

Time Value of Money

In this chapter you learn to convert cash flows occurring in one time period to an equivalent cash flow occurring at another time period or into a uniform series of cash flows occurring over successive periods. Understanding the time value of money is a prerequisite to understanding debt financing and how to compare two or more financial options, which are the topics of Chapters 16, 17, and 18. Additionally, you learn how to adjust interest rates for inflation.

Suppose that I were to offer to give you \$1,000 today or \$1,000 a year from now. Which would you take? Most people would take the \$1,000 today because a dollar today is worth more than a dollar tomorrow. Money's value is based not only on the amount of money received but also on when the money is received. In the case of the \$1,000, the amount of money received is the same; however, the amounts are received at different points in time. Each of you could have taken the \$1,000 today, invested it in the bank and at the end of the year had the original \$1,000 plus the interest paid by the bank for the use of your money. If the bank were paying an interest rate of 5% compounded annually, the original \$1,000 investment would have earned \$50 ($\$1,000 \times 0.05$) interest for a total of \$1,050 by the end of the first year. Therefore, \$1,000 received today would be equivalent, although not equal in amount, to \$1,050 received a year from now based on an interest rate of 5% compounded annually.

Now suppose that I were to offer to give you \$1,000 today or \$1,050 a year from now. Which would you take? First, we notice that the amounts of money received are different. The receipt a year from now is \$50 greater. However, these receipts are equivalent when compared at an interest rate of 5% compounded annually because \$1,000 invested today at an interest rate of 5% compounded annually would earn \$50 interest and be worth \$1,050 a year from now, which is the same as the payment a year from now.

EQUIVALENCE

Equivalent cash flows are cash flows that produce the same result over a specific period of time. In the example on the previous page you would have the same amount of cash at the end of the first year regardless of whether you chose to receive the \$1,000 today and invest it at an interest rate of 5% compounded annually or you chose to receive the \$1,050 a year from now. Equivalence is a function of the following:

1. The size of the cash flows,
2. The timing of the cash flows, and
3. The interest rate.

When analyzing cash flows it is important to take into account not only the amount or size of the cash flows but also the timing of the cash flows. To simplify the comparison of cash flows at different time periods, it is useful to convert the cash flows into their equivalent cash flows at a specific point in time based on a specified interest rate. The point in time may be now, at some time in the future, or a series of periodic cash flows over a specified period of time (for example, monthly or annual cash flows). By converting the cash flows to a specific point in time at a given interest rate, the timing of the cash flows and interest rate become fixed, leaving equivalence a function of the size of the cash flows. The cash flows then can be directly compared. For the purpose of determining equivalence, cash flows that occur in the same period of time may be added or subtracted.

A number of formulas have been developed to convert cash flows, which occur at different points in time, into their equivalent cash flows at some specific point or points in time. These formulas use the following variables:

P = the present value of a cash flow or the value of a cash flow at the present time. In the case of the \$1,000, its present value is \$1,000. When used in the equivalence formulas, the present value occurs at the beginning of the year, which is the same point in time as the end of the previous year. The present value may be used to refer to any point in time prior to a cash flow or series of cash flows for which we want to determine the equivalence of the cash flow or cash flows. This point may be now, in the past, or in the future.

F = the future value of a cash flow or the value of a cash flow at some specific point in the future. In the case of the \$1,000, its future value in one year at an interest rate of 5% compounded annually is \$1,050. The future value occurs at the end of the year. The future value may be used to refer to any point in time after a cash flow or series of cash flows for which we want to determine the equivalence of the cash flow or cash flows. The future value may also occur concurrently with the last cash flow in a uniform series of cash flows.

i = the periodic interest rate or interest rate for one period. The period may be any specified amount of time with years, quarters, and months being the most common. The period must be the same as the compounding period for the interest rate. In the case of the \$1,000, the periodic interest rate was 5% and the period was one year. The compounding period was also one year.

n = the number of interest compounding periods of time. The length of the periods used to determine n must be the same as the length of the periods for the periodic interest rate. If the length of the compounding period is measured in months, then n equals the number of months. In the case of the \$1,000, the number of periods was one and is measured in years.

A = the value of a single cash flow in a uniform series of cash flows that occurs at the end of each period and continues for n number of periods. These cash flows must be uniform or equal in amount (not equivalent), the cash flows must occur at the end of each and every period in the series, and the length of the periods must be the same length as the length of the periods for the periodic interest rate.

SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR

Returning to the case of the \$1,000, we saw that at the end of the first year the principal or original amount of money deposited in the bank had earned \$50 ($\$1,000 \times 0.05$) interest for a total of \$1,050. If this amount were to be left in the bank for an additional year it would have earned \$52.50 ($\$1,050 \times 0.05$) interest during the second year for a total of \$1,102.50 at the end of the second year. During the second year, interest was earned on the interest paid during the first year. This is known as compound interest. Compound interest is where interest is paid on the interest from the previous periods. With compound interest, interest is due at the end of each period. If the interest is not paid each period, then interest accrues on the unpaid interest. During the second year \$2.50 ($\50×0.05) of interest accrued on the unpaid interest from the first year. The future value at year n of a single cash flow is calculated by the following equation:

$$F = P(1 + i)^n \quad (15-1)$$

This formula is known as the single-payment compound-amount factor. The derivation of Eq. (15-1) is found in Appendix D. The single-payment compound-amount factor converts a present value into a future value at a specified rate of interest and is shown in Figure 15-1.

Eq. (15-1) may also be written using shorthand notation as follows:

$$F = P(F/P, i, n) \quad (15-2)$$

where

$$(F/P, i, n) = (1 + i)^n$$

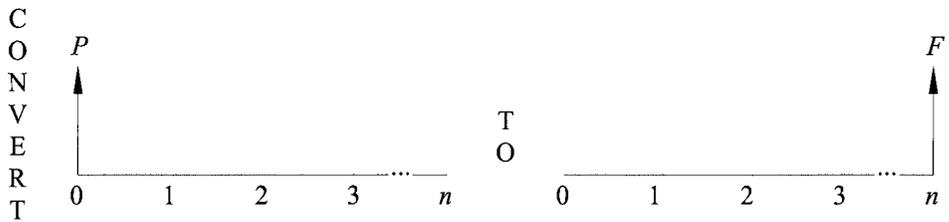


FIGURE 15-1 Single-Payment Compound-Amount Factor

The shorthand notation for the single-payment compound-amount factor and other cash-flow conversion formulas are written in the following standard format:

(Value Desired/Value to be Converted, Interest Rate, Number of Periods)

hence $(F/P, i, n)$ is used to get the future value from the present value using a periodic interest rate of i and n number of periods.

Values for $(F/P, i, n)$ for standard periodic interest rates and periods are found in Appendix E. To find a value one must first find the table in the appendix that corresponds to the interest rate. In the case of a 10% interest rate you would use Table E-16. The tables in Appendix E are read by finding the appropriate formula in the column headings and row that corresponds to the number of periods in the left column. Figure 15-2 shows the value for single-payment compound-amount factor $(F/P, i, n)$ for five periods at an interest rate of 10%, which is 1.6105.

The advantage of using the tables in Appendix E over Eq. (15-1) is that the math is less complex. The disadvantages are that the tables are quite limited in the number of interest rates and periods that are available and the tables may introduce rounding errors into the solutions.

