

Chapter 24

TERM STRUCTURE: INTEREST RATE MODELS*

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Abstract

Interest movement models are important to financial modeling because they can be used for valuing any financial instruments whose values are affected by interest rate movements. Specifically, we can classify the interest rate movement models into two categories: equilibrium models and no-arbitrage models. The equilibrium models emphasize the equilibrium concept. However, the no-arbitrage models argue that the term-structure movements should satisfy the no-arbitrage condition. The arbitrage-free interest rate model is an extension of the Black–Scholes model to value interest rate derivatives. The model valuation is assured to be consistent with the observed yield curve in valuing interest rate derivatives and providing accurate pricing of interest rate contingent claims. Therefore, it is widely used for portfolio management and other capital market activities.

Keywords: lognormal versus normal movements; mean reversion; interest correlation; term structure volatility; Cox, Ingersoll and Ross model; Åsicek model; Brennan and Schwartz two-factor model; Ho and Lee model; Black, Derman, and Toy model; Hull and White model

24.1. Introduction

There are many examples of interest rate derivatives that are actively traded in over-the-counter markets and in organized exchanges. Caps, floors, Treasury bond options, Treasury bond futures options, Euro-dollar futures options, and swaption are just some examples of this important class of derivatives in our financial markets. They are classified as “interest rate derivatives” because their stochastic movements are directly related to the interest rate movements in a way that is analogous to the stock option price that moves in step with the underlying stock price.

We first present an empirical analysis of historical yield curve movements, which conveys its relationship to interest rate models. Then we provide an overview of the interest rate models.

24.2. Interest Rate Movements: Historical Experiences

Interest rate movements refer to the uncertain movements of the Treasury spot yield curve. Each STRIPS bond is considered a security. When the daily closing price is reported, the bond’s yield-to-maturity can be calculated. The observed Treasury

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spot yield curve is the scattered plot of the yield to maturity against the maturity for all the STRIPS bonds. Since the spot yield curve is a representation of the time value of money, and the time value of money is related to the time-to-horizon in a continuous fashion, the scattered plots should be a continuous curve. Hence, we call the scattered plot a yield curve.

What are the dynamics of the spot yield curve? Let us consider the behavior of spot yield curve movements in relation to interest rate levels, historically. The monthly spot yield curves from the beginning of 1994 until the end of 2001 are depicted in the figure below.

As Figure 24.1 shows, the spot yield curves can take on a number of shapes. When the yields of the bonds increase with the bonds' maturities, the yield curve is said to be upward sloping. Conversely, when the yield decreases with maturity, the spot curve is called downward sloping. Although not shown in Figure 24.1, the early 1980s displayed a yield curve that was downward sloping. In 1998, the yield curve was level or flat. In the early part of 2001, the yield curve was humped, with the yields reaching the peak at the one-year maturity. Historically, the spot yield curve has changed its shape as well as the level continually.

The yield curve movement is concerned with the change of the yield curve shape over a relatively

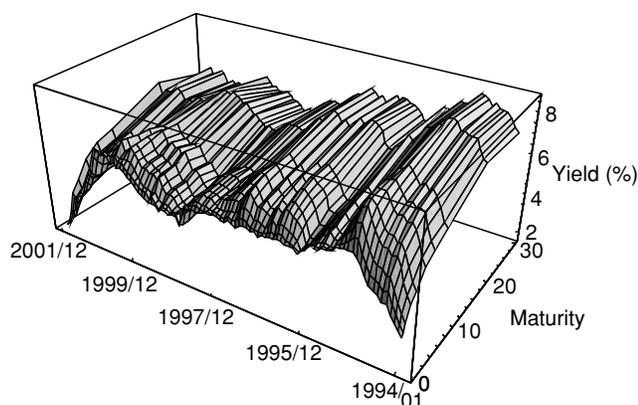


Figure 24.1. A time-series diagram of monthly spot yield curve movements (1994.01 ~ 2001.12). *Data Source:* <http://www.economagic.com/>

