%? (**10 years**) How long would they last if you earned the 7% but limited your withdrawals to \$7,000 per year? (**forever**)

Your rich uncle named you as the beneficiary of his life insurance policy. The insurance company gives you a choice of \$100,000 today or a 12-year annuity of \$12,000 at the end of each year. What rate of return is the insurance company offering? (**6.11%**)

Assume that you just inherited an annuity that will pay you \$10,000 per year for 10 years, with the first payment being made today. A friend of your mother offers to give you \$60,000 for the annuity. If you sell it to him, what rate of return will your mother's friend earn on the investment? (**13.70%**) If you think a "fair" rate of return would be 6%, how much should you ask for the annuity? (**\$78,016.92**)

## 4.11 PERPETUITIES

In the previous section we dealt with annuities whose payments continue for a specific number of periods—for example, \$100 per year for 10 years. However, some securities promise to make payments forever. For example, in the mid-1700s the British government issued some bonds that never matured and whose proceeds were used to pay off other British bonds. Since this action consolidated the government's debt, the new bonds were called "consols." The term stuck, and now any bond that promises to pay interest perpetually is called a **consol**, or a perpetuity. The interest rate on the consols was 2.5%, so a consol with a face value of \$1,000 would pay \$25 per year in perpetuity.<sup>11</sup>

<sup>11</sup>The consols actually pay interest in pounds, but we discuss them in dollar terms for simplicity.

## Using the Internet for Personal Financial Planning

People continually face important financial decisions that require an understanding of the time value of money. Should we buy or lease a car? How much and how soon should we begin to save for our children's education? How expensive a house can we afford? Should we refinance our home mortgage? How much must we save each year if we are to retire comfortably?

The answers to these questions are often complicated, and they depend on a number of factors, such as projected housing and education costs, interest rates, inflation, expected family income, and stock mar-

ket returns. Hopefully, after completing this chapter, you will have a better idea of how to answer such questions. Note, though, that a number of online resources are available to help with financial planning. A good place to start is <http://www.smartmoney.com>. *Smartmoney* is a personal finance magazine produced by the publishers of *The Wall Street Journal*. If you go to *Smartmoney's* Web site you will find a section entitled "Tools." This section has a number of financial calculators, spreadsheets, and descriptive materials that cover a wide range of personal finance issues.

A consol, or **perpetuity**, is simply an annuity whose promised payments extend out forever. Since the payments go on forever, you can't apply the step-by-step approach. However, it's easy to find the PV of a perpetuity with the following formula:<sup>12</sup>

$$\text{PV of a perpetuity} = \frac{\text{PMT}}{I} \quad (4-9)$$

We can use Equation 4-9 to find the value of a British consol with a face value of \$1,000 that pays \$25 per year in perpetuity. The answer depends on the interest rate being earned on investments of comparable risk at the time the consol is being valued. Originally, the "going rate" as established in the financial marketplace was 2.5%, so originally the consol's value was \$1,000:

$$\text{Consol's value}_{\text{Originally}} = \$25/0.025 = \$1,000$$

The annual payment is still \$25 today, but the going interest rate has risen to about 5.2%, causing the consol's value to fall to \$480.77:

$$\text{Consol's value}_{\text{Today}} = \$25/0.052 = \$480.77$$

Note, though, that if interest rates decline in the future, say to 2%, then the value of the consol will rise to \$1,250.00:

$$\text{Consol's value if rates decline to 2\%} = \$25/0.02 = \$1,250.00$$

These examples demonstrate an important point: *When interest rates change, the prices of outstanding bonds also change, but inversely to the change in rates. Thus, bond prices decline if rates rise, and prices increase if rates fall. This holds for all bonds, both consols and those with finite maturities.* We will discuss this point in more detail in Chapter 5, where we cover bonds in depth.

### Self-Test

What is the present value of a perpetuity that pays \$1,000 per year, beginning 1 year from now, if the appropriate interest rate is 5%? (**\$20,000**) What would the value be if

<sup>12</sup>See *Web Extension 4B* on the textbook's Web site for a derivation of the perpetuity formula.

the annuity began its payments immediately? (**\$21,000**) (*Hint: Just add the \$1,000 to be received immediately to the formula value of the annuity.*)

Do bond prices move directly or inversely with interest rates—that is, what happens to the value of a bond if interest rates increase or decrease?

## 4.12 UNEVEN, OR IRREGULAR, CASH FLOWS

The definition of an annuity includes the term *constant payment*—in other words, annuities involve a set of identical payments over a given number of periods. Although many financial decisions do involve constant payments, many others involve cash flows that are **uneven** or **irregular**. For example, the dividends on common stocks are typically expected to increase over time, and investments in capital equipment almost always generate cash flows that vary from year to year. Throughout the book, we use the term *payment* (PMT) in situations where the cash flows are constant and thus an annuity is involved; we use the term *cash flow* ( $CF_t$ ), where the  $t$  designates the period in which the particular cash flow occurs, if the cash flows are irregular.

There are two important classes of uneven cash flows: (1) those in which the cash flow stream consists of a series of annuity payments plus an additional final lump sum in Year  $N$ , and (2) all other uneven streams. Bonds are an instance of the first type, while stocks and capital investments illustrate the second type. Here's an example of each type.

### Stream 1. Annuity plus additional final payment:

Periods	0	1	2	3	4	5
	$i = 12\%$					
Cash flows	\$0	\$100	\$100	\$100	\$100	\$100 \$1,000 <u>\$1,100</u>

### Stream 2. Irregular cash flows:

Periods	0	1	2	3	4	5
	$i = 12\%$					
Cash flows	\$0	\$100	\$300	\$300	\$300	\$500

Equation 4-10 can be used, following the step-by-step procedure, to find the PV of either stream. However, as we shall see, the solution process differs significantly for the two types.

$$PV = \frac{CF_1}{(1+I)^1} + \frac{CF_2}{(1+I)^2} + \dots + \frac{CF_N}{(1+I)^N} = \sum_{t=1}^N \frac{CF_t}{(1+I)^t} \quad (4-10)$$

### Annuity Plus Additional Final Payment

First, consider Stream 1 and notice that it is a 5-year, 12%, ordinary annuity plus a final payment of \$1,000. We can find the PV of the annuity, find the PV of the final payment, and then sum them to get the PV of the stream. Financial calculators are programmed to do this for us—we use all five time value of money (TVM) keys, entering the data for the four known values as shown below, and then pressing the PV key to get the answer, \$927.90:

<b>Inputs:</b>	5	12	100	1000	
	<b>N</b>	<b>I/YR</b>	<b>PV</b>	<b>PMT</b>	<b>FV</b>
<b>Output:</b>			-927.90		

Similarly, we could use *Excel's* PV function,  $=PV(I,N,PMT,FV) = PV(0.12,5,100,1000) = -\$927.90$ . Note that the process is similar to that for annuities, except we now have a nonzero value for FV.

### Irregular Cash Flow Stream

Now consider the irregular stream, which is analyzed in Figure 4-7. The top section shows the basic time line, which contains the inputs, and we first use the step-by-step approach to find  $PV = \$1,016.35$ . Note that we show the PV of each cash flow directly below the cash flow, and then we sum these PVs to find the PV of the stream. This setup saves space as compared with showing the individual PVs in a column, and it is also transparent and thus easy to understand.

Now consider the financial calculator approach. The cash flows don't form an annuity, so you can't use the annuity feature on the calculator. You could, of course, use the calculator in the step-by-step procedure, but financial calculators have a feature—the cash flow register—that allows you to find the present value more efficiently. First, you input the individual cash flows, in chronological



**FIGURE 4-7** Present Value of an Irregular Cash Flow Stream

	A	B	C	D	E	F	G
470	<u>Step-by-step:</u>						
471	Interest rate	= I =	12%				
472	Periods:	0	1	2	3	4	5
473	----- ----- ----- ----- ----- -----						
474	CF Time Line:	\$0.00	\$100.00	\$300.00	\$300.00	\$300.00	\$500.00
475	PVs of the CFs:		\$89.29	\$239.16	\$213.53	\$190.66	\$283.71
476							
477	$\Sigma C477:G477 =$	\$1,016.35	= Sum of the individual PVs = PV of the irregular CF stream.				
478							
479	Here we put the PVs of each individual CF under the CF itself and then summed them to find the PV of						
480	the entire stream, rather than show them all in Column C as was done in Figure 4-6. This setup takes						
481	up less space and also makes the calculations quite transparent, which is useful, especially when the						
482	table must be explained to people who did not develop it. People appreciate transparency and clarity.						
483							
484	<u>Calculator:</u>	You could enter the cash flows into the cash flow register of a financial calculator, enter I/YR, and then press the NPV key to find the answer.					\$1,016.35
485	<u>Excel Function:</u>	Fixed inputs:	NPV = =NPV(0.12,100,300,300,300,500)				\$1,016.35
486		Cell references:	NPV = =NPV(C471,C474:G474)				\$1,016.35
487							
488	Our Excel formula ignores the initial cash flow (in Year 0). When entering a cash flow range, Excel						
489	assumes that the first value occurs at the end of the first year. As we will see later, if there is an initial						
490	cash flow, it must be added separately to complete the NPV formula result. Notice too that you can						
491	enter cash flows one-by-one, but if the cash flows appear in consecutive cells, you can enter the cell						
492	range, as we did here.						

order, into the cash flow register.<sup>13</sup> Cash flows are designated  $CF_0$ ,  $CF_1$ ,  $CF_2$ ,  $CF_3, \dots$ ,  $CF_N$ . Next, you enter the interest rate,  $I$ . At this point, you have substituted in all the known values of Equation 4-10, so when you press the NPV key you get the PV of the stream. The calculator finds the PV of each cash flow and sums them to find the PV of the entire stream. To input the cash flows for this problem, enter 0 (because  $CF_0 = 0$ ), 100, 300, 300, 300, and 500 in that order into the cash flow register, enter  $I=12$ , and then press NPV to obtain the answer, \$1,016.35.