Using Eq. (15-1) we find that by the end of ten years the \$1,000 would have grown to \$1,628.90. The calculation of this future value is as follows:

$$F_{10} = \$1,000(1 + 0.05)^{10} = \$1,628.89$$

Alternately, we may use Eq. (15-2) to solve for the future value at the end of the ten years as follows:

$$F_{10} = \$1,000(F/P, 5, 10) = \$1,000(1.6289) = \$1,628.90$$

The minor difference in the future value for year 10 is due to rounding that occurs in the tables located in Appendix E.

These equations are useful when we want to find the future value of a specific amount of money invested today, as in the case of the \$1,000. It may also be used to find the future value of money invested at anytime.

Example 15-1: What is the future value six years from now of a \$1,000 cash flow that occurs one year from now using a periodic interest rate of 10% compounded annually?

Table E-16
Interest Factors for 10.00%

<i>n</i>	Single Payment		Uniform Series			
	Compound-Amount Factor	Present-Worth Factor	Compound-Amount Factor	Sinking-Fund Factor	Present-Worth Factor	Capital-Recovery Factor
	Convert <i>P</i> to <i>F</i> (<i>F/P, i, n</i>)	Convert <i>F</i> to <i>P</i> (<i>P/F, i, n</i>)	Convert <i>A</i> to <i>F</i> (<i>F/A, i, n</i>)	Convert <i>F</i> to <i>A</i> (<i>A/F, i, n</i>)	Convert <i>A</i> to <i>P</i> (<i>P/A, i, n</i>)	Convert <i>P</i> to <i>A</i> (<i>A/P, i, n</i>)
1	1.1000	0.9091	1.0000	1.0000	0.9091	1.1000
2	1.2100	0.8264	2.1000	0.4762	1.7355	0.5762
3	1.3310	0.7513	3.3100	0.3021	2.4869	0.4021
4	1.4641	0.6830	4.6410	0.2155	3.1699	0.3155
5	1.6105	0.6209	6.1051	0.1638	3.7908	0.2638
6	1.7716	0.5645	7.7156	0.1296	4.3553	0.2296
7	1.9487	0.5132	9.4872	0.1054	4.8684	0.2054
8	2.1436	0.4665	11.4359	0.0874	5.3349	0.1874
9	2.3579	0.4241	13.5795	0.0736	5.7590	0.1736
10	2.5937	0.3855	15.9374	0.0627	6.1446	0.1627
11	2.8531	0.3505	18.5312	0.0540	6.4951	0.1540
12	3.1384	0.3186	21.3843	0.0468	6.8137	0.1468
13	3.4523	0.2897	24.5227	0.0408	7.1034	0.1408
14	3.7975	0.2633	27.9750	0.0357	7.3667	0.1357
15	4.1772	0.2394	31.7725	0.0315	7.6061	0.1315
16	4.5950	0.2176	35.9497	0.0278	7.8237	0.1278
17	5.0545	0.1978	40.5447	0.0247	8.0216	0.1247
18	5.5599	0.1799	45.5992	0.0219	8.2014	0.1219
19	6.1159	0.1635	51.1591	0.0195	8.3649	0.1195
20	6.7275	0.1486	57.2750	0.0175	8.5136	0.1175
21	7.4002	0.1351	64.0025	0.0156	8.6487	0.1156
22	8.1403	0.1228	71.4027	0.0140	8.7715	0.1140
23	8.9543	0.1117	79.5430	0.0126	8.8832	0.1126
24	9.8497	0.1015	88.4973	0.0113	8.9847	0.1113
25	10.8347	0.0923	98.3471	0.0102	9.0770	0.1102
26	11.9182	0.0839	109.1818	0.0092	9.1609	0.1092
27	13.1100	0.0763	121.0999	0.0083	9.2372	0.1083
28	14.4210	0.0693	134.2099	0.0075	9.3066	0.1075
29	15.8631	0.0630	148.6309	0.0067	9.3696	0.1067
30	17.4494	0.0573	164.4940	0.0061	9.4269	0.1061
35	28.1024	0.0356	271.0244	0.0037	9.6442	0.1037
40	45.2593	0.0221	442.5926	0.0023	9.7791	0.1023
45	72.8905	0.0137	718.9048	0.0014	9.8628	0.1014

FIGURE 15-2 Table E-16 from Appendix E

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CONVERTING A PRESENT VALUE TO A FUTURE VALUE USING EXCEL

Example 15-1 may be set up in a spreadsheet as shown in the following figure:

	A	B
1	Interest Rate	10.00%
2	Periods	5
3	Uniform Series	0.00
4	Present Value	1,000.00
5	Future Value	-1,610.51

To set up this spreadsheet, the following formulas, text, and values need to be entered into it:

	A	B
1	Interest Rate	0.1
2	Periods	5
3	Uniform Series	0
4	Present Value	1000
5	Future Value	=FV(B1,B2,B3,B4)

The spreadsheet uses the FV function to calculate the future value. The FV function is written as

$$=FV(\text{rate}, \text{nper}, \text{pmt}, \text{pv}, \text{type})$$

where

rate = periodic interest rate

nper = number of periods

pmt = uniform series of cash flows

pv = cash flow at the present time

type = 1 for payment at the beginning of the period and 0 for payment at the end of the period (defaults to 0 if left blank)

To convert a present value into a future value, the uniform series (pmt) is set to zero and the type is left blank. The future value will always carry the sign opposite from the present value.

Solution: To solve this problem, a period with a length of one year is used because interest rate compounds annually. We would also use n equal to 5 periods because the \$1,000 was invested for five years (6 years – 1 year). Substituting these values into Eq. (15-1) we get the following:

$$F = \$1,000(1 + 0.10)^5 = \$1,610.51$$

Alternately, Eq. (15-2) may be used to solve this problem as follows:

$$F = \$1,000(F/P, 10, 5) = \$1,000(1.6105) = \$1,610.50$$

Again, the difference in these two solutions is due to rounding that occurs in the tables of Appendix E.

The future value at the end of six years of the \$1,000 invested a year from now is \$1,610.51. Similarly, had we invested \$1,000 one year from now in a bank account earning 10% compounded annually, by the end of the sixth year, the \$1,000 would have been invested for five years and would have grown to \$1,610.51, earning \$610.51 (\$1,610.51 – \$1,000.00) of interest.

This equation may also be used when the compounding period is months. To do this the periodic interest rate must be compounded monthly and n would represent the number of months.

Example 15-2: What is the future value a year from now of a \$1,000 cash flow that occurs today using a periodic interest rate of 1% compounded monthly?

Solution: To solve this problem, a period with a length of one month is used because the interest rate compounds monthly. We would also use n equal to 12 periods, the number of months in a year. Substituting these values into Eq. (15-1) we get the following:

$$F = \$1,000(1 + 0.01)^{12} = \$1,126.83$$

Alternately, Eq. (15-2) may be used to solve this problem as follows:

$$F = \$1,000(F/P, 1, 12) = \$1,000(1.1268) = \$1,126.80$$

Again, the difference in these two solutions is due to rounding that occurs in the tables of Appendix E.

The future value of the \$1,000 one year from now is \$1,126.83. Similarly, \$1,000 invested for one year, at an interest rate of 1% compounded monthly, would grow to \$1,126.83 and would earn \$126.83 (\$1,126.83 – \$1,000.00) of interest.