short time interval, say, one month. Describing yield curve movements is slightly more complicated than describing a stock movement. To describe the movement of stocks, we can decompose the stock movement into two parts: the expected drift or expected returns and the uncertain movement. The model is represented by:

$$dS = \mu S dt + \sigma S dZ \quad (24.1)$$

where dS represents a small movement for a short time interval dt . μ is called the instantaneous returns of the stock, σ is the instantaneous standard deviation (or volatility) of the stock. dZ represents a small uncertain movement specified by a normal distribution. The mean and the standard deviation of the normal distribution is 0 and \sqrt{dt} , respectively. The first term is called the drift term. It represents the expected movement of the stock price. If the first term is zero, then the future stock price is *expected* to remain the same as the present observed price. Of course, the realized stock price in the future can deviate from the initial stock price because of the uncertain stock price movement specified by the second term. The random term dZ can be viewed as a unit of risk, a normal distribution over an (infinitely) short time interval. The coefficient of the dZ term represents the volatility of the process. If this coefficient is zero, then the process has no risk, and the stock price movement has no uncertainty.

But to specify the movement of the yield curve, in a way that is similar to Equation (24.1), is more problematic. Since a yield curve is determined by all the U.S. STRIPS bonds, the movement of the yield curve should be represented by the movements of all the bond prices. But the movements of all the bond prices are not independent of each other. They have to be correlated. The following empirical evidence may suggest how the yield curve movements may be best specified.

24.2.1. Lognormal Versus Normal Movements

The movements (often referred to as the *dynamics*) of each interest rate of the spot yield curve can be

specified as we have done for a stock. We can rewrite Equation (24.1), replacing the stock price with a rate that is the yield to maturity of a zero coupon bond of a specific maturity “ t ”. Thus we have:

$$dr = \mu(r,t)r dt + r\sigma dZ \quad (24.2)$$

When a t year rate is assumed to follow the process specified by Equation (24.2), we say that the interest rate follows a *lognormal process* and Equation (24.2) is called a *lognormal model*. In comparing Equation (24.2) with Equation (24.1), note that the drift term of the interest rate model is any function of the short-term interest rate r and time, while the lognormal model for stock tends to assume that the instantaneous stock return is a constant number. Therefore, the research literature of interest rate models has somewhat abused the language in calling Equation (24.2) a lognormal model. The important point is that, in a lognormal process, the volatility term is proportional to the interest rate level $r(t)$. When the interest rate level is high, we experience high interest rate volatility. When the interest rate level is low, we experience low interest rate volatility.

There is an alternative specification of the interest rate process, which research literature calls the *normal process*. In the normal process, the volatility is independent of the interest rate level, and it is given below:

$$dr = \mu(r,t)dt + \sigma dz \quad (24.3)$$

Equation (24.3) is called the *normal model*. Note that the distinction made between the lognormal model and the normal model depends only on the volatility term and not on the drift term. For a normal model, the interest rate fluctuates with a volatility independent of the interest rate level over a short time interval. For a lognormal model, the interest rate has a volatility related to the interest rate level, in particular, when the volatility becomes arbitrarily small as interest rate level approaches zero. This way, the interest rates can never become negative. And a lognormal process is written as:

$$\frac{dr}{r} = \mu(r,t)dt + \sigma dZ \quad (24.3a)$$

Based on historical observations, the yield curve movements have been shown to be both normal and lognormal depending on the interest rate levels. Which model is more appropriate to describe interest rate movements, the normal or lognormal model? We need to evaluate the model from an empirical perspective. Using U.S. historical interest rates, the squared change of the interest rate over a one-month period could be plotted against the interest rate level. Then we can see that the interest rate volatility has no relationship between the interest rate levels. If there were a positive relationship, we would see the higher volatility values related to higher interest rates. This result is consistent with Cheyette (1997), where he shows that the positive correlation between the interest rate volatility and the interest rate level is weak when the interest rate level is below 10 percent. However, when interest rate level was high in the late 1970s and early 1980s, the interest rate volatility was also high then, showing positive correlations only during that period.

24.2.2. Interest Rate Correlations

We have discussed the dynamics of interest rates. Now, let us consider the co-movements of interest rates. Do interest rates move together in steps, such that they all rise or fall together?