Two points should be noted. First, when dealing with the cash flow register, the calculator uses the term “NPV” rather than “PV.” The N stands for “net,” so NPV is the abbreviation for “net present value,” which is simply the net present value of a series of positive and negative cash flows, including any cash flow at time zero. The NPV function will be used extensively when we get to capital budgeting, where  $CF_0$  is generally the cost of the project.

The second point to note is that repeated cash flows with identical values can be entered into the cash flow register more efficiently by using the  $N_j$  key. In this illustration, you would enter  $CF_0=0$ ,  $CF_1=100$ ,  $CF_2=300$ ,  $N_j=3$  (which tells the calculator that the 300 occurs 3 times), and  $CF_5=500$ .<sup>14</sup> Then enter  $I=12$ , press the NPV key, and 1,016.35 will appear in the display. Also, note that numbers entered into the cash flow register remain in the register until they are cleared. Thus, if you previously worked a problem with eight cash flows, then moved to one with only four cash flows, the calculator would simply add the cash flows from the second problem to those of the first problem, and you would get an incorrect answer. Therefore, you must be sure to clear the cash flow register before starting a new problem.

Spreadsheets are especially useful for solving problems with uneven cash flows. You enter the cash flows in the spreadsheet as shown in Figure 4-7 on Row 474. To find the PV of these cash flows without going through the step-by-step process, you would use the NPV function. First put the cursor on the cell where you want the answer to appear, Cell G486, click Financial, scroll down to NPV, and click OK to get the dialog box. Then enter C471 (or 0.12) for Rate and enter either the individual cash flows or the range of cells containing the cash flows, C474:G474, for Value 1. Be very careful when entering the range of cash flows. With a financial calculator, you begin by entering the Time-0 cash flow. With *Excel*, you do *not* include the Time-0 cash flow; instead, you begin with the Year-1 cash flow. Now, when you click OK, you get the PV of the stream, \$1,016.35. Note that you can use the PV function if the payments are constant, but you must use the NPV function if the cash flows are not constant. Finally, note that *Excel* has a major advantage over financial calculators in that you can see the cash flows, which makes it easy to spot data entry errors. With a calculator, the numbers are buried in the machine, making it harder to check your work.

---

<sup>13</sup>We cover the calculator mechanics in the tutorial, and we discuss the process in more detail in Chapter 10, where we use the NPV calculation to analyze proposed projects. If you don't know how to use the cash flow register of your calculator, you should go to our tutorial or your calculator manual, learn the steps, and be sure you can make this calculation. You will have to know how to do it eventually, and now is a good time to learn.

<sup>14</sup>On some calculators, instead of entering  $CF_5 = 500$ , you enter  $CF_3 = 500$ , because this is the next cash flow *different* from 300.

## Self-Test

Could you use Equation 4-3, once for each cash flow, to find the PV of an uneven stream of cash flows?

What is the present value of a 5-year ordinary annuity of \$100 plus an additional \$500 at the end of Year 5 if the interest rate is 6%? (**\$794.87**) How would the PV change if the \$100 payments occurred in Years 1 through 10 and the \$500 came at the end of Year 10? (**\$1,015.21**)

What is the present value of the following uneven cash flow stream: \$0 at Time 0, \$100 at the end of Year 1 (or at Time 1), \$200 at the end of Year 2, \$0 at the end of Year 3, and \$400 at the end of Year 4—assuming the interest rate is 8%? (**\$558.07**) Would a “typical” common stock provide cash flows more like an annuity or more like an uneven cash flow stream?

## 4.13 FUTURE VALUE OF AN UNEVEN CASH FLOW STREAM

The future value of an uneven cash flow stream (sometimes called the **terminal**, or **horizon**, **value**) is found by compounding each payment to the end of the stream and then summing the future values:

$$\begin{aligned} FV &= CF_0(1+I)^N + CF_1(1+I)^{N-1} + CF_2(1+I)^{N-2} + \dots + CF_{N-1}(1+I) + CF_N \\ &= \sum_{t=0}^N CF_t(1+I)^{N-t} \end{aligned} \quad (4-11)$$

The future value of our illustrative uneven cash flow stream is \$1,791.15, as shown in Figure 4-8.

Most financial calculators have a net future value (NFV) key which, after the cash flows and interest rate have been entered, can be used to obtain the future value of an uneven cash flow stream. If your calculator doesn't have the NFV feature, you can first find the net present value of the stream, then find its net future value as  $NFV = NPV(1+I)^N$ . In the illustrative problem, we find  $PV = 1,016.35$  using the cash flow register and  $I=12$ . Then we use the TVM register, entering  $N=5$ ,  $I=12$ ,  $PV = -1016.35$ , and  $PMT = 0$ . When we press FV, we find  $FV = 1,791.15$ , which is

**FIGURE 4-8** Future Value of an Irregular Cash Flow Stream

	A	B	C	D	E	F	G
503	<u>Step-by-step:</u>						
504	Periods:	0	1	2	3	4	5
505	Interest rate	=	I	=	12%		
506		----- ----- ----- ----- -----					
507	CF Time Line:	\$0	\$100	\$300	\$300	\$300	\$500
508	FV of each CF:	\$0.00	\$157.35	\$421.48	\$376.32	\$336.00	\$500
509		Sum of the Cash Flows' FVs = FV of the stream =					\$1,791.15
510							
511	<u>Calculator:</u>	You could enter the cash flows into the cash flow register of a financial					\$1,791.15
512							
513	<u>Excel:</u>	Step 1. Find NPV:			=NPV(C505,C507:G507)		\$1,016.35
514		Step 2. Compound NPV to find NFV:			=FV(C505,G504,0,-G513)		\$1,791.15



the same as the value shown on the time line in Figure 4-8. As Figure 4-8 also shows, this same procedure can be used with *Excel*.

### Self-Test

What is the future value of this cash flow stream: \$100 at the end of 1 year, \$150 after 2 years, and \$300 after 3 years, assuming the appropriate interest rate is 15%? (**\$604.75**)

## 4.14 SOLVING FOR I WITH IRREGULAR CASH FLOWS

Before financial calculators and spreadsheets existed, it was *extremely difficult* to find I if the cash flows were uneven. However, with spreadsheets and financial calculators it's easy to find I. If you have an *annuity plus a final lump sum*, you can input values for N, PV, PMT, and FV into the calculator's TVM registers and then press the I/YR key. Here's the setup for Stream 1 from Section 4.12, assuming we must pay \$927.90 to buy the asset:

Inputs:	5	-927.90	100	1000
	N	I/YR	PMT	FV
Output:		12.00		

The rate of return on the \$927.90 investment is 12%.

Finding the interest rate for an *irregular cash flow stream* with a calculator is a bit more complicated. Figure 4-9 shows Stream 2 from Section 4.12, assuming a required investment of  $CF_0 = -\$1,000$ . First, note that there is no simple step-by-step method for finding the rate of return—finding the rate for this investment requires a trial-and-error process, which is terribly time consuming. Therefore, we really need a financial calculator or a spreadsheet. With a calculator, we would enter the CFs into the cash flow register and then press the IRR key to get the answer. IRR stands for “internal rate of return,” and it is the rate of return the investment provides. The investment is the cash flow at Time 0, and it must be entered as a negative number. When we enter those cash flows in the calculator's cash flow register and press the IRR key, we get the rate of return on the \$1,000 investment, 12.55%. Finally, note that once you have entered the cash flows in the calculator's register, you can find both the investment's net present value (NPV) and its internal rate of return. For investment decisions, we typically want both of these numbers. Therefore, we generally enter the data once and then find both the NPV and the IRR.

You would get the same answer using *Excel*'s IRR function, as shown in Figure 4-9. Notice that when using the IRR—unlike using the NPV function—you must include all cash flows, including the Time-0 cash flow.

**FIGURE 4-9** IRR of an Uneven Cash Flow Stream

	A	B	C	D	E	F	G	
547	Periods:	0	1	2	3	4	5	
548								
549	CF Time Line:	-\$1,000	\$100	\$300	\$300	\$300	\$500	
550	Calculator:	You could enter the cash flows into the cash flow register of a financial calculator and then press the IRR key to find the answer.						12.55%
551	Excel IRR Function:	Cell references:			IRR =	=IRR(B549:G549)		12.55%

## Self-Test

An investment costs \$465 now and is expected to produce cash flows of \$100 at the end of each of the next 4 years, plus an extra lump-sum payment of \$200 at the end of the fourth year. What is the expected rate of return on this investment? (9.05%)

An investment costs \$465 and is expected to produce cash flows of \$100 at the end of Year 1, \$200 at the end of Year 2, and \$300 at the end of Year 3. What is the expected rate of return on this investment? (11.71%)

## 4.15 SEMIANNUAL AND OTHER COMPOUNDING PERIODS

In most of our examples thus far, we assumed that interest is compounded once a year, or annually. This is **annual compounding**. Suppose, however, that you put \$1,000 into a bank that pays a 6% annual interest rate but credits interest each 6 months. This is **semiannual compounding**. If you leave your funds in the account, how much would you have at the end of 1 year under semiannual compounding? Note that you will receive \$60 of interest for the year, but you will receive \$30 of it after only 6 months and the other \$30 at the end of the year. You will earn interest on the first \$30 during the second 6 months, so you will end the year with more than the \$60 you would have had under annual compounding. You would be even better off under quarterly, monthly, weekly, or daily compounding. Note also that virtually all bonds pay interest semiannually; most stocks pay dividends quarterly; most mortgages, student loans, and auto loans involve monthly payments; and most money fund accounts pay interest daily. Therefore, it is essential that you understand how to deal with nonannual compounding.