SINGLE-PAYMENT PRESENT-WORTH FACTOR

Conversely, we can find the present value of some future cash flow by solving Eq. (15-1) for P . Solving for P we get the following:

$$P = F/(1 + i)^n \tag{15-3}$$

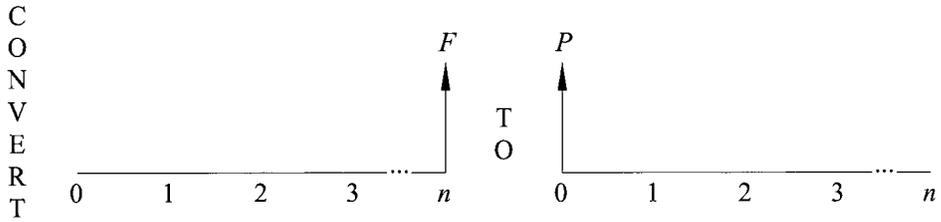


FIGURE 15-3 Single-Payment Present-Worth Factor

This equation is known as the single-payment present-worth factor. The single-payment present-worth factor converts a future value into a present value at a specified rate of interest and is shown in Figure 15-3.

Eq. (15-3) may also be written using shorthand notation as follows:

$$P = F(P/F, i, n) \tag{15-4}$$

where

$$(P/F, i, n) = 1/(1 + i)^n$$

Values for $(P/F, i, n)$ for standard periodic interest rates and periods are found in Appendix E.

These equations are useful when we want to find out how much money needs to be set aside today in an interest-bearing account to make some payment in the future.

Example 15-3: What is the present value of \$1,000 received five years from now using a periodic interest rate of 10% compounded annually?

Solution: For this problem, the periodic interest rate is 10%, the period is one year, and the number of periods is 5. Substituting these values into Eq. (15-3) we get the following:

$$P = \$1,000/(1 + 0.10)^5 = \$620.92$$

Alternately, Eq. (15-4) may be used to solve this problem as follows:

$$P = \$1,000(P/F, 10, 5) = \$1,000(0.6209) = \$620.90$$

Again, the difference in these two solutions is due to rounding that occurs in the tables of Appendix E.

The present value of \$1,000 received five years from now is \$620.92. Similarly, had we set aside \$620.92 today in a bank account earning an interest rate of 10% compounded annually, in five years it would have grown to \$1,000 and earned \$379.08 (\$1,000 – \$620.92) of interest.

Once a series of cash flows have been converted into their equivalent cash flows at a specific point in time, whether now or at some future time, they may be added to each other as shown in Example 15-4.

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CONVERTING A FUTURE VALUE TO A PRESENT VALUE USING EXCEL

Example 15-3 may be set up in a spreadsheet as shown in the following figure:

	A	B
1	Interest Rate	10.00%
2	Periods	5
3	Uniform Series	0.00
4	Future Value	1,000.00
5	Present Value	-620.92

To set up this spreadsheet, the following formulas, text, and values need to be entered into it:

	A	B
1	Interest Rate	0.1
2	Periods	5
3	Uniform Series	0
4	Future Value	1000
5	Present Value	=PV(B1,B2,B3,B4)

The spreadsheet uses the PV function to calculate the present value. The PV function is written as

$$=PV(\text{rate}, \text{nper}, \text{pmt}, \text{fv}, \text{type})$$

where

rate = periodic interest rate

nper = number of periods

pmt = uniform series of cash flows

fv = cash flow at the end of the specified number of periods (nper)

type = 1 for payment at the beginning of the period and 0 for payment at the end of the period (defaults to 0 if left blank)

To convert a future value into a present value, the uniform series (pmt) is set to zero and the type is left blank. The present value will always carry the sign opposite from the future value.

Example 15-4: What is the present value of \$1,000 cash flow one year from now and \$1,000 cash flow two years from now using a 10% compounded annually interest rate?

Solution: To solve this example Eq. (15-3) will be used to convert both future cash flows into their present values. These values are as follows:

$$P_1 = \$1,000/(1 + 0.10)^1 = \$909.09$$

$$P_2 = \$1,000/(1 + 0.10)^2 = \$826.45$$

After finding the present values for these two future cash flows, they may be added together:

$$P = \$909.09 + \$826.45 = \$1,735.54$$

The present value of the two future cash flows is \$1,735.54. Similarly, had we deposited \$1,735.54 in a bank account earning 10% interest compounded annually, we could have withdrawn a \$1,000 at the end of the first year and at the end of the second year there would have been \$1,000 remaining in the bank account. During the first year the \$1,735.54 in the bank account would have earned \$173.55 ($\$1,735.54 \times 0.10$) in interest. After the interest is paid and the \$1,000 is withdrawn, there would have remained \$909.09 ($\$1,735.54 + \$173.55 - \$1,000.00$) in the account. During the second year the \$909.09 would have earned \$90.91 ($\909.09×0.10) in interest, leaving \$1,000 ($\$909.09 + \90.91) in the account to be withdrawn at the end of the second year.

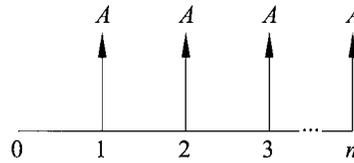
UNIFORM-SERIES COMPOUND-AMOUNT FACTOR

When faced with a long series of uniform cash flows, such as found in a thirty-year mortgage, it would be quite time consuming to calculate the present or future value for each individual cash flow and then add them together. To simplify these calculations, four formulas have been developed to deal with uniform cash flows. The first formula, the uniform-series compounding-amount factor, converts a series of uniform cash flows into a future value. For a series of cash flows to be uniform:

1. The cash flows must be uniform or equal in amount (not equivalent),
2. The cash flows must occur at the end of each and every period in the series,
3. The length of the periods in the series must be the same duration as the compounding period for the periodic interest rate, and
4. The length of the periods in the series must be the same length.

In a uniform series, the first cash flow occurs at the end of the first period, the second at the end of the second period, . . . , and the n th cash flow at the end of the n th period. The future value is calculated at the end of the n th period and

FIGURE 15-4 Uniform Series of Cash Flows



occurs at the same time as the last cash flow in the series. A uniform series of cash flows is shown in Figure 15-4.

Suppose we were to deposit \$1,000 a year for the next three years in a bank account with an interest rate of 8% compounded annually. The deposits are to be made at the end of each year. How much money would be in the account at the end of the third year?

The first deposit will be made at the end of the first year and will accrue interest during the second and third years or for two periods. Using Eq. (15-1), the future value of the first cash flow is as follows:

$$F_1 = \$1,000(1 + 0.08)^2 = \$1,166.40$$

The second deposit will be made at the end of the second year and accrue interest during the third year or for one period. Using Eq. (15-1) the future value of the second cash flow is as follows:

$$F_2 = \$1,000(1 + 0.08)^1 = \$1,080.00$$

The third and final deposit will be made at the end of the third year, the same point in time for which we are calculating the future value; therefore, the third deposit will not earn any interest and its future value will be equivalent to the amount of the deposit. The future value for the third year is as follows:

$$F_3 = \$1,000(1 + 0.08)^0 = \$1,000.00$$

Summing the future values we get the following:

$$F = \$1,166.40 + \$1,080.00 + \$1,000.00 = \$3,246.40$$

Alternately, this future value may be calculated by the following equation:

$$F = A[(1 + i)^n - 1]/i \quad (15-5)$$

This equation is known as the uniform-series compound-amount factor. The derivation for Eq. (15-5) is found in Appendix D. The uniform-series compounding-amount factor converts a uniform series of cash flows into a future value at a specified rate of interest and is shown in Figure 15-5.