While the yield curve in principle can take many shapes historically, all the interest rates along the yield curve are positively correlated. But the interest rates do not shift by the same amount. The co-movements of the interest rates can be investigated by evaluating the correlations of the interest rates, as presented in Table 24.1.

The results show that all the correlations are positive, which suggests that all the interest rates tend to move in the same direction. The long rates, which are the interest rates with terms over 10 years, are highly correlated, meaning that the segment of the yield curve from a 10- to 30-year range

Table 24.1. Correlation matrix of the interest rates

	0.25	0.5	1	2	3	5	7	10	20	30
0.25	1.000	0.936	0.837	0.701	0.630	0.533	0.443	0.377	0.087	0.083
0.5	0.936	1.000	0.938	0.832	0.770	0.675	0.587	0.509	0.224	0.154
1	0.837	0.938	1.000	0.940	0.895	0.816	0.731	0.654	0.379	0.291
2	0.701	0.832	0.940	1.000	0.989	0.950	0.898	0.832	0.573	0.426
3	0.630	0.770	0.895	0.989	1.000	0.980	0.945	0.887	0.649	0.493
5	0.533	0.675	0.816	0.950	0.980	1.000	0.982	0.946	0.736	0.595
7	0.443	0.587	0.731	0.898	0.945	0.982	1.000	0.976	0.821	0.670
10	0.377	0.509	0.654	0.832	0.887	0.946	0.976	1.000	0.863	0.750
20	0.087	0.224	0.379	0.573	0.649	0.736	0.821	0.863	1.000	0.867
30	0.083	0.154	0.291	0.426	0.493	0.595	0.670	0.750	0.867	1.000

tends to move up and down together. The interest rates that are closer together along the yield curve have higher correlations.

24.2.3. Term Structure of Volatilities

Interest rate volatility is not the same for all interest rates along the yield curve. By convention, based on the lognormal model, the uncertainty of an interest rate is measured by the annualized standard deviation of the proportional change in a bond yield over a time interval (dt). For example, if the time interval is a one-month period, then dt equals $1/12$ year. This measure is called the interest rate volatility and it is denoted by $\sigma(t, T)$, the volatility of the T -th year rate at time t . More precisely, the volatility is the standard deviation of the proportional change in rate over a short time interval, and it is given by:

$$\sigma(t, T) = \text{Std} \left(\frac{\Delta r(t, T)}{r(t, T)} \right) / \sqrt{\Delta t} \quad (24.4)$$

where $r(t, T)$ is the yield-to-maturity of the zero-coupon bond with time-to-maturity T at time t and $\text{Std}(\cdot)$ is a standard deviation over dt . We can

relate Equation (24.4) to (24.3a) by the following algebraic manipulations. For a small time step, Equation (24.3a) can be written as:

$$\frac{\Delta r(t, T)}{r(t, T)} \cong \mu \Delta t + \sigma(t, T) \Delta Z$$

For sufficiently small Δt , we have:

$$\sigma \left(\frac{\Delta r(t, T)}{r(t, T)} \right) \cong \sigma(t, T) \sqrt{\Delta t}$$

Rearranging the terms, we can express σ as Equation (24.4) requires. Similarly, based on the normal model, the term structure of volatilities is given by

$$\sigma(t, T) = \sigma(\Delta r(t, T)) / \sqrt{\Delta t} \quad (24.5)$$

The relationship of the volatilities with respect to the maturity is called *the term structure of volatilities*. The interest rate volatilities can be estimated using historical monthly data ($\Delta t = 1/12$). Below is the standard deviation of the rates for 0.25, 0.5, 1, 2, 3, 5, 7, 10, 20, 30 years.

The historical term structure of volatilities shows that the short-term rates tend to have higher volatilities than the long-term rates, falling from

Table 24.2. Historical term structure of volatilities; $\sigma(\Delta r(t)/r(t)) \cdot \sqrt{12}$

0.25	0.5	1	2	3	5	7	10	20	30
0.1906	0.1908	0.1872	0.1891	0.1794	0.1632	0.1487	0.1402	0.1076	0.1137

19.06 percent for the 0.25-year rate to 11.37 percent for the 30-year rate. The empirical results suggest that we cannot think of interest rate volatility as one number. The volatility has to depend on the term of the interest rate in question.