### Types of Interest Rates

When we move beyond annual compounding, we must deal with the following four types of interest rates:

- Nominal annual rates, given the symbol  $I_{\text{NOM}}$
- Annual percentage rates, termed **APR** rates
- Periodic rates, denoted as  $I_{\text{PER}}$
- Effective annual rates, given the symbol **EAR** or **EFF%**

**Nominal (or Quoted) Rate,  $I_{\text{NOM}}$ .**<sup>15</sup> This is the rate quoted by banks, brokers, and other financial institutions. So, if you talk with a banker, broker, mortgage lender, auto finance company, or student loan officer about rates, the nominal rate is the one he or she will normally quote you. However, to be meaningful, the quoted nominal rate must also include the number of compounding periods per year. For example, a bank might offer you a CD at 6% compounded daily, while a credit union might offer 6.1% compounded monthly.

*Note that the nominal rate is never shown on a time line, and it is never used as an input in a financial calculator (except when compounding occurs only once a year). If more frequent compounding occurs, you must use periodic rates.*

**Periodic Rate,  $I_{\text{PER}}$ .** This is the rate charged by a lender or paid by a borrower each period. It can be a rate per year, per 6 months (semiannually), per quarter, per month, per day, or per any other time interval. For example, a bank might charge 1.5% per month on

<sup>15</sup>The term *nominal rate* as it is used here has a different meaning than the way it was used in Chapter 1. There, nominal interest rates referred to stated market rates as opposed to real (zero-inflation) rates. In this chapter, the term *nominal rate* means the stated, or quoted, annual rate as opposed to the effective annual rate, which we explain later. In both cases, though, *nominal* means *stated*, or *quoted*, as opposed to some sort of adjusted rate.

its credit card loans, or a finance company might charge 3% per quarter on installment loans.

We find the periodic rate as follows:

$$\text{Periodic rate } I_{\text{PER}} = I_{\text{NOM}}/M \quad (4-12)$$

where  $I_{\text{NOM}}$  is the nominal annual rate and  $M$  is the number of compounding periods per year. Thus, a 6% nominal rate with semiannual payments results in a periodic rate of

$$\text{Periodic rate } I_{\text{PER}} = 6\%/2 = 3.00\%.$$

If only one payment is made per year then  $M = 1$ , in which case the periodic rate would equal the nominal rate:  $6\%/1 = 6\%$ .

*The periodic rate is the rate shown on time lines and used in calculations.*<sup>16</sup> To illustrate, suppose you invest \$100 in an account that pays a nominal rate of 12%, compounded quarterly, or 3% per period. How much would you have after 2 years if you leave the funds on deposit? First, here is the time line for the problem:



To find the FV, we would use this modified version of Equation 4-1:

$$FV_N = PV(1 + I_{\text{PER}})^{\text{Number of periods}} = PV\left(1 + \frac{I_{\text{NOM}}}{M}\right)^{MN} \quad (4-13)$$

$$= \$100\left(1 + \frac{0.12}{4}\right)^{4 \times 2} = \$100(1 + 0.03)^8 = \$126.68.$$

With a financial calculator, we find the FV using these inputs:  $N = 4 \times 2 = 8$ ,  $I = 12/4 = 3$ ,  $PV = -100$ , and  $PMT = 0$ . The result is again  $FV = \$126.68$ .<sup>17</sup>

<b>Inputs:</b>	8	3	-100	0	
	<b>N</b>	<b>I/YR</b>	<b>PV</b>	<b>PMT</b>	<b>FV</b>
<b>Output:</b>					126.68

<sup>16</sup>The only exception is in cases where (1) annuities are involved and (2) the payment periods do not correspond to the compounding periods. In such cases—for example, if you are making quarterly payments into a bank account to build up a specified future sum but the bank pays interest on a daily basis—then the calculations are more complicated. For such problems, the simplest procedure is to determine the periodic (daily) interest rate by dividing the nominal rate by 365 (or by 360 if the bank uses a 360-day year), then compound each payment over the exact number of days from the payment date to the terminal point, and then sum the compounded payments to find the future value of the annuity. This is what is generally done in the real world, because with a computer it's a simple process.

<sup>17</sup>Most financial calculators have a feature that allows you to set the number of payments per year and then use the nominal annual interest rate. However, students tend to make fewer errors when using the periodic rate with their calculators set for one payment per year (i.e., per period), so this is what we recommend. Note also that a normal time line cannot be used unless you use the periodic rate.

**Effective (or Equivalent) Annual Rate (EAR or EFF%).** This is the annual (interest once a year) rate that produces the same final result as compounding at the periodic rate for  $M$  times per year. The EAR, also called EFF% (for effective percentage rate), is found as follows:<sup>18</sup>

$$\text{EAR} = \text{EFF}\% = \left(1 + \frac{I_{\text{NOM}}}{M}\right)^M - 1.0 \quad (4-14)$$

Here  $I_{\text{NOM}}/M$  is the periodic rate and  $M$  is the number of periods per year. If a bank would lend you money at a nominal rate of 12%, compounded quarterly, then the EFF% rate would be 12.5509%:

$$\begin{aligned} \text{Rate on bank loan: EFF}\% &= (1 + 0.03)^4 - 1.0 = (1.03)^4 - 1.0 \\ &= 1.125509 - 1.0 = 0.125509 = 12.5509\% \end{aligned}$$

To see the importance of the EFF%, suppose that—as an alternative to the bank loan—you could borrow on a credit card that charges 1% per month. Would you be better off using the bank loan or credit card loan? *To answer this question, the cost of each alternative must be expressed as an EFF%. We just saw that the bank loan’s effective cost is 12.5509%. The cost of the credit card loan, with monthly payments, is slightly higher, 12.6825%:*

$$\begin{aligned} \text{Credit card loan: EFF}\% &= (1 + 0.01)^{12} - 1.0 = (1.01)^{12} - 1.0 \\ &= 1.126825 - 1.0 = 0.126825 = 12.6825\% \end{aligned}$$

This result is logical: Both loans have the same 12% nominal rate, yet you would have to make the first payment after only one month on the credit card versus three months under the bank loan.

The EFF% rate is rarely used in calculations. *However, it must be used to compare the effective costs of different loans or rates of return on different investments when payment periods differ*, as in our example of the credit card versus a bank loan.

## The Result of Frequent Compounding

What would happen to the future value of an investment if interest were compounded annually, semiannually, quarterly, or some other less-than-annual period? Because interest will be earned on interest more often, you should expect higher future values the more frequently compounding occurs. Similarly, you should expect the effective annual rate to increase with more frequent compounding. As Figure 4-10 shows, these results do occur—the future value and the EFF% do increase as the frequency of compounding increases. Notice that the biggest increase in FV (and in EFF%) occurs when compounding goes from annual to semiannual, and notice also that moving from monthly to daily compounding has a relatively small impact. Although Figure 4-10 shows daily compounding as the smallest interval, it is possible to compound even more frequently. At the limit, compounding can occur **continuously**. This is explained in *Web Extension 4C* on the textbook’s Web site.

<sup>18</sup>You could also use the “interest conversion feature” of a financial calculator. Most financial calculators are programmed to find the EFF% or, given the EFF%, to find the nominal rate; this is called “interest rate conversion.” You enter the nominal rate and the number of compounding periods per year, then press the EFF% key to find the effective annual rate. However, we generally use Equation 4-14 because it’s easy and because using the equation reminds us of what we are really doing. If you do use the interest rate conversion feature on your calculator, don’t forget to reset your settings afterward. Interest conversion is discussed in our calculator tutorials.

## Truth in Lending: What Loans Really Cost

Congress passed the Consumer Credit Protection Act in 1968. The Truth in Lending provisions in the Act require banks and other lenders to disclose the **annual percentage rate (APR)** they are charging. For example, suppose you plan to buy a fancy TV set that costs \$3,000, and the store offers you credit for one year at an “add-on” quoted rate of 8%. Here we first find the total dollars of interest by multiplying the \$3,000 you are borrowing times 8%, getting \$240. This interest is then added to the \$3,000 cost of the TV, resulting in a total loan of \$3,240. The total loan is divided by 12 to get the monthly payments:  $\$3,240/12 = \$270$  per month, with the first payment made at the time of purchase. Therefore, we have a 12-month annuity due with payments of \$270. Is your cost really the 8% that you were quoted?