It is important to note that the future value is calculated at the same point in time as the final cash flow for the uniform series, period n .

Eq. (15-5) may also be written using shorthand notation as follows:

$$F = A(F/A, i, n) \quad (15-6)$$

where

$$(F/A, i, n) = [(1 + i)^n - 1]/i$$

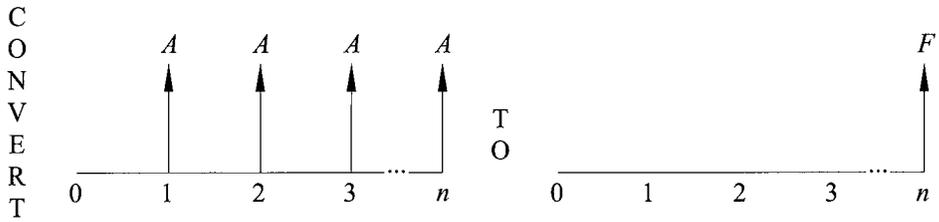


FIGURE 15-5 Uniform-Series Compound-Amount Factor

Values for $(F/P, i, n)$ for standard periodic interest rates and periods are found in Appendix E.

Using Eq. (15-5) to determine the future value of the three \$1,000 payment cash flows, we get:

$$F = \$1,000[(1 + 0.08)^3 - 1]/0.08 = \$3,246.40$$

Eq. (15-5) produces the same result as solving for the future value for each of the cash flows and then adding them together; however, Eq. (15-5) is much quicker.

Example 15-5: What is the future value ten years from now of ten \$500 cash flows using a periodic interest rate of 9% compounded annually? The cash flows occur at the end of each year and begin a year from now.

Solution: For this problem, the annual cash flows equal \$500, the periodic interest rate equals 9%, and the number of periods is 10. The future value is calculated as follows using Eq. (15-5):

$$F = \$500[(1 + 0.09)^{10} - 1]/0.09 = \$7,596.46$$

Alternately, Eq. (15-6) may be used to solve this problem as follows:

$$F = \$500(F/A, 9, 10) = \$500(15.1929) = \$7,596.45$$

Again, the difference in these two solutions is due to rounding that occurs in the tables of Appendix E.

The future value of this series of \$500 cash flows is \$7,596.46. Similarly, \$500 invested at the end of the year for the next ten years in a bank account earning 9% compounded annually would have grown to \$7,596.46 at the end of the tenth year, which is the time of the tenth deposit.

UNIFORM-SERIES SINKING-FUND FACTOR

Solving Eq. (15-5) for A we get the following:

$$A = Fi/[(1 + i)^n - 1] \tag{15-7}$$

This equation is known as the uniform-series sinking-fund factor. Equation (15-7) is based on the cash flows being uniform and occurring at the end of

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CONVERTING A UNIFORM SERIES TO A FUTURE VALUE USING EXCEL

Example 15-5 may be solved using the spreadsheet from Sidebar 15-1, as shown in the following figure:

	A	B
1	Interest Rate	9.00%
2	Periods	10
3	Uniform Series	500.00
4	Present Value	0.00
5	Future Value	-7,596.46

To convert a uniform series into a future value, the present value is set to zero and the type is left blank. The future value will always carry the opposite sign from the uniform series.

each period in the series. The uniform-series sinking-fund factor converts a future value to a series of annual values at a specified rate of interest and is shown in Figure 15-6.

It is important to note that the last cash flow for the uniform series occurs at the same point in time as the future value, period n .

Eq. (15-7) may also be written using shorthand notation as follows:

$$A = F(A/F, i, n) \tag{15-8}$$

where

$$(A/F, i, n) = i / [(1 + i)^n - 1]$$

Values for $(A/F, i, n)$ for standard periodic interest rates and periods are found in Appendix E.

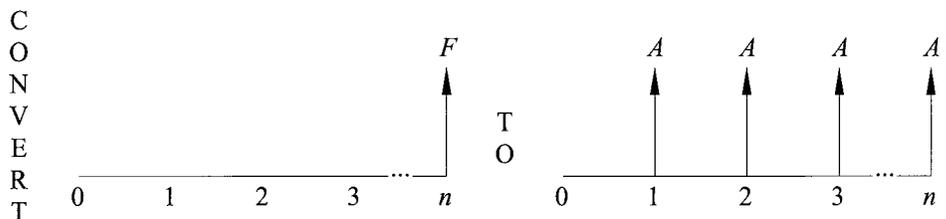


FIGURE 15-6 Uniform-Series Sinking-Fund Factor

S I D E B A R 1 5 - 4

CONVERTING A FUTURE VALUE TO A UNIFORM SERIES USING EXCEL

Example 15-6 may be set up in a spreadsheet as shown in the following figure:

	A	B
1	Interest Rate	10.00%
2	Periods	20
3	Present Value	0.00
4	Future Value	100,000.00
5	Uniform Series	-1,745.96

To set up this spreadsheet, the following formulas, text, and values need to be entered into it:

	A	B
1	Interest Rate	0.1
2	Periods	20
3	Present Value	0
4	Future Value	100000
5	Uniform Series	=PMT(B1,B2,B3,B4)

The spreadsheet uses the PMT function to calculate the uniform series. The PMT function is written as

$$=PMT(\text{rate}, \text{nper}, \text{pv}, \text{fv}, \text{type})$$

where

rate = periodic interest rate

nper = number of periods

pv = cash flow at the present time

fv = cash flow at the end of the specified number of periods (nper)

type = 1 for payment at the beginning of the period and 0 for payment at the end of the period (defaults to 0 if left blank)

To convert a future value into a uniform series, the present value is set to zero and the type is left blank. The uniform series will always carry the opposite sign from the future value.

Example 15-6: What uniform series of cash flows is equivalent to a \$100,000 cash flow twenty years from now if the uniform cash flows occur at the end of the year for the next twenty years and the periodic interest rate is 10% compounded annually?

Solution: For this problem the number of periods is 20, the periodic interest rate is 10%, and the future value is \$100,000. By substituting these amounts into Eq. (15-7) we get the following:

$$A = \$100,000(0.10)/[(1 + 0.10)^{20} - 1] = \$1,745.96$$

Alternately, Eq. (15-8) may be used to solve this problem as follows:

$$A = \$100,000(A/F, 10, 20) = \$100,000(0.0175) = \$1,750.00$$

Again, the difference in these two solutions is due to rounding that occurs in the tables of Appendix E.

A uniform series of \$1,750 cash flows is equivalent to \$100,000 twenty years from now at a 10% interest rate compounded annually. Similarly, had we invested \$1,745.96 a year from now at a periodic interest rate of 10% compounded annually and an additional \$1,745.96 were invested each year thereafter for a total of twenty years at the same interest rate, at the end of the twentieth year there would be \$100,000 in the account. Of this, \$34,919.20 ($\$1,745.96 \times 20$) would have been deposited over the years and \$65,080.80 ($\$100,000.00 - \$34,919.20$) would have been earned as interest.