24.2.4. Mean Reversion

Thus far the discussion focuses on the volatility term of the dynamics of the interest rates. Now we investigate the drift term of interest rate movements. Research tends to argue that the yield curve cannot follow a random walk like a stock, as in Equation (24.1). The yields of the Treasury bonds cannot rise and fall with the expected drift, yet to be constant or at a certain fixed proportion to the interest rate level. Since the nominal interest rate, which is what we are concerned with here, is decomposed into the real interest rate and the expected inflation rate as stated in the Fisher equation, the movements of the nominal rates can be analyzed by considering the movements of the real rates and the inflation rate. One may argue that the real rate cannot follow a random walk because the real rate is related to all the individuals' time value of money in real terms. We tend to think the real interest rate is quite stable and that the real rate does not follow a random walk like a stock. To the extent that we believe the government seeks to control the inflation rate of an economy, the inflation rate cannot follow a random walk either. Therefore, we cannot assume that the (nominal) interest rate follows a random walk.

One may conclude that the interest rates tend to fall when the interest rates are high. Conversely, the interest rates tend to rise when interest rates are low. This is a somewhat imprecise description of a yield curve behavior, but we will provide a more precise description of this behavior later in the chapter, where we will provide alternative interest rate models in specifying this behavior. Research literature calls the dynamics that describe this behavior of interest rates a *mean reversion process*.

24.3. Equilibrium Models

Interest rate models seek to specify the interest rate movements such that we can develop a pricing methodology for an interest rate option.

24.3.1. The Cox-Ingersoll-Ross Model

The Cox, Ingersoll, and Ross (CIR) (1985) interest rate model is based on the productive processes of an economy. According to the model, every individual has to make the decision of consuming and investing with their limited capital. Investing in the productive process may lead to higher consumption in the following period, but it would sacrifice consumption today. The individual must determine the optimal trade off.

Now assume that the individual can also borrow and lend capital to another individual. Each person has to make economic choices. The interest rates reach the market equilibrium rate when no one needs to borrow or lend. The model can explain the interest rate movements in terms of an individual's preferences for investment and consumption as well as the risks and returns of the productive processes of the economy.

As a result of the analysis, the model can show how the short-term interest rate is related to the risks of the productive processes of the economy. Assuming that an individual requires a premium on the long-term rate (called term premium), the model continues to show how the short-term rate can determine the entire term structure of interest rates and the valuation of interest rate contingent claims.

The CIR model

$$dr = a(b - r)dt + \sigma\sqrt{r} dZ \quad (24.6)$$

Cox et al. (1985) offer one of the earlier attempts at modeling interest rate movements. The proposed equilibrium model extends from economic principles of interest rates. It assumes mean reversion of interest rates. As we have discussed in the previous section, mean reversion of interest rates

means that when the short-term interest rate (r) is higher than the long-run interest rates (b), the short-term rate would fall adjusting gradually to the long-run interest rate. Conversely, when the short-term interest rate is lower than the long-run interest rate, the short-term rate would rise gradually to the long-run interest rate. Note that the long-run interest rate is not the long-term interest rate. Long-term interest rates continuously make stochastic movements, while the long-run interest rate is a theoretical construct, hypothesizing that the economy has a constant long-run interest rate that interest rates converge to over time. The constant (a) determines the speed of this adjustment. If the constant (a) is high/low, the adjustment rate to the long-term rate would be high/low. The CIR model is a lognormal model since the interest rate volatility is positively related to the interest rate level. The classification of lognormal and normal is based on the uncertain movement of the interest rate over a short period of time as described above.

24.3.2. The Vasicek Model

The second model is called the Vasicek model (1977). This model is similar to the CIR model such that the model assumes that all interest rate contingent claims are based on short-term interest rates. The only difference is that the volatility is not assumed to be dependent on the interest rate level, and therefore it is a normal model.