To find the APR, you first set your calculator to Begin Mode, then enter  $N = 12$ ,  $PV = 3000$ ,  $PMT = -270$ , and  $FV = 0$ . Then, when you press the I/YR key, you get the periodic rate, 1.4313%. You then multiply by 12 to get the APR, 17.1758%. You could also find the EFF%, which is 18.5945%. We show these calculations using both the calculator and *Excel*, along with a time line that helps us visualize what’s happening, in the chapter’s *Excel* Tool Kit.

The 17.1758% APR that the dealer is required to report is a much better indicator of the loan’s cost than the 8% nominal rate, but it still does not reflect the true cost, which is the 18.5945% effective annual rate. Thus, buying the TV on time would really cost you 18.5945%. If you don’t know what’s happening when you buy on time or borrow, you may pay a lot more than you think!

**FIGURE 4-10** Effect on \$100 of Compounding More Frequently Than Once a Year

	A	B	C	D	E	F	G
	Frequency of Compounding	Nominal Annual Rate	Number of periods per year (M) <sup>a</sup>	Periodic Interest Rate	Effective Annual Rate <sup>b</sup>	Future Value <sup>c</sup>	Percentage increase in FV
566	Annual	12%	1	12.0000%	12.0000%	\$112.00	
567	Semiannual	12%	2	6.0000%	12.3600%	\$112.36	0.32%
568	Quarterly	12%	4	3.0000%	12.5509%	\$112.55	0.17%
570	Monthly	12%	12	1.0000%	12.6825%	\$112.68	0.12%
571	Daily	12%	365	0.0329%	12.7475%	\$112.75	0.06%
572							
573	<sup>a</sup> We used 365 days per year in the calculations.						
574	<sup>b</sup> The EFF% is calculated using text Equation 4-14.						
575	<sup>c</sup> The Future Value is calculated using text Equation 4-1.						

### Self-Test

Would you rather *invest* in an account that pays a 7% nominal rate with annual compounding or with monthly compounding? If you *borrowed* at a nominal rate of 7%, would you rather make annual or rather monthly payments? Why?

What is the *future value* of \$100 after 3 years if the appropriate interest rate is 8%, compounded annually? (**\$125.97**) Compounded monthly? (**\$127.02**)

What is the *present value* of \$100 due in 3 years if the appropriate interest rate is 8%, compounded annually? (**\$79.38**) Compounded monthly? (**\$78.73**)

Define the following terms: “annual percentage rate, or APR”; “effective annual rate, or EFF%”; and “nominal interest rate,  $I_{\text{NOM}}$ .”

A bank pays 5% with daily compounding on its savings accounts. Should it advertise the nominal or effective rate if it is seeking to attract new deposits?

Credit card issuers must by law print their annual percentage rate on their monthly statements. A common APR is 18%, with interest paid monthly. What is the EFF% on such a loan? (**19.56%**)

Some years ago banks weren't required to reveal the rate they charged on credit cards. Then Congress passed a "truth in lending" law that required them to publish their APR rate. Is the APR rate really the "most truthful" rate, or would the EFF% be even "more truthful"?

## 4.16 FRACTIONAL TIME PERIODS<sup>19</sup>

Thus far we have assumed that payments occur at either the beginning or the end of periods, but not *within* periods. However, we occasionally encounter situations that require compounding or discounting over fractional periods. For example, suppose you deposited \$100 in a bank that pays a nominal rate of 10%, compounded daily, based on a 365-day year. How much would you have after 9 months? The answer of \$107.79 is found as follows:<sup>20</sup>

$$\begin{aligned}\text{Periodic rate} &= I_{\text{PER}} = 0.10/365 = 0.000273973 \text{ per day} \\ \text{Number of days} &= (9/12)(365) = 0.75(365) \\ &= 273.75 \text{ days, rounded to 274} \\ \text{Ending amount} &= \$100(1.000273973)^{274} = \$107.79\end{aligned}$$

Now suppose that instead you borrow \$100 at a nominal rate of 10% per year, *simple interest*, which means that interest is not earned on interest. If the loan is outstanding for 274 days (or 9 months), how much interest would you have to pay? The interest owed is equal to the principal multiplied by the interest rate times the number of periods. In this case, the number of periods is equal to a fraction of a year:  $N = 274/365 = 0.7506849$ .

$$\text{Interest owed} = \$100(10\%)(0.7506849) = \$7.51$$

Another approach would be to use the daily rate rather than the annual rate and thus to use the exact number of days rather than the fraction of the year:

$$\text{Interest owed} = \$100(0.000273973)(274) = \$7.51$$

You would owe the bank a total of \$107.51 after 274 days. This is the procedure most banks actually use to calculate interest on loans, except that they generally require borrowers to pay the interest on a monthly basis rather than after 274 days; this more frequent compounding raises the EFF% and thus the total amount of interest paid.

### Self-Test

Suppose a company borrowed \$1 million at a rate of 9%, simple interest, with interest paid at the end of each month. The bank uses a 360-day year. How much interest would the firm have to pay in a 30-day month? (**\$7,500.00**) What would the interest be if the bank used a 365-day year? (**\$7,397.26**)

Suppose you deposited \$1,000 in a credit union that pays 7% with daily compounding and a 365-day year. What is the EFF%? (**7.250098%**) How much could you withdraw after 7 months, assuming this is 7/12 of a year? (**\$1,041.67**)

<sup>19</sup>This section is interesting and useful, but relatively technical. It can be omitted, at the option of the instructor, without loss of continuity.

<sup>20</sup>We assume that these 9 months constitute 9/12 of a year. Also, bank deposit and loan contracts specifically state whether they are based on a 360-day or a 365-day year. If a 360-day year is used, then the daily rate is higher, so the effective rate is also higher. Here we assumed a 365-day year. Finally, note that banks' computers, like *Excel*, have built-in calendars, so they can calculate the exact number of days.

Note also that banks often treat such loans as follows. (1) They require monthly payments, and they figure the interest for the month as the periodic rate multiplied by the beginning-of-month balance times the number of days in the month. This is called "simple interest." (2) The interest for the month is either added to the next beginning of month balance, or else the borrower must actually pay the earned interest. In this case, the EFF% is based on 12 compounding periods, not 365 as is assumed in our example.

## 4.17 AMORTIZED LOANS

An extremely important application of compound interest involves loans that are paid off in installments over time. Included are automobile loans, home mortgage loans, student loans, and many business loans. A loan that is to be repaid in equal amounts on a monthly, quarterly, or annual basis is called an **amortized loan**.<sup>21</sup> For example, suppose a company borrows \$100,000, with the loan to be repaid in 5 equal payments at the end of each of the next 5 years. The lender charges 6% on the balance at the beginning of each year.

Here's a picture of the situation:



Our task is to find the amount of the payment, PMT, such that the sum of their PVs equals the amount of the loan, \$100,000:

$$\$100,000 = \frac{\text{PMT}}{(1.06)^1} + \frac{\text{PMT}}{(1.06)^2} + \frac{\text{PMT}}{(1.06)^3} + \frac{\text{PMT}}{(1.06)^4} + \frac{\text{PMT}}{(1.06)^5} = \sum_{t=1}^5 \frac{\text{PMT}}{(1.06)^t}$$

It is possible to solve the annuity formula, Equation 4-7, for PMT, but it is much easier to use a financial calculator or spreadsheet. With a financial calculator, we insert values as shown below to get the required payments, \$23,739.64.

Inputs:	5	6	100000	0
	N	I/YR	PV	FV
Output:			-23739.64	

With *Excel*, you would use the PMT function: **=PMT(I,N,PV,FV) = PMT(0.06, 5,100000,0) = -\$23,739.64**. Thus, we see that the borrower must pay the lender \$23,739.64 per year for the next 5 years.

Each payment will consist of two parts—part interest and part repayment of principal. This breakdown is shown in the **amortization schedule** given in Figure 4-11. The interest component is relatively high in the first year, but it declines as the loan balance decreases. For tax purposes, the borrower would deduct the interest component while the lender would report the same amount as taxable income. Over the 5 years, the lender will earn 6% on its investment and also recover the amount of its investment.

### Self-Test

Consider again the example in Figure 4-11. If the loan were amortized over 5 years with 60 equal monthly payments, how much would each payment be, and how would the first payment be divided between interest and principal? **(Each payment would be \$1,933.28; the first payment would have \$500 of interest and \$1,433.28 of principal repayment.)**

Suppose you borrowed \$30,000 on a student loan at a rate of 8% and now must repay it in three equal installments at the end of each of the next 3 years. How large would your payments be, how much of the first payment would represent

<sup>21</sup>The word *amortized* comes from the Latin *mors*, meaning “death,” so an amortized loan is one that is “killed off” over time.