UNIFORM-SERIES PRESENT-WORTH FACTOR

To find the present value of a uniform series of annual cash flows we could use Eq. (15-5) to find the future value and then use Eq. (15-3) to convert the future value to a present value. By substituting Eq. (15-5) into Eq. (15-3) we get the following:

$$\begin{aligned} P &= F/(1 + i)^n \quad \text{where } F = A[(1 + i)^n - 1]/i \\ P &= A[(1 + i)^n - 1]/[i(1 + i)^n] \end{aligned} \quad (15-9)$$

This equation is known as the uniform-series present-worth factor. The uniform-series present-worth factor converts a series of annual cash flows into a present value at a specified rate of interest and is shown in Figure 15-7.

It is important to note that the present value occurs one period prior to the first cash flow in the uniform series. Because the beginning of a period is the same point in time as the end of the previous period, the present value occurs at the beginning of the period containing the first cash flow in the uniform series.

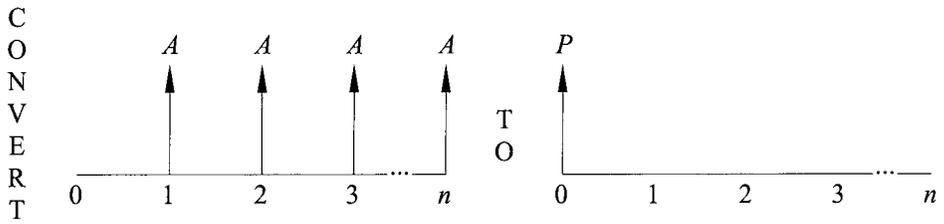


FIGURE 15-7 Uniform-Series Present-Worth Factor

Eq. (15-9) may also be written using shorthand notation as follows:

$$P = A(P/A, i, n) \tag{15-10}$$

where

$$(P/A, i, n) = [(1 + i)^n - 1] / [i(1 + i)^n]$$

Values for $(P/A, i, n)$ for standard periodic interest rates and periods are found in Appendix E.

Example 15-7: What is the present value of five \$500 cash flows that occur at the end of each year for the next five years at a periodic interest rate of 6% compounded annually? The first cash flow occurs a year from now, the second cash flow occurs two years from now, . . . , and the fifth cash flow occurs five years from now.

S I D E B A R 1 5 - 5

CONVERTING A UNIFORM SERIES TO A PRESENT VALUE USING EXCEL

Example 15-7 may be solved using the spreadsheet from Sidebar 15-2 as shown in the following figure:

	A	B
1	Interest Rate	6.00%
2	Periods	5
3	Uniform Series	500.00
4	Future Value	0.00
5	Present Value	-2,106.18

To convert a uniform series into a present value, the future value is set to zero and the type is left blank. The present value will always carry the sign opposite from the uniform series.

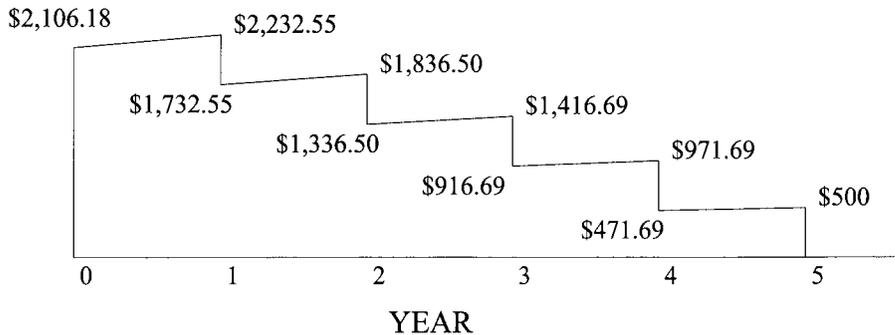


FIGURE 15-8 Value of Bank Account

Solution: For this problem, the annual cash flow is \$500, the number of periods is 5, and the periodic interest rate is 6%. Substituting these values into Eq. (15-9) we get the following:

$$P = \$500[(1 + 0.06)^5 - 1]/[0.06(1 + 0.06)^5] = \$2,106.18$$

Alternately, Eq. (15-10) may be used to solve this problem as follows:

$$P = \$500(P/A, 6, 5) = \$500(4.2124) = \$2,106.20$$

Again, the difference in these two solutions is due to rounding that occurs in the tables of Appendix E.

The present value of these cash flows is \$2,106.20. Similarly, had we invested \$2,106.20 in a bank account earning 6% interest compounded annually, we could have withdrawn \$500 a year for the next five years with the withdrawals occurring at the end of the first, second, third, fourth, and fifth years. During the five-year existence of the bank account the original \$2,106.18 would have earned \$393.82 ($5 \times \$500.00 - \$2,106.18$) of interest, allowing annual withdrawals of \$500 each. The interest earned on the money remains in the account each year. The interest decreases each year because the amount in the bank account decreases each year as withdrawals are made. The value of the bank account over time is shown in Figure 15-8.

UNIFORM-SERIES CAPITAL-RECOVERY FACTOR

Solving Eq. (15-9) for A we get the following:

$$A = P[i(1 + i)^n]/[(1 + i)^n - 1] \quad (15-11)$$

This equation is known as the uniform-series capital-recovery factor. The uniform-series capital-recovery factor converts a present value into a uniform series of annual values at a specified rate of interest and is shown in Figure 15-9.

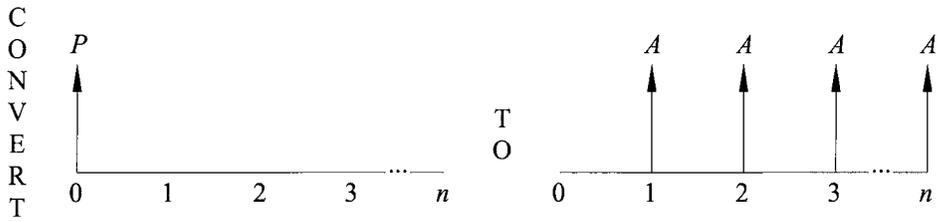


FIGURE 15-9 Uniform-Series Capital-Recovery Factor

It is important to note that the first cash flow for the uniform series occurs one year after the present value.

Eq. (15-11) may also be written using shorthand notation as follows:

$$A = P(A/P, i, n) \tag{15-12}$$

where

$$(A/P, i, n) = [i(1 + i)^n] / [(1 + i)^n - 1]$$

Values for $(A/P, i, n)$ for standard periodic interest rates and periods are found in Appendix E.

Example 15-8: What uniform series of cash flows is equivalent to a \$10,000 cash flow occurring today if the uniform series of cash flows occur at the end of each month for the next five years and the periodic interest rate is 1% compounded monthly?

Solution: For this problem, the present value is \$10,000, the number of periods is sixty months, and the periodic interest rate is 1%. Substituting these values into Eq. (15-11) we get the following:

$$A = \$10,000 [0.01(1 + 0.01)^{60}] / [(1 + 0.01)^{60} - 1] = \$222.44$$

Alternately, Eq. (15-12) may be used to solve this problem as follows:

$$A = \$10,000(A/P, 1, 60) = \$10,000(0.0222) = \$222.00$$

Again, the difference in these two solutions is due to rounding that occurs in the tables of Appendix E.