The Vasicek model

$$dr = a(b - r)dt + \sigma dZ, \quad (a > 0) \quad (24.7)$$

These models assume that there is only one source of risk and the models are referred to as *one-factor models*. This assumption implies that all bond prices depend on the movements of the rate (r), and that all bond prices move in tandem because of their dependence on one factor. At first, this assumption seems to be unrealistic because, as we have discussed, the yield curve seems to have many degrees of freedom in its movements, and therefore, how can we confine our yield curve to exhibit a one-factor movement?

24.3.3. The Brennan and Schwartz Two-Factor Model

For many purposes the one-factor model may not be appropriate to use as valuation models. An interest rate spread option is one example that a one-factor model may not be adequate to value. The values of some securities depend on the changing interest rate spreads between the two-year rate and the ten-year rate. The one-factor model assumes that all the interest rates that move in tandem would eliminate the risk of the spread between the two-year and the ten-year rates.

One extension asserts that all the bond prices of all maturities are generated by the short-term interest rate and a long-term rate – the long-term rate being the consol bond, which has no maturity and whose rate represents the long-term rate. Different versions of the two-factor models have been proposed in the following papers: Brennan and Schwartz (1982), Richard (1978), and Longstaff and Schwartz (1992). The Brennan and Schwartz model is given below:

$$\begin{aligned} dr &= a_1 + b_1(l - r)dt + r\sigma_1 dZ \\ dl &= l(a_2 + b_2r + c_2l)dt + l\sigma_2 dW \end{aligned} \quad (24.8)$$

where r is the short-term rate and l is the consol rate, and where a consol bond is a bond that pays a fixed coupon periodically into the future on a notional amount with no maturity. σ_1 and σ_2 are the standard deviations of the short-term and consol rate, respectively. dZ and dW represent the risks which may be correlated. All the parameters a_1 , b_1 and a_2 , b_2 , c_2 are estimated from the historical data.

24.4. Arbitrage-Free Models

From the standard economic theory perspective, arbitrage-free modeling takes a departure from the CIR approach. The main point of the departure is sacrificing the economic theory in providing a model of the term structure of interest rates for a more accurate tool for valuing securities. Since

the yield curve measures the agents' time value of money, the standard economic theory relates the interest rate movements to the dynamics of the economy. By way of contrast, arbitrage-free modeling assumes the yield curve follows a random movement much like the model used to describe a stock price movement. We can show that stock prices are assumed to be random and such an assumption does not incorporate the modeling of the agent's behavior and the economy.

24.4.1. The Ho–Lee model

Ho–Lee (1986) takes a different approach in modeling yield curve movements as compared to CIR and Åsicek. The arbitrage-free interest rate model uses the relative valuation concepts of the Black–Scholes model. This concept of relative valuation becomes a more complex concept to accept in the interest rate theory. Arbitrage-free modeling, like the Black–Scholes model, argues that the valuation of interest rate contingent claims is based solely on the yield curve. Economic research focuses on understanding the inferences made from the yield curve shape and its movements. The arbitrage-free model omits all these fundamental issues, apparently ignoring part of the economic theory behind interest rate research. The model assumes that the yield curve moves in a way that is consistent with the arbitrage-free condition.

Let us assume that there is a perfect capital market in a discrete time world. But this time, the binomial model is applied to the yield curve movements. We assume:

- (1) Given the initial spot yield curve, the binomial lattice model requires that the yield curve can move only up and down.
- (2) The one period interest rate volatility (the instantaneous volatility) is the same in all states of the world.
- (3) There is no arbitrage opportunity in any state of the world (at any node point on the binomial lattice).

Assumption (1) is a technical construct of the risk model. Assumption (2) is made simply for this example. This assumption can be altered. Assumption (3) is the most interesting and important, called the “arbitrage-free condition”. This arbitrage-free condition imposes constraints on the yield curve movements.