FIGURE 4-11 Loan Amortization Schedule, \$100,000 at 6% for 5 Years

	A	B	C	D	E	F
644	Amount borrowed:		\$100,000			
645	Years:		5			
646	Rate:		6%			
647	PMT:		\$23,739.64 = PMT(C646,C645,-C644)			
		Beginning Amount (1)	Payment (2)	Interest <sup>a</sup> (3)	Repayment of Principal <sup>b</sup> (2) - (3) = (4)	Ending Balance (1) - (4) = (5)
648	Year					
649	1	\$100,000.00	\$23,739.64	\$6,000.00	\$17,739.64	\$82,260.36
650	2	\$82,260.36	\$23,739.64	\$4,935.62	\$18,804.02	\$63,456.34
651	3	\$63,456.34	\$23,739.64	\$3,807.38	\$19,932.26	\$43,524.08
652	4	\$43,524.08	\$23,739.64	\$2,611.44	\$21,128.20	\$22,395.89
653	5	\$22,395.89	\$23,739.64	\$1,343.75	\$22,395.89	\$0.00
654	<sup>a</sup> Interest in each period is calculated by multiplying the loan balance at the beginning of the year by the interest rate. Therefore, interest in Year 1 is \$100,000(0.06) = \$6,000; in Year 2 it is \$82,260.36(0.06) = \$4,935.62; and so on.					
655	<sup>b</sup> Repayment of principal is the \$23,739.64 annual payment minus the interest charges for the year, \$17,739.64 for Year 1.					
656						
657						
658						

interest and how much would be principal, and what would your ending balance be after the first year? (**PMT = \$11,641.01; interest = \$2,400; principal = \$9,241.01; balance at end of Year 1 = \$20,758.99**)

## 4.18 GROWING ANNUITIES<sup>22</sup>

Normally, an annuity is defined as a series of *constant* payments to be received over a specified number of periods. However, the term **growing annuity** is used to describe a series of payments that grow at a constant rate.

### Example 1: Finding a Constant Real Income

Growing annuities are often used in the area of financial planning, where a prospective retiree wants to determine the maximum constant *real*, or *inflation-adjusted*, withdrawals that he or she can make over a specified number of years. For example, suppose a 65-year-old is contemplating retirement, expects to live for another 20 years, has a \$1 million nest egg, expects the investments to earn a nominal annual rate of 6%, expects inflation to average 3% per year, and wants to withdraw a constant *real* amount annually over the next 20 years so as to maintain a constant standard of living. If the first withdrawal is to be made today, what is the amount of that initial withdrawal?

This problem can be solved in three ways. (1) Set up a spreadsheet model that is similar to an amortization table, where the account earns 6% per year, withdrawals rise at the 3% inflation rate, and *Excel's* Goal Seek function is used to find the initial inflation-adjusted withdrawal. A zero balance will be shown at the end of the twentieth year. (2) Use a financial calculator, where we first calculate the real rate of return,



<sup>22</sup>This section is interesting and useful, but relatively technical. It can be omitted, at the option of the instructor, without loss of continuity.


**THE GLOBAL ECONOMIC CRISIS**

### An Accident Waiting to Happen: Option Reset Adjustable Rate Mortgages

Option reset adjustable rate mortgages (ARMs) give the borrower some choices regarding the initial monthly payment. One popular option ARM allowed borrowers to make a monthly payment equal to only half of the interest due in the first month. Because the monthly payment was less than the interest charge, the loan balance grew each month. When the loan balance exceeded 110% of the original principal, the monthly payment was reset to fully amortize the now-larger loan at the prevailing market interest rates.

Here's an example. Someone borrows \$325,000 for 30 years at an initial rate of 7%. The interest accruing in the first month is  $(7\%/12)(\$325,000) = \$1,895.83$ . Therefore, the initial monthly payment is  $50\%(\$1,895.83) = \$947.92$ . Another \$947.92 of deferred interest is added to the loan balance, taking it up to  $\$325,000 + \$947.92 = \$325,947.82$ . Because the loan is now larger, interest in the second month is higher, and both interest and the loan balance will continue to rise each month. The first month after the loan balance exceeds  $110\%(\$325,000) = \$357,500$ , the contract calls for the payment to be reset so as to fully amortize the loan at the then-prevailing interest rate.

First, how long would it take for the balance to exceed \$357,500? Consider this from the lender's perspective: the lender initially pays out \$325,000, receives \$947.92 each month, and then would receive a payment of \$357,500 if the loan were payable when the balance hit that amount, with interest accruing at a 7% annual rate and with monthly compounding. We enter these values into a financial calculator:  $I = 7\%/12$ ,

$PV = -325000$ ,  $PMT = 947.92$ , and  $FV = 357500$ . We solve for  $N = 31.3$  months, rounded up to 32 months. Thus, the borrower will make 32 payments of \$947.92 before the ARM resets.

The payment after the reset depends upon the terms of the original loan and the market interest rate at the time of the reset. For many borrowers, the initial rate was a lower-than-market "teaser" rate, so a higher-than-market rate would be applied to the remaining balance. For this example, we will assume that the original rate wasn't a teaser and that the rate remains at 7%. Keep in mind, though, that for many borrowers the reset rate was higher than the initial rate. The balance after the 32nd payment can be found as the future value of the original loan and the 32 monthly payments, so we enter these values in the financial calculator:  $N = 32$ ,  $I = 7\%/12$ ,  $PMT = 947.92$ ,  $PV = -325000$ , and then solve for  $FV = \$358,242.84$ . The number of remaining payments to amortize the \$358,242.84 loan balance is  $360 - 32 = 328$ , so the amount of each payment is found by setting up the calculator as:  $N = 328$ ,  $I = 7\%/12$ ,  $PV = 358242.84$  and  $FV = 0$ . Solving, we find that  $PMT = \$2,453.94$ .

Even if interest rates don't change, the monthly payment jumps from \$947.92 to \$2,453.94 and would increase even more if interest rates were higher at the reset. This is exactly what happened to millions of American homeowners who took out option reset ARMs in the early 2000s. When large numbers of resets began in 2007, defaults ballooned. The accident caused by option reset ARMs didn't wait very long to happen!

adjusted for inflation, and use it for I/YR when finding the payment for an annuity due. (3) Use a relatively complicated and obtuse formula to find this same amount.<sup>23</sup> We will focus on the first two approaches.

<sup>23</sup>For example, the formula used to find the payment of a growing annuity due is shown below. If  $g$  = annuity growth rate and  $r$  = nominal rate of return on investment, then

$$PVIF \text{ of a growing annuity due} = PVIFGA_{\text{Due}} = \{1 - [(1 + g)/(1 + r)]^N\}[(1 + r)/(r - g)]$$

$$PMT = PV/PVIFGA_{\text{Due}}$$

where PVIF denotes "present value interest factor." Similar formulas are available for growing ordinary annuities.

We illustrate the spreadsheet approach in the chapter model, *Ch04 Tool Kit.xls*. The spreadsheet model provides the most transparent picture of what's happening, since it shows the value of the retirement portfolio, the portfolio's annual earnings, and each withdrawal over the 20-year planning horizon—especially if you include a graph. A picture is worth a thousand numbers, and graphs make it easy to explain the situation to people who are planning their financial futures.

To implement the calculator approach, we first find the expected *real* rate of return, where  $r_r$  is the real rate of return and  $r_{\text{NOM}}$  the nominal rate of return. The real rate of return is the return that we would see if there were no inflation. We calculate the real rate as:

$$\begin{aligned} \text{Real rate} = r_r &= [(1 + r_{\text{NOM}})/(1 + \text{Inflation})] - 1.0 && (4-15) \\ &= [1.06/1.03] - 1.0 = 0.029126214 = 2.9126214\% \end{aligned}$$

Using this real rate of return, we solve the annuity due problem exactly as we did earlier in the chapter. We set the calculator to Begin Mode, after which we input  $N=20$ ,  $I/YR = \text{real rate}=2.9126214$ ,  $PV=-1,000,000$ , and  $FV=0$ ; then we press PMT to get \$64,786.88. This is the amount of the initial withdrawal at Time 0 (today), and future withdrawals will increase at the inflation rate of 3%. These withdrawals, growing at the inflation rate, will provide the retiree with a constant real income over the next 20 years—provided the inflation rate and the rate of return do not change.

In our example we assumed that the first withdrawal would be made immediately. The procedure would be slightly different if we wanted to make end-of-year withdrawals. First, we would set the calculator to End Mode. Second, we would enter the same inputs into the calculator as just listed, including the real interest rate for  $I/YR$ . The calculated PMT would be \$66,673.87. However, that value is in beginning-of-year terms, and since inflation of 3% will occur during the year, we must make the following adjustment to find the inflation-adjusted initial withdrawal:

$$\begin{aligned} \text{Initial end-of-year withdrawal} &= \$66,673.87(1 + \text{Inflation}) \\ &= \$66,673.87(1.03) \\ &= \$68,674.09. \end{aligned}$$

Thus the first withdrawal at the *end* of the year would be \$68,674.09; it would grow by 3% per year; and after the 20th withdrawal (at the end of the 20th year) the balance in the retirement fund would be zero.

We also demonstrate the solution for this end-of-year payment example in *Ch04 Tool Kit.xls*. There we set up a table showing the beginning balance, the annual withdrawals, the annual earnings, and the ending balance for each of the 20 years. This analysis confirms the \$68,674.09 initial end-of-year withdrawal derived previously.

## Example 2: Initial Deposit to Accumulate a Future Sum

As another example of growing annuities, suppose you need to accumulate \$100,000 in 10 years. You plan to make a deposit in a bank now, at Time 0, and then make 9 more deposits at the beginning of each of the following 9 years, for a total of 10 deposits. The bank pays 6% interest, you expect inflation to be 2% per year, and you plan to increase your annual deposits at the inflation rate. How much must you deposit initially? First, we calculate the real rate:

$$\text{Real rate} = r_r = [1.06/1.02] - 1.0 = 0.0392157 = 3.9215686\%$$

Next, since inflation is expected to be 2% per year, in 10 years the target \$100,000 will have a real value of

$$\$100,000/(1 + 0.02)^{10} = \$82,034.83.$$

Now we can find the size of the required initial payment by setting a financial calculator to the Begin Mode and then inputting  $N = 10$ ,  $I/YR = 3.9215686$ ,  $PV = 0$ , and  $FV = 82,034.83$ . Then, when we press the PMT key, we get  $PMT = -6,598.87$ . Thus, a deposit of \$6,598.87 made at time 0 and growing by 2% per year will accumulate to \$100,000 by Year 10 if the interest rate is 6%. Again, this result is confirmed in the chapter's *Tool Kit*. The key to this analysis is to express I/YR, FV, and PMT in real, not nominal, terms.