The uniform series of cash flows is \$222.00 per year. Similarly, had we borrowed \$10,000 today at an interest rate of 1% compounded monthly and paid it off in sixty monthly cash flows occurring at the end of each month, we would have paid \$13,320, of which \$3,320 (\$13,320 - \$10,000) was interest.

S I D E B A R 1 5 - 6

CONVERTING A PRESENT VALUE TO A UNIFORM SERIES USING EXCEL

Example 15-8 may be solved using the spreadsheet from Sidebar 15-4 as shown in the following figure:

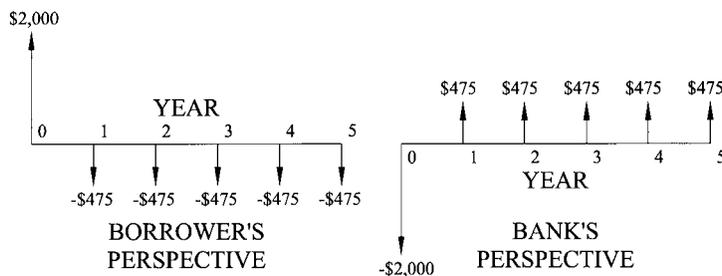
	A	B
1	Interest Rate	1.00%
2	Periods	60
3	Present Value	10,000.00
4	Future Value	0.00
5	Uniform Series	-222.44

To convert a present value into a uniform series, the future value is set to zero and the type is left blank. The uniform series will always carry the sign opposite from the present value.

CASH FLOW DIAGRAMS

To visualize the cash flows it is often helpful to draw a cash flow diagram. In a cash flow diagram the periods are represented along the horizontal axis and the cash flows are drawn as vertical arrows next to the period in which they occur. Arrows drawn in the up direction represent cash receipts and arrows drawn in the down direction represent cash disbursements. Figure 15-10 shows the cash flow diagram for a \$2,000 loan with an interest rate of 6% compounded annually with the principal being paid back over five years, with a single payment at the end of each year. The cash flows are drawn from both the borrower's perspective and the bank's perspective.

FIGURE 15-10 Cash Flow Diagrams



These two cash flows are identical except for the direction that the cash is flowing. When the borrower borrows money from the bank, the borrower sees a receipt of cash, whereas the bank sees a disbursement of cash. When the borrower makes the annual payments, the bank sees a cash receipt and the borrower sees a cash disbursement. When reading a cash flow diagram it is important to know from which perspective the diagram is being drawn. Unless noted, all cash flow diagrams in this book are drawn from the construction company's perspective.

COMPLEX CASH FLOWS

Equivalent values for all cash flows can be calculated by using the equations in this chapter. Complex cash flows may require that multiple equations be used and their results added together.

Example 15-9: Determine the future value of a \$2000 cash flow that occurs today and an additional \$1,000 cash flow that occurs at the end of each of the next ten years using a periodic interest rate of 7% compounded annually.

Solution: The cash flow for this example is shown in Figure 15-11.

The future value of today's cash flow is calculated by using Eq. (15-1) as follows:

$$F_0 = \$2,000(1 + 0.07)^{10} = \$3,934.30$$

The future value of the cash flows for years 1 through 10 is calculated by using Eq. (15-5) as follows:

$$F_{1-10} = \$1,000[(1 + 0.07)^{10} - 1]/0.07 = \$13,816.45$$

Summing these two future values we get the following:

$$F = \$3,934.30 + \$13,816.45 = \$17,750.75$$

The future value is \$17,750.75. Similarly, if we deposited the cash flows in Figure 15-11 in a bank account earning 7% interest compounded annually,

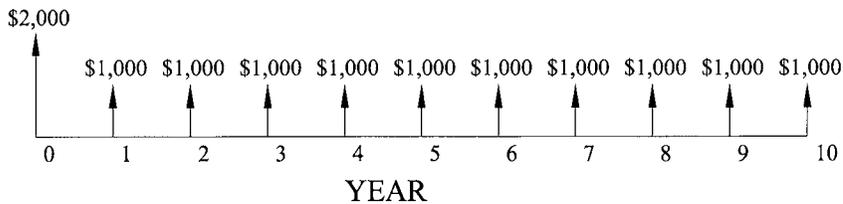


FIGURE 15-11 Cash Flow for Example 15-9

at the end of the tenth year there would be \$17,750.75 in the bank account, which includes the original \$12,000.00 of principal plus \$5,750.75 in interest.

Example 15-10: Determine the future value at the end of August for the following cash flows using a periodic interest rate of 1.5% compounded monthly:

MONTH	CASH FLOW (\$)
March	10,000
April	15,000
May	25,000
June	21,000
July	17,000

These cash flows occur at the end of the respective months.

Solution: The cash flows for Example 15-10 are shown in Figure 15-12. The number of compounding periods for each cash flow is shown in Table 15-1.

Using Eq. (15-1) we can calculate the future values of the money borrowed during each of the months. These calculations are as follows:

$$\begin{aligned} \text{March} &= \$10,000(1 + 0.015)^5 = \$10,772.84 \\ \text{April} &= \$15,000(1 + 0.015)^4 = \$15,920.45 \\ \text{May} &= \$25,000(1 + 0.015)^3 = \$26,141.96 \\ \text{June} &= \$21,000(1 + 0.015)^2 = \$21,634.72 \\ \text{July} &= \$17,000(1 + 0.015)^1 = \$17,255.00 \end{aligned}$$

FIGURE 15-12 Cash Flow for Example 15-10

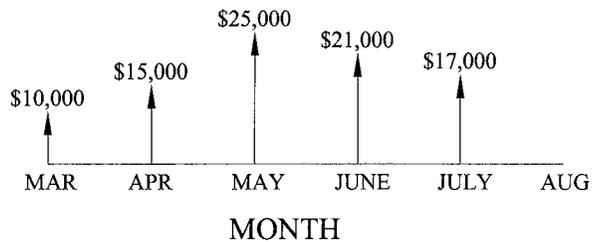


TABLE 15-1 Number of Compounding Periods

CASH FLOW	PERIODS
March	5
April	4
May	3
June	2
July	1

Adding the values together we get the following:

$$\begin{aligned}
 F &= \$10,772.84 + \$15,920.45 + \$26,141.96 + \$21,634.72 \\
 &\quad + \$17,255.00 \\
 F &= \$91,724.97
 \end{aligned}$$

The future value of the cash flows is \$91,724.97. Similarly, if a contractor had borrowed these amounts at the end of the months indicated at a periodic interest rate of 1.5% compounded monthly, the contractor would have to pay back the \$88,000 borrowed plus \$3,724.97 interest for a total of \$91,724.97.

Individual items in a cash flow may be treated as two separate cash flows to simplify the calculations. Figure 15-13 shows a nonuniform series of cash flows for which the present value needs to be calculated.

To simplify the calculations, the cash flow occurring in the third year may be treated as two \$1,000 cash flows as shown in Figure 15-14.

The present value of the five \$1,000 cash flows is determined by using Eq. (15-9) and the present value of the additional \$1,000 cash flow that occurs in year 3 may be calculated by using Eq. (15-3). The present value of the entire cash flow is determined by adding these two present values together.

Example 15-11: Determine the present value of the cash flow shown in Figure 15-13 using a periodic interest rate of 12% compounded annually.