Thus far it seems that the extension is directly from the Black–Scholes model. But there is one problem: interest rate is not a security. We cannot buy and sell the one-period rate, though we can invest in the rate as the risk-free rate. Moreover, we cannot use the one-period rate to form an arbitrage argument as the Black–Scholes model does with stock, since the one-period rate is the risk-free rate, which obviously cannot be the “underlying asset” as well. In equity option, the stock is both the underlying instrument as well as the risk source or the risk driver.

A. Arbitrage-free hedging: The conceptual extension of the interest rate arbitrage-free model from the Black–Scholes model is to introduce the short-term interest rate as the risk source (or risk drive or state of the world). The Black–Scholes model's risk neutral argument requires an underlying security and the risk-free rate. However, in the interest rate model, the risk-free rate is the risk source. One condition we want to impose on the interest rate movement is arbitrage-free, that is, the interest rate movements do not allow any possible arbitrage opportunity in holding a portfolio of bonds at any time. Research shows that the interest rate movements are arbitrage-free if the following two conditions hold (Harrison and Kreps 1979): (1) all the bonds at any time and state of the world have a risk-neutral expected return of the prevailing one period rate and (2) any bond on the initial yield curve has the risk-neutral expected return of the one-period interest rate of the initial yield curve. That is, for an interest rate movement to be arbitrage-free, there must be a probability assigned to each node of a tree such that all interest rate contingent claims have an expected “risk-free return,” which is the one-period rate. Note that this probability is the “risk

neutral,” where the market probability can be quite different.

B. Recombining condition: For tractability of the model, we require the discount function to recombine in a binomial lattice. This requirement is similar to the Black–Scholes model. Namely, the yield curve making an up movement and then a down movement must have the same value as the yield curve that makes a down movement and then an up movement. The difference between the yield curve movement and the stock movement is that we need the entire discount function (or the yield curve), and not just one bond price, to be identical when they recombine.

Under these restrictions, we can derive all the possible solutions. Let us consider the simplest solution for us to gain insight into these arbitrage-free models. Suppose the spot yield curve is flat. The spot curve can shift in a parallel fashion up and down. The binomial lattice represented is called “normal” (or arithmetic) because the parallel shift of the curve is a fixed amount and not a proportion of the value at the node. The movements of the discount function can be represented by the binomial movements.

The purpose of the arbitrage-free model is not to determine the yield curve from any economic theory or to hypothesize that the yield curve should take on particular shapes. The arbitrage-free model takes the yield curve (or the discount function) as given, and then hypothesizes the yield curve (or the discount function) movements in order to relatively value other interest rate derivatives. Using a dynamic hedging argument similar to the Black–Scholes model, the argument shows that we can assume the local expectation hypothesis to hold: the expected return of all the bonds over each time step is the risk-free rate, the one-period interest rate.

The Ho–Lee model is similar to the M̄sicek model in that they are both normal models. The main difference of course is that the Ho–Lee model is specified to fit the yield curve, whereas the M̄sicek model is developed to model the term structure of

interest rates. For this reason, the M̄sicek model has the unobservable parameter called term premium, and the yield curve derived from the M̄sicek model is not the same as the observed yield curve in general. Unlike the M̄sicek model, the arbitrage-free interest rate model does not require the term premium, which cannot be directly observed. Instead, the arbitrage-free interest model only requires the given observed yield curve to value bonds. Hence, the theoretical bond prices would be the same as those observed.

Specifically, let the initial discount function, prices of zero-coupon bonds with a face value of \$ and with maturity T , be denoted by $P(T)$. The discount function $P(T)$, for example, may be observed from the STRIPS market. The yield of the bond $P(T)$ is denoted by $r(T)$. Let σ be the volatility of the interest rate. Interest rate volatility may be estimated from historical data. Then the price of a one-period bond $P_i^n(1)$ in time n and state i on the binomial lattice is given by:

$$P_i^n(1) = 2 \left[\frac{P(n+1)}{P(n)} \right] \cdot \frac{\delta^i}{(1+\delta^n)} \quad (24.9)$$

where

$$\begin{aligned} P_i^n(1) &= \text{a one-period bond price at time period } n \\ &\quad \text{and state } i, \\ \delta &= e^{-2r(1)\sigma}, \\ \sigma &= \text{Std.} \left(\frac{\Delta r(1)}{r(1)} \right). \end{aligned}$$

$-0.5 \ln \delta$ is the standard deviation of the change of the interest rate over each step size, while σ is the standard deviation of the proportional change of the interest rate.