#### Self-Test

Differentiate between a “regular” and a “growing” annuity.

What three methods can be used to deal with growing annuities?

If the nominal interest rate is 10% and the expected inflation rate is 5%, what is the expected real rate of return? (**4.7619%**)

## Summary

Most financial decisions involve situations in which someone makes a payment at one point in time and receives money later. Dollars paid or received at two different points in time are different, and this difference is dealt with using *time value of money (TVM) analysis*.

- **Compounding** is the process of determining the **future value (FV)** of a cash flow or a series of cash flows. The compounded amount, or future value, is equal to the beginning amount plus interest earned.
- Future value of a single payment =  $FV_N = PV(1 + I)^N$ .
- **Discounting** is the process of finding the **present value (PV)** of a future cash flow or a series of cash flows; discounting is the reciprocal, or reverse, of compounding.
- Present value of a payment received at the end of Time  $N = PV = \frac{FV_N}{(1 + I)^N}$ .
- An **annuity** is defined as a series of equal periodic payments (PMT) for a specified number of periods.
- An annuity whose payments occur at the *end* of each period is called an **ordinary annuity**.
- Future value of an (ordinary) annuity  $FVA_N = PMT \left[ \frac{(1 + I)^N}{I} - \frac{1}{I} \right]$ .
- Present value of an (ordinary) annuity  $PVA_N = PMT \left[ \frac{1}{I} - \frac{1}{I(1 + I)^N} \right]$ .
- If payments occur at the *beginning* of the periods rather than at the end, then we have an **annuity due**. The PV of each payment is larger, because each payment is discounted back one year less, so the PV of the annuity is also larger. Similarly, the FV of the annuity due is larger because each payment is compounded for an extra year. The following formulas can be used to convert the PV and FV of an ordinary annuity to an annuity due:

$$PVA_{\text{due}} = PVA_{\text{ordinary}}(1 + I)$$

$$FVA_{\text{due}} = FVA_{\text{ordinary}}(1 + I)$$

- A **perpetuity** is an annuity with an infinite number of payments.

$$\text{Value of a perpetuity} = \frac{\text{PMT}}{I}$$

- To find the PV or FV of an uneven series, find the PV or FV of each individual cash flow and then sum them.
- If you know the cash flows and the PV (or FV) of a cash flow stream, you can **determine its interest rate**.
- When compounding occurs more frequently than once a year, the nominal rate must be converted to a periodic rate, and the number of years must be converted to periods:

$$\begin{aligned} \text{Periodic rate}(I_{\text{PER}}) &= \text{Nominal annual rate} \div \text{Periods per year} \\ \text{Number of Periods} &= \text{Years} \times \text{Periods per year} \end{aligned}$$

The periodic rate and number of periods is used for calculations and is shown on time lines.

- If you are comparing the costs of alternative loans that require payments more than once a year, or the rates of return on investments that pay interest more than once a year, then the comparisons should be based on **effective** (or **equivalent**) **rates** of return. Here is the formula:

$$\text{EAR} = \text{EFF\%} = \left(1 + \frac{I_{\text{NOM}}}{M}\right)^M - 1.0$$

- The general equation for finding the future value of a current cash flow (PV) for any number of compounding periods per year is

$$FV_N = PV(1 + I_{\text{PER}})^{\text{Number of periods}} = PV\left(1 + \frac{I_{\text{NOM}}}{M}\right)^{MN}$$

where

$I_{\text{NOM}}$  = Nominal quoted interest rate

$M$  = Number of compounding periods per year

$N$  = Number of years

- An **amortized loan** is one that is paid off with equal payments over a specified period. An **amortization schedule** shows how much of each payment constitutes interest, how much is used to reduce the principal, and the unpaid balance at the end of each period. The unpaid balance at Time  $N$  must be zero.
- A **“Growing Annuity”** is a stream of cash flows that grows at a constant rate for a specified number of years. The present and future values of growing annuities can be found with relatively complicated formulas or, more easily, with an *Excel* model.
- *Web Extension 4A* explains the **tabular approach**.
- *Web Extension 4B* provides derivations of the annuity formulas.
- *Web Extension 4C* explains **continuous compounding**.

## Questions

- (4-1) Define each of the following terms:
- PV; I; INT;  $FV_N$ ;  $PVA_N$ ;  $FVA_N$ ; PMT; M;  $I_{NOM}$
  - Opportunity cost rate
  - Annuity; lump-sum payment; cash flow; uneven cash flow stream
  - Ordinary (or deferred) annuity; annuity due
  - Perpetuity; consol
  - Outflow; inflow; time line; terminal value
  - Compounding; discounting
  - Annual, semiannual, quarterly, monthly, and daily compounding
  - Effective annual rate (EAR or EFF%); nominal (quoted) interest rate; APR; periodic rate
  - Amortization schedule; principal versus interest component of a payment; amortized loan
- (4-2) What is an *opportunity cost rate*? How is this rate used in discounted cash flow analysis, and where is it shown on a time line? Is the opportunity rate a single number that is used to evaluate all potential investments?
- (4-3) An *annuity* is defined as a series of payments of a fixed amount for a specific number of periods. Thus, \$100 a year for 10 years is an annuity, but \$100 in Year 1, \$200 in Year 2, and \$400 in Years 3 through 10 does *not* constitute an annuity. However, the entire series *does contain* an annuity. Is this statement true or false?
- (4-4) If a firm's earnings per share grew from \$1 to \$2 over a 10-year period, the *total growth* would be 100%, but the *annual growth rate* would be *less than* 10%. True or false? Explain.
- (4-5) Would you rather have a savings account that pays 5% interest compounded semi-annually or one that pays 5% interest compounded daily? Explain.

## Self-Test Problems

Solutions Appear in Appendix A

- (ST-1) Assume that 1 year from now you plan to deposit \$1,000 in a savings account that pays a nominal rate of 8%.
- Future Value
- If the bank compounds interest annually, how much will you have in your account 4 years from now?
  - What would your balance be 4 years from now if the bank used quarterly compounding rather than annual compounding?
  - Suppose you deposited the \$1,000 in 4 payments of \$250 each at the end of Years 1, 2, 3, and 4. How much would you have in your account at the end of Year 4, based on 8% annual compounding?
  - Suppose you deposited 4 equal payments in your account at the end of Years 1, 2, 3, and 4. Assuming an 8% interest rate, how large would each of your payments have to be for you to obtain the same ending balance as you calculated in part a?
- (ST-2) Assume that 4 years from now you will need \$1,000. Your bank compounds interest at an 8% annual rate.
- Time Value of Money



- How much must you deposit 1 year from now to have a balance of \$1,000 at Year 4?
- If you want to make equal payments at the end of Years 1 through 4 to accumulate the \$1,000, how large must each of the 4 payments be?
- If your father were to offer either to make the payments calculated in part b (\$221.92) or to give you a lump sum of \$750 one year from now, which would you choose?
- If you will have only \$750 at the end of Year 1, what interest rate, compounded annually, would you have to earn to have the necessary \$1,000 at Year 4?
- Suppose you can deposit only \$186.29 each at the end of Years 1 through 4, but you still need \$1,000 at the end of Year 4. What interest rate, with annual compounding, is required to achieve your goal?
- To help you reach your \$1,000 goal, your father offers to give you \$400 one year from now. You will get a part-time job and make 6 additional deposits of equal amounts each 6 months thereafter. If all of this money is deposited in a bank that pays 8%, compounded semiannually, how large must each of the 6 deposits be?
- What is the effective annual rate being paid by the bank in part f?

**(ST-3)**  
Effective Annual  
Rates

Bank A pays 8% interest, compounded quarterly, on its money market account. The managers of Bank B want its money market account's effective annual rate to equal that of Bank A, but Bank B will compound interest on a monthly basis. What nominal, or quoted, rate must Bank B set?

## Problems

Answers Appear in Appendix B

### EASY PROBLEMS 1–8

- (4-1)** If you deposit \$10,000 in a bank account that pays 10% interest annually, how much will be in your account after 5 years?  
Future Value of a Single Payment
- (4-2)** What is the present value of a security that will pay \$5,000 in 20 years if securities of equal risk pay 7% annually?  
Present Value of a Single Payment
- (4-3)** Your parents will retire in 18 years. They currently have \$250,000, and they think they will need \$1 million at retirement. What annual interest rate must they earn to reach their goal, assuming they don't save any additional funds?  
Interest Rate on a Single Payment
- (4-4)** If you deposit money today in an account that pays 6.5% annual interest, how long will it take to double your money?  
Number of Periods of a Single Payment
- (4-5)** You have \$42,180.53 in a brokerage account, and you plan to deposit an additional \$5,000 at the end of every future year until your account totals \$250,000. You expect to earn 12% annually on the account. How many years will it take to reach your goal?  
Number of Periods for an Annuity
- (4-6)** What is the future value of a 7%, 5-year ordinary annuity that pays \$300 each year? If this were an annuity due, what would its future value be?  
Future Value: Ordinary Annuity versus Annuity Due
- (4-7)** An investment will pay \$100 at the end of each of the next 3 years, \$200 at the end of Year 4, \$300 at the end of Year 5, and \$500 at the end of Year 6. If other investments of equal risk earn 8% annually, what is this investment's present value? Its future value?  
Present and Future Value of an Uneven Cash Flow Stream



- (4-8) Annuity Payment and EAR You want to buy a car, and a local bank will lend you \$20,000. The loan would be fully amortized over 5 years (60 months), and the nominal interest rate would be 12%, with interest paid monthly. What is the monthly loan payment? What is the loan's EFF%?