Solution: The present value of the five \$1,000 cash flows is determined by using Eq. (15-9) as follows:

$$P_{1-5} = \$1,000[(1 + 0.12)^5 - 1]/[0.12(1 + 0.12)^5] = \$3,604.78$$

The present value for the additional \$1,000 cash flow that occurs in year 3 is calculated by using Eq. (15-3) as follows:

$$P_3 = \$1,000/(1 + 0.12)^3 = \$711.78$$

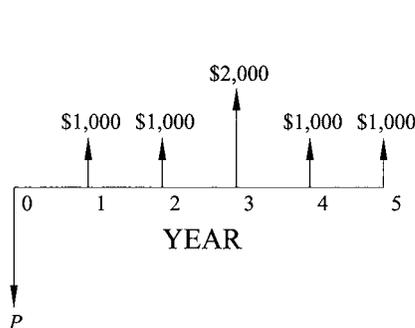


FIGURE 15-13 Cash Flow

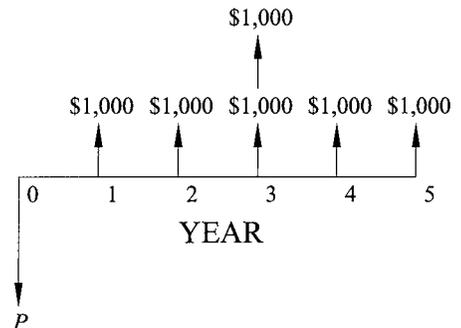


FIGURE 15-14 Equivalent Cash Flow for Figure 15-13

SIDEBAR 15 - 7

CONVERTING AN NON-UNIFORM SERIES TO A PRESENT VALUE USING EXCEL

Example 15-12 may be set up in a spreadsheet as shown in the following figure:

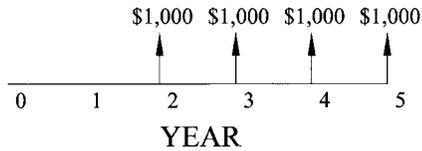
	A	B	C
1	Interest Rate	12.00%	
2			
3	Period	Amount	PV
4	0	0.00	0.00
5	1	1,000.00	-892.86
6	2	1,000.00	-797.19
7	3	2,000.00	-1,423.56
8	4	1,000.00	-635.52
9	5	1,000.00	-567.43
10		Total	-4,316.56

To set up this spreadsheet, the following formulas, text, and values need to be entered into it:

	A	B	C
1	Interest Rate	0.12	
2			
3	Period	Amount	PV
4	0	0	=PV(\$B\$1,A4,0,B4)
5	1	1000	=PV(\$B\$1,A5,0,B5)
6	2	1000	=PV(\$B\$1,A6,0,B6)
7	3	2000	=PV(\$B\$1,A7,0,B7)
8	4	1000	=PV(\$B\$1,A8,0,B8)
9	5	1000	=PV(\$B\$1,A9,0,B9)
10		Total	=SUM(C4:C9)

The spreadsheet uses the PV function, discussed in Sidebar 15-2, to convert each cash flow to its present value. The present values for each year are then summed using the SUM function. See Appendix B for more information on the SUM function. Both positive and negative cash flows may be used for the annual cash flows.

FIGURE 15-15 Cash Flow



The present value for the entire cash flow is determined by adding the two above present values together as follows:

$$P = \$3,604.78 + \$711.78 = \$4,316.56$$

The present value of the cash flow shown in Figure 15-13 at an interest rate of 12% compounded annually is \$4,316.56.

The calculation of the present value, the future value, or the annual values for a complex cash flow is independent of the steps taken to calculate the equivalent cash flow. Figure 15-15 shows a cash flow for which the present value needs to be calculated.

At least two of the formulas in this chapter must be used to calculate the present value. The present value could be calculated by taking the present value of each of the individual cash flows using Eq. (15-3).

Alternately, the cash flows could be treated as a uniform series for which the future value in year 5 could be calculated by using Eq. (15-5). This would create the equivalent cash flow shown in Figure 15-16.

The present value could then be calculated by taking the present value of the equivalent cash flow shown in Figure 15-16 using Eq. (15-3).

Alternately, the cash flows could be treated as a uniform series for which the present value in year 1 could be calculated by using Eq. (15-9). This would create the equivalent cash flow shown in Figure 15-17.

The present value could then be calculated by taking the present value of the equivalent cash flow shown in Figure 15-17 using Eq. (15-3).

Example 15-12: Determine the present value of the cash flow shown in Figure 15-15 using an interest rate of 8% compounded annually.

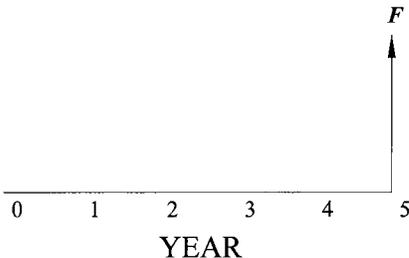


FIGURE 15-16 Equivalent Cash Flow for Figure 15-15

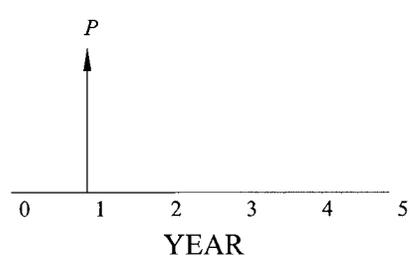


FIGURE 15-17 Equivalent Cash Flow for Figure 15-14

Solution: The present value of each of the cash flows are calculated by using Eq. (15-3) as follows:

$$P_2 = \$1,000/(1 + 0.08)^2 = \$857.34$$

$$P_3 = \$1,000/(1 + 0.08)^3 = \$793.83$$

$$P_4 = \$1,000/(1 + 0.08)^4 = \$735.03$$

$$P_5 = \$1,000/(1 + 0.08)^5 = \$680.58$$

Summing these cash flows we get the following:

$$P = \$857.34 + \$793.83 + \$735.03 + \$680.58 = \$3,066.78$$

Alternately, the future value in year 5 of the four \$1,000 cash flows may be determined by using Eq. (15-5), using four periods as follows:

$$F_5 = \$1,000[(1 + 0.08)^4 - 1]/0.08 = \$4,506.11$$

Next, the present value is calculated by taking the present value of the future value in year 5 using Eq. (15-3) as follows:

$$P = \$4,506.11/(1 + 0.08)^5 = \$3,066.78$$

Alternately, the value of the four \$1,000 cash flows may be determined by using Eq. (15-9). Year 1 is regarded as the present time for the purposes of this calculation. The calculations are as follows:

$$P_1 = \$1,000[(1 + 0.08)^4 - 1]/[0.08(1 + 0.08)^4] = \$3,312.13$$

The value of the four \$1,000 cash flows in year 1 is still one year into the future and is treated as a future value when converting the present value in year 1 to the present value in year 0. The present value is calculated by taking the present value in year 1 and substituting it into Eq. (15-3) as the future value as follows:

$$P = \$3,312.13/(1 + 0.08)^1 = \$3,066.79$$

We see that in all three cases the present value is the same, other than rounding errors. This is because when the calculations are completed we have performed the same mathematical calculations on the cash flows.