While Equation (24.9) provides the bond price for one period at any state i and time n , the model also has closed form solutions for bonds with any maturity at any node point on the lattice.

The basic idea of the derivation is quite simple, though the manipulation of the algebra is somewhat laborious. To derive the model, we need to determine the close form solution for $P_i^n(T)$, the

price of a T year zero-coupon bond, at time n and state i , such that, under the risk-neutral probability 0.5, the expected return of a zero-coupon bond with any maturity, at any node point, equals the one-period risk-free rate. That is:

$$P_i^n(T) = 0.5P_i^n(1)\{P_{i+1}^{n+1}(T-1) + P_{i+1}^{n+1}(T-1)\} \tag{24.10}$$

and we need to satisfy the initial observed yield curve condition:

$$P(T) = 0.5P(1)\{P_0^1(T-1) + P_1^1(T-1)\} \tag{24.11}$$

The above equations hold for any i , n , and T . Then the model is assumed to be arbitrage-free in that all bonds have the expected returns and the bond pricing consistent with the initial spot yield curve (or the discount function $P(T)$).

Equation (24.9) specifies the one-period bond price (and hence the one-period interest rate) on each node of the binomial lattice. For this reason, we say that the model is an interest rate model, as the model specifies how the short-term interest rate movements are projected into the future.

We can show that once we can specify the one-period rate on a lattice, we can determine all the bond prices at each node point on the lattice by a backward substitution procedure similar to that used by the Black-Scholes model.

We can define the one period rate to be

$$r_i^n(1) = -\ln P_i^n(1) \tag{24.12}$$

Using Equation (24.12), we see that the $r_i^n(1)$ can be expressed in three terms:

$$r_i^n(1) = \ln \frac{P(n)}{P(n+1)} + \ln(0.5(\delta^{-(n/2)} + \delta^{n/2})) + \left(\frac{n}{2} - i\right) \ln \delta \tag{24.13}$$

The first term is the one-period forward rate. That means that under the arbitrage-free interest rate movement model, we can think of the movement of the short-term rate as based on the forward rates. When there is no interest rate

uncertainty, ($\delta = 1$), both the second and third terms are equal to zero, and therefore, the one-period forward rates define the future spot rate arbitrage-free movements.

The last term specifies the cumulative upward and downward shifts of the rates after n periods. It is important to note that the sizes of all the shifts are the same, $\ln \delta$. That means the interest rate risk is independent of the level of interest rate, and the interest rate follows a normal distribution.

The second term is more difficult to explain as well as important. Let us consider a two-year bond. Assume that the yield curve is flat at 10 percent. The bond price is therefore 0.826446. After one year, the interest rate shifts to 20 percent or 0 percent with equal probability, just to exaggerate the problem a little bit. The expected price of the bond is now

$$0.916607 \left(= \frac{1}{2} \times \left(\frac{1}{1.2} \right) + \frac{1}{2} \times \left(\frac{1}{1.0} \right) \right).$$

The expected return of the bond over the first year is 10.9095 ($= (0.916607/0.826446) - 1$). Therefore, even with a yield curve that is flat at 10 percent, the yield curve makes the shifts of up or down with the same probability, and the expected return of the bond exceeds 10 percent. The reason is straightforward: When the interest rate moves, the bond price does not move in step with the interest rates. This is simply a matter of bond arithmetic of the yield calculation, where the yield is in the denominator. We can show that bonds have positive convexity. When the yield curve makes a parallel shift up or down with equal probability, the expected bond price is higher than the prevailing bond price. After all, it is the positive convexity of a bond that motivates the barbell trades.