## INTERMEDIATE PROBLEMS

9-29

- (4-9) Present and Future Values of Single Cash Flows for Different Periods Find the following values, *using the equations*, and then work the problems using a financial calculator to check your answers. Disregard rounding differences. (*Hint:* If you are using a financial calculator, you can enter the known values and then press the appropriate key to find the unknown variable. Then, without clearing the TVM register, you can “override” the variable that changes by simply entering a new value for it and then pressing the key for the unknown variable to obtain the second answer. This procedure can be used in parts b and d, and in many other situations, to see how changes in input variables affect the output variable.)

- An initial \$500 compounded for 1 year at 6%
- An initial \$500 compounded for 2 years at 6%
- The present value of \$500 due in 1 year at a discount rate of 6%
- The present value of \$500 due in 2 years at a discount rate of 6%

- (4-10) Present and Future Values of Single Cash Flows for Different Interest Rates Use both the TVM equations and a financial calculator to find the following values. See the Hint for Problem 4-9.
- An initial \$500 compounded for 10 years at 6%
  - An initial \$500 compounded for 10 years at 12%
  - The present value of \$500 due in 10 years at a 6% discount rate
  - The present value of \$500 due in 10 years at a 12% discount rate

- (4-11) Time for a Lump Sum to Double To the closest year, how long will it take \$200 to double if it is deposited and earns the following rates? [*Notes:* (1) See the Hint for Problem 4-9. (2) This problem cannot be solved exactly with some financial calculators. For example, if you enter  $PV = -200$ ,  $PMT = 0$ ,  $FV = 400$ , and  $I = 7$  in an HP-12C and then press the N key, you will get 11 years for part a. The correct answer is 10.2448 years, which rounds to 10, but the calculator rounds up. However, the HP-10B gives the exact answer.]
- 7%
  - 10%
  - 18%
  - 100%

- (4-12) Future Value of an Annuity Find the *future value* of the following annuities. The first payment in these annuities is made at the *end* of Year 1, so they are *ordinary annuities*. (*Notes:* See the Hint for Problem 4-9. Also, note that you can leave values in the TVM register, switch to Begin Mode, press FV, and find the FV of the annuity due.)
- \$400 per year for 10 years at 10%
  - \$200 per year for 5 years at 5%
  - \$400 per year for 5 years at 0%
  - Now rework parts a, b, and c assuming that payments are made at the *beginning* of each year; that is, they are *annuities due*.

(4-13)

Present Value of an Annuity

Find the *present value* of the following *ordinary annuities* (see the Notes to Problem 4-12).

- \$400 per year for 10 years at 10%
- \$200 per year for 5 years at 5%
- \$400 per year for 5 years at 0%
- Now rework parts a, b, and c assuming that payments are made at the *beginning* of each year; that is, they are *annuities due*.

(4-14)

Uneven Cash Flow Stream

Find the present values of the following cash flow streams. The appropriate interest rate is 8%. (*Hint*: It is fairly easy to work this problem dealing with the individual cash flows. However, if you have a financial calculator, read the section of the manual that describes how to enter cash flows such as the ones in this problem. This will take a little time, but the investment will pay huge dividends throughout the course. Note that, when working with the calculator's cash flow register, you must enter  $CF_0 = 0$ . Note also that it is quite easy to work the problem with *Excel*, using procedures described in the Chapter 4 *Tool Kit*.)

Year	Cash Stream A	Cash Stream B
1	\$100	\$300
2	400	400
3	400	400
4	400	400
5	300	100

- What is the value of each cash flow stream at a 0% interest rate?

(4-15)

Effective Rate of Interest

Find the interest rate (or rates of return) in each of the following situations.

- You *borrow* \$700 and promise to pay back \$749 at the end of 1 year.
- You *lend* \$700 and receive a promise to be paid \$749 at the end of 1 year.
- You borrow \$85,000 and promise to pay back \$201,229 at the end of 10 years.
- You borrow \$9,000 and promise to make payments of \$2,684.80 at the end of each of the next 5 years.

(4-16)

Future Value for Various Compounding Periods

Find the amount to which \$500 will grow under each of the following conditions.

- 12% compounded annually for 5 years
- 12% compounded semiannually for 5 years
- 12% compounded quarterly for 5 years
- 12% compounded monthly for 5 years

(4-17)

Present Value for Various Compounding Periods

Find the present value of \$500 due in the future under each of the following conditions.

- 12% nominal rate, semiannual compounding, discounted back 5 years
- 12% nominal rate, quarterly compounding, discounted back 5 years
- 12% nominal rate, monthly compounding, discounted back 1 year

(4-18)

Future Value of an Annuity for Various Compounding Periods

Find the future values of the following ordinary annuities.

- FV of \$400 each 6 months for 5 years at a nominal rate of 12%, compounded semiannually
- FV of \$200 each 3 months for 5 years at a nominal rate of 12%, compounded quarterly

- c. The annuities described in parts a and b have the same total amount of money paid into them during the 5-year period, and both earn interest at the same nominal rate, yet the annuity in part b earns \$101.75 more than the one in part a over the 5 years. Why does this occur?

(4-19)

Effective versus Nominal Interest Rates

Universal Bank pays 7% interest, compounded annually, on time deposits. Regional Bank pays 6% interest, compounded quarterly.

- Based on effective interest rates, in which bank would you prefer to deposit your money?
- Could your choice of banks be influenced by the fact that you might want to withdraw your funds during the year as opposed to at the end of the year? In answering this question, assume that funds must be left on deposit during an entire compounding period in order for you to receive any interest.

(4-20)

Amortization Schedule

- Set up an amortization schedule for a \$25,000 loan to be repaid in equal installments at the end of each of the next 5 years. The interest rate is 10%.
- How large must each annual payment be if the loan is for \$50,000? Assume that the interest rate remains at 10% and that the loan is still paid off over 5 years.
- How large must each payment be if the loan is for \$50,000, the interest rate is 10%, and the loan is paid off in equal installments at the end of each of the next 10 years? This loan is for the same amount as the loan in part b, but the payments are spread out over twice as many periods. Why are these payments not half as large as the payments on the loan in part b?

(4-21)

Growth Rates

Sales for Hanebury Corporation's just-ended year were \$12 million. Sales were \$6 million 5 years earlier.

- At what rate did sales grow?
- Suppose someone calculated the sales growth for Hanebury in part a as follows: "Sales doubled in 5 years. This represents a growth of 100% in 5 years; dividing 100% by 5 results in an estimated growth rate of 20% per year." Explain what is wrong with this calculation.

(4-22)

Expected Rate of return

Washington-Pacific invested \$4 million to buy a tract of land and plant some young pine trees. The trees can be harvested in 10 years, at which time W-P plans to sell the forest at an expected price of \$8 million. What is W-P's expected rate of return?

(4-23)

Effective Rate of Interest

A mortgage company offers to lend you \$85,000; the loan calls for payments of \$8,273.59 at the end of each year for 30 years. What interest rate is the mortgage company charging you?

(4-24)

Required Lump-Sum Payment

To complete your last year in business school and then go through law school, you will need \$10,000 per year for 4 years, starting next year (that is, you will need to withdraw the first \$10,000 one year from today). Your rich uncle offers to put you through school, and he will deposit in a bank paying 7% interest a sum of money that is sufficient to provide the 4 payments of \$10,000 each. His deposit will be made today.

- How large must the deposit be?
- How much will be in the account immediately after you make the first withdrawal? After the last withdrawal?

**(4-25)** Repaying a Loan While Mary Corens was a student at the University of Tennessee, she borrowed \$12,000 in student loans at an annual interest rate of 9%. If Mary repays \$1,500 per year, then how long (to the nearest year) will it take her to repay the loan?

**(4-26)** Reaching a Financial Goal You need to accumulate \$10,000. To do so, you plan to make deposits of \$1,250 per year—with the first payment being made a year from today—into a bank account that pays 12% annual interest. Your last deposit will be less than \$1,250 if less is needed to round out to \$10,000. How many years will it take you to reach your \$10,000 goal, and how large will the last deposit be?

**(4-27)** Present Value of a Perpetuity What is the present value of a perpetuity of \$100 per year if the appropriate discount rate is 7%? If interest rates in general were to double and the appropriate discount rate rose to 14%, what would happen to the present value of the perpetuity?

**(4-28)** PV and Effective Annual Rate Assume that you inherited some money. A friend of yours is working as an unpaid intern at a local brokerage firm, and her boss is selling securities that call for 4 payments of \$50 (1 payment at the end of each of the next 4 years) plus an extra payment of \$1,000 at the end of Year 4. Your friend says she can get you some of these securities at a cost of \$900 each. Your money is now invested in a bank that pays an 8% nominal (quoted) interest rate but with quarterly compounding. You regard the securities as being just as safe, and as liquid, as your bank deposit, so your required effective annual rate of return on the securities is the same as that on your bank deposit. You must calculate the value of the securities to decide whether they are a good investment. What is their present value to you?