If we were to write a mathematical equation for converting the cash flow to a future value in year 5 and then convert it to its present value in year 0 for a generic annual value and periodic interest rate we would get the following:

$$P = F_5/(1 + i)^5 \quad \text{where } F_5 = A[(1 + i)^4 - 1]/i$$

Combining these two equations we get the following:

$$P = \{A[(1 + i)^4 - 1]/i\}/(1 + i)^5 = A[(1 + i)^4 - 1]/[i(1 + i)^5]$$

If we were to write a mathematical equation for converting the cash flow to a present value in year 1 and then convert it to its present value in year 0 for a generic annual value and periodic interest rate we would get the following:

$$P = P_1/(1 + i)^1 \quad \text{where } P_1 = A[(1 + i)^4 - 1]/[i(1 + i)^4]$$

Combining these two equations we get the following:

$$P = \{A[(1 + i)^4 - 1]/[i(1 + i)^4]\}/(1 + i)^1$$

$$P = A[(1 + i)^4 - 1]/[i(1 + i)^5]$$

We see that in both cases the resulting mathematical equation is the same.

FIND UNKNOWN PERIODIC INTEREST RATES

In some cases we may have cash receipts that are equivalent to cash disbursements at some unknown interest rate. The interest rate may be found by using Equations (15-1) through (15-12) to write an algebraic equation for the equivalence of the cash flows. The algebraic equation may then be solved for i . This is relatively simple when dealing with a single cash disbursement and single cash receipt.

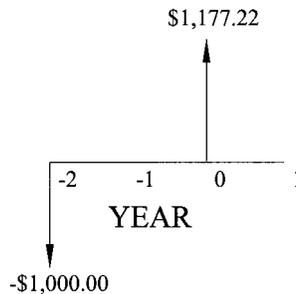
Example 15-13: At what periodic interest rate is a \$1,000 cash disbursement occurring two years ago equivalent to cash receipt of \$1,177.22 occurring today? The periodic interest rate is compounded annually.

Solution: The cash flow for this example is shown in Figure 15-18.

Because the interest compounds annually and the cash disbursement occurs two years before the cash receipt, the number of periods is two. For the purposes of this example, the time of the cash disbursement may be regarded as the present time and today may be regarded as the future time. Substituting the cash disbursement as the present value and the cash receipt as the future value into Eq. (15-1) we get the following:

$$\$1,177.22 = \$1,000.00(1 + i)^2$$

FIGURE 15-18 Cash Flow for Example 15-13



Solving for i we get the following:

$$\begin{aligned}(1 + i)^2 &= \$1,177.22/\$1,000.00 \\ 1 + i &= (\$1,177.22/\$1,000.00)^{0.5} \\ i &= (\$1,177.22/\$1,000.00)^{0.5} - 1 = 0.085\end{aligned}$$

The periodic interest rate at which the cash receipts are equivalent to the cash disbursements is 8.5% compounded annually.

If the algebraic equation for the equivalence of the cash flows can be reduced to Eq. (15-6), Eq. (15-8), Eq. (15-10), or Eq. (15-12) the interest rate may be approximated by solving for $(F/A, i, n)$, $(A/F, i, n)$, $(P/A, i, n)$, or $(A/P, i, n)$ and estimating the value of i from the tables in Appendix E.

Example 15-14: At what periodic interest rate is a \$1,000 cash receipt occurring at the beginning of year 1 equivalent to two \$550 cash disbursements, one occurring at the end of year 1 and the second occurring at the end of year 2? The periodic interest rate is compounded annually.

Solution: The cash flow for the problem is shown in Figure 15-19. Note that the beginning of year 1 is the same as the end of year 0.

This cash flow corresponds to Eq. (15-12) and is written as follows:

$$\$550.00 = \$1,000.00(A/P, i, 2)$$

Solving for $(A/P, i, n)$ we get the following:

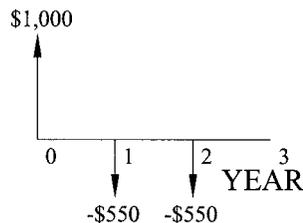
$$\begin{aligned}(A/P, i, 2) &= \$550.00/\$1,000.00 \\ (A/P, i, 2) &= 0.5500\end{aligned}$$

Using the tables located in Appendix E we see that the value 0.5500 falls between $(A/P, 6, 2)$, which has a value of 0.5454, and $(A/P, 7, 2)$, which is 0.5531. The interest rate may be approximated by linearly interpolating between these two values as follows:

$$i = (7 - 6)(0.5500 - 0.5454)/(0.5531 - 0.5454) + 6 = 6.597\%$$

The periodic interest rate at which the cash receipts are equivalent to the cash disbursements is 6.6% compounded annually.

FIGURE 15-19 Cash Flow for Example 15-14



Cash flows that cannot be solved algebraically must be solved by trial and error. To do this, an algebraic equation may be set up with a known value on the left side of the equation and the unknown interest rate on the right side of the equation. The interest rate is changed until the right side of the equation equals the known value on the left side of the equation. Modern spreadsheet applications make solving trial-and-error problems easy.

Example 15-15: At what periodic interest rate is a \$1,000 cash receipt occurring at the beginning of year 1 equivalent to a \$500 cash disbursement occurring at the end of year 1 and a \$600 cash disbursement occurring at the end of year 2? The periodic interest rate is compounded annually.

Solution: The cash flow for the problem is shown in Figure 15-20. Note that the beginning of year 1 is the same as the end of year 0.

Using Eq. (15-3) the present values for the two cash disbursements may be expressed algebraically as follows:

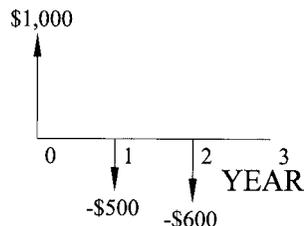
$$P_1 = \$500.00/(1 + i)^1$$

$$P_2 = \$600.00/(1 + i)^2$$

By trial and error we find i equals 6.3943. The periodic interest rate at which the cash receipts are equivalent to the cash disbursements is 6.394% compounded annually.

When the cash disbursements are equivalent to the cash receipts, they are equivalent regardless of the period of time chosen for the analysis. If receipts and disbursements are equivalent in the present time at a specified interest rate, they will also be equivalent at any point in the future or past at the same interest rate. Therefore, equivalence is independent of the time period chosen to evaluate equivalence. In Example 15-15 the \$1,000 cash receipt was equivalent to the cash disbursements—the \$500 cash disbursement at the end of year 1 and the \$600 cash disbursement at the end of year 2—at an interest rate of 6.394% compounded annually. If we were to calculate the value of the cash

FIGURE 15-20 Cash Flow for Example 15-15



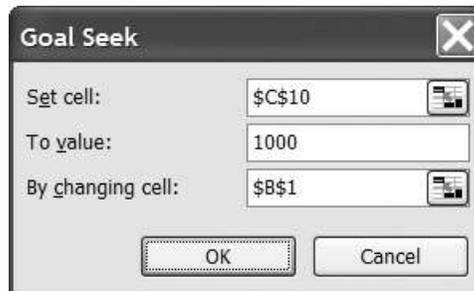
SIDEBAR 15 - 8

FINDING AN UNKNOWN INTEREST RATE FOR A PRESENT VALUE AND A SERIES OF CASH FLOWS USING EXCEL