Since bonds have positive convexity, if the interest rate shifts up or down by the same amount (with equal probability) relative to the forward rate, the expected returns of the bonds would exceed the one-period interest rate. To maintain the arbitrage-free condition, such that the local

expectation hypothesis holds, we require the interest rate to shift higher in both up and down movements, so that the expected bonds' returns are equal to the one-period interest rate. That is, the interest rate movements must be adjusted upwards to correct for this convexity effect. This correction is the second term. Note that the second term, the convexity adjustment term, increases with the volatility as one may expect.

Thus far, we have discussed a set of interest rate models that exhibit normal distributions, which does not reflect the relationship between the interest rate uncertain movements and the interest rate levels. The lognormal model ensures that the interest rate uncertain movement increases or decreases with interest rate level. In particular, when interest rates continue to fall, the interest rate movement will continue to become smaller. In this case, the interest rates cannot become negative, while the normal model often has scenarios where the interest rates can become negative. An example of a lognormal model is the Black–Derman–Toy model.

24.4.2. The Black–Derman–Toy Model

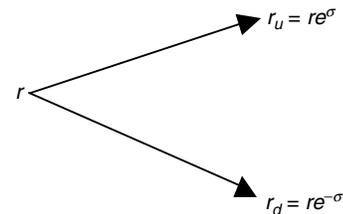
The Black–Derman–Toy (BDT) (1990) model is a binomial lattice model. This model assumes that the short-term interest rate follows a lognormal process. The model does not have a closed form solution and can best be explained by describing the procedure to construct the short-term interest rate movements.

The Black–Derman–Toy model uses a recombining lattice to determine a lognormal interest rate model. Further, the model can take the initial term structure of interest rate as input, as well as the term structure of volatilities, as in the extended Ho–Lee model. The model is specified by an iterative construction that can be best illustrated with an example:

As inputs to the model, we begin with the given term structure of interest rates and term structure of forward volatilities:

Maturity (years)	1	2	3	4	5
Yield (%)	6.0	7.0	8.0	9.5	10.0
Forward volatility (%)	15.0	14.0	13.0	11.0	

On the lattice, initially we have a one-period rate, say, 6 percent. The lognormal model is determined by the following random walk at a node:



Note that, using the definition of r_u and r_d , we know

$$r_u = r_d e^{2\sigma}. \quad (24.14)$$

Step 1. Construct the lowest short-term rate for each period in the lattice.

These rates are r , $r \cdot \exp[-\sigma(1)]\mu(1)$, $r \cdot \exp[-\sigma(2)]\mu(2)$. Note that we do not know μ , the only parameter unknown at this point.

Step 2. Specify the short-term rates at all the nodes using Equation (24.14).

We need to iteratively calculate the rate r_u , applying Equation (24.14) repeatedly.

Step 3. Determine μ by a “bootstrap” approach.

Search for the value $\mu(1)$ such that a two-year bond, given by the discount function $P(T)$, can be priced according to the market. Then, we determine $\mu(2)$ such that $\mu(2)$ can price the three-year bond exactly according to the observed (or given) three-year bond price. This iterative procedure, called the bootstrap approach, can determine the lattice as desired.

We calculate the short rates by following the BDT procedure given the yields and the instantaneous forward volatilities in the table above.

24.4.3. The Hull–White Model

The Hull–White model (1990) is a normal model that has an explicit term to capture the mean reversion of interest rates. It is similar to the Åsicek model with the difference of being arbitrage-free. This approach enables the model to capture the term structure of volatilities by adjusting the adjustment rate of the short-term rate to the long-term equilibrium rate. The lattice model they propose is not a binomial model but a trinomial model. The trinomial model enables the model to adjust for the speed of adjustment and it can be constructed such that the model has no negative interest rates in all scenarios.

The Hull–White model can also be extended to a two-factor model (1994) that is arbitrage-free in a form similar to the Brennan and Schwartz model. Specifically, the model is specified by two simultaneous equations:

$$dr = [\theta(t) + u - ar]dt + \sigma_1 dW \quad (24.15)$$

$$du = -bu dt + \sigma_2 dZ \quad (24.16)$$

In this case, the short-term rate makes partial adjustments to the long-term rate, while the long-term rate follows a random movement. Using normal model properties, these models can derive closed form solutions for many derivatives in the continuous time formulation.

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