**(4-29)** Loan Amortization Assume that your aunt sold her house on December 31, and to help close the sale she took a second mortgage in the amount of \$10,000 as part of the payment. The mortgage has a quoted (or nominal) interest rate of 10%; it calls for payments every 6 months, beginning on June 30, and is to be amortized over 10 years. Now, 1 year later, your aunt must inform the IRS and the person who bought the house about the interest that was included in the two payments made during the year. (This interest will be income to your aunt and a deduction to the buyer of the house.) To the closest dollar, what is the total amount of interest that was paid during the first year?

CHALLENGING PROBLEMS  
30–34

**(4-30)** Loan Amortization Your company is planning to borrow \$1 million on a 5-year, 15%, annual payment, fully amortized term loan. What fraction of the payment made at the end of the second year will represent repayment of principal?

- (4-31)** Nonannual Compounding
- It is now January 1. You plan to make a total of 5 deposits of \$100 each, one every 6 months, with the first payment being made *today*. The bank pays a nominal interest rate of 12% but uses *semiannual* compounding. You plan to leave the money in the bank for 10 years. How much will be in your account after 10 years?
  - You must make a payment of \$1,432.02 in 10 years. To get the money for this payment, you will make 5 equal deposits, beginning today and for the following 4 quarters, in a bank that pays a nominal interest rate of 12% with *quarterly compounding*. How large must each of the 5 payments be?

**(4-32)** Nominal Rate of return Anne Lockwood, manager of Oaks Mall Jewelry, wants to sell on credit, giving customers 3 months to pay. However, Anne will have to borrow from her bank to carry the accounts receivable. The bank will charge a nominal rate of 15% and will

compound monthly. Anne wants to quote a nominal rate to her customers (all of whom are expected to pay on time) that will exactly offset her financing costs. What nominal annual rate should she quote to her credit customers?

**(4-33)**  
Required Annuity  
Payments

Assume that your father is now 50 years old, that he plans to retire in 10 years, and that he expects to live for 25 years after he retires—that is, until age 85. He wants his first retirement payment to have the same purchasing power at the time he retires as \$40,000 has today. He wants all of his subsequent retirement payments to be equal to his first retirement payment. (Do not let the retirement payments grow with inflation: Your father realizes that the real value of his retirement income will decline year by year after he retires.) His retirement income will begin the day he retires, 10 years from today, and he will then receive 24 additional annual payments. Inflation is expected to be 5% per year from today forward. He currently has \$100,000 saved up; and he expects to earn a return on his savings of 8% per year with annual compounding. To the nearest dollar, how much must he save during each of the next 10 years (with equal deposits being made at the end of each year, beginning a year from today) to meet his retirement goal? (*Note:* Neither the amount he saves nor the amount he withdraws upon retirement is a growing annuity.)

**(4-34)**  
Growing Annuity  
Payments

You want to accumulate \$1 million by your retirement date, which is 25 years from now. You will make 25 deposits in your bank, with the first occurring *today*. The bank pays 8% interest, compounded annually. You expect to get annual raises of 3%, which will offset inflation, and you will let the amount you deposit each year also grow by 3% (i.e., your second deposit will be 3% greater than your first, the third will be 3% greater than the second, etc.). How much must your first deposit be if you are to meet your goal?

## SPREADSHEET PROBLEM

**(4-35)**

Build a Model: The  
Time Value of Money



Start with the partial model in the file *Cb04 P35 Build a Model.xls* from the textbook's Web site. Answer the following questions, using a spreadsheet model to do the calculations.

- Find the FV of \$1,000 invested to earn 10% annually 5 years from now. Answer this question first by using a math formula and then by using the *Excel* function wizard.
- Now create a table that shows the FV at 0%, 5%, and 20% for 0, 1, 2, 3, 4, and 5 years. Then create a graph with years on the horizontal axis and FV on the vertical axis to display your results.
- Find the PV of \$1,000 due in 5 years if the discount rate is 10% per year. Again, work the problem with a formula and also by using the function wizard.
- A security has a cost of \$1,000 and will return \$2,000 after 5 years. What rate of return does the security provide?
- Suppose California's population is 30 million people and its population is expected to grow by 2% per year. How long would it take for the population to double?
- Find the PV of an ordinary annuity that pays \$1,000 at the end of each of the next 5 years if the interest rate is 15%. Then find the FV of that same annuity.
- How would the PV and FV of the above annuity change if it were an annuity due rather than an ordinary annuity?
- What would the FV and PV for parts a and c be if the interest rate were 10% with *semiannual* compounding rather than 10% with *annual* compounding?



- i. Find the PV and FV of an investment that makes the following end-of-year payments. The interest rate is 8%.

Year	Payment
1	\$100
2	200
3	400

- j. Suppose you bought a house and took out a mortgage for \$50,000. The interest rate is 8%, and you must amortize the loan over 10 years with equal end-of-year payments. Set up an amortization schedule that shows the annual payments and the amount of each payment that repays the principal and the amount that constitutes interest expense to the borrower and interest income to the lender.
1. Create a graph that shows how the payments are divided between interest and principal repayment over time.
  2. Suppose the loan called for 10 years of monthly payments, 120 payments in all, with the same original amount and the same nominal interest rate. What would the amortization schedule show now?

### Mini Case

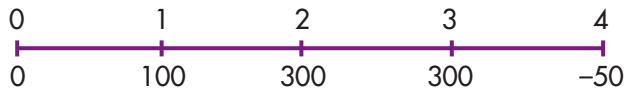
Assume that you are nearing graduation and have applied for a job with a local bank. As part of the bank's evaluation process, you have been asked to take an examination that covers several financial analysis techniques. The first section of the test addresses discounted cash flow analysis. See how you would do by answering the following questions.

- a. Draw time lines for (1) a \$100 lump sum cash flow at the end of Year 2, (2) an ordinary annuity of \$100 per year for 3 years, and (3) an uneven cash flow stream of -\$50, \$100, \$75, and \$50 at the end of Years 0 through 3.
- b.
  1. What's the *future value* of an initial \$100 after 3 years if it is invested in an account paying 10% annual interest?
  2. What's the *present value* of \$100 to be received in 3 years if the appropriate interest rate is 10%?
- c. We sometimes need to find out how long it will take a sum of money (or anything else) to grow to some specified amount. For example, if a company's sales are growing at a rate of 20% per year, how long will it take sales to double?
- d. If you want an investment to double in 3 years, what interest rate must it earn?
- e. What's the difference between an ordinary annuity and an annuity due? What type of annuity is shown below? How would you change the time line to show the other type of annuity?



- f.
  1. What's the future value of a 3-year ordinary annuity of \$100 if the appropriate interest rate is 10%?
  2. What's the present value of the annuity?
  3. What would the future and present values be if the annuity were an annuity due?

- g. What is the present value of the following uneven cash flow stream? The appropriate interest rate is 10%, compounded annually.



- h. 1. Define the stated (quoted) or nominal rate  $I_{\text{NOM}}$  as well as the periodic rate  $I_{\text{PER}}$ .
2. Will the future value be larger or smaller if we compound an initial amount more often than annually—for example, every 6 months, or *semiannually*—holding the stated interest rate constant? Why?
3. What is the future value of \$100 after 5 years under 12% annual compounding? Semiannual compounding? Quarterly compounding? Monthly compounding? Daily compounding?
4. What is the effective annual rate (EAR or EFF%)? What is the EFF% for a nominal rate of 12%, compounded semiannually? Compounded quarterly? Compounded monthly? Compounded daily?
- i. Will the effective annual rate ever be equal to the nominal (quoted) rate?
- j. 1. Construct an amortization schedule for a \$1,000, 10% annual rate loan with 3 equal installments.
2. During Year 2, what is the annual interest expense for the borrower, and what is the annual interest income for the lender?
- k. Suppose that on January 1 you deposit \$100 in an account that pays a nominal (or quoted) interest rate of 11.33463%, with interest added (compounded) daily. How much will you have in your account on October 1, or 9 months later?
- l. 1. What is the value at the end of Year 3 of the following cash flow stream if the quoted interest rate is 10%, compounded semiannually?



2. What is the PV of the same stream?
3. Is the stream an annuity?
4. An important rule is that you should never show a nominal rate on a time line or use it in calculations unless what condition holds? (*Hint:* Think of annual compounding, when  $I_{\text{NOM}} = \text{EFF}\% = I_{\text{PER}}$ .) What would be wrong with your answers to parts (1) and (2) if you used the nominal rate of 10% rather than the periodic rate,  $I_{\text{NOM}}/2 = 10\%/2 = 5\%$ ?
- m. Suppose someone offered to sell you a note calling for the payment of \$1,000 in 15 months. They offer to sell it to you for \$850. You have \$850 in a bank time deposit that pays a 6.76649% nominal rate with daily compounding, which is a 7% effective annual interest rate, and you plan to leave the money in the bank unless you buy the note. The note is not risky—you are sure it will be paid on schedule. Should you buy the note? Check the decision in three ways: (1) by comparing your future value if you buy the note versus leaving your money in the bank; (2) by comparing the PV of the note with your current bank account; and (3) by comparing the EFF% on the note with that of the bank account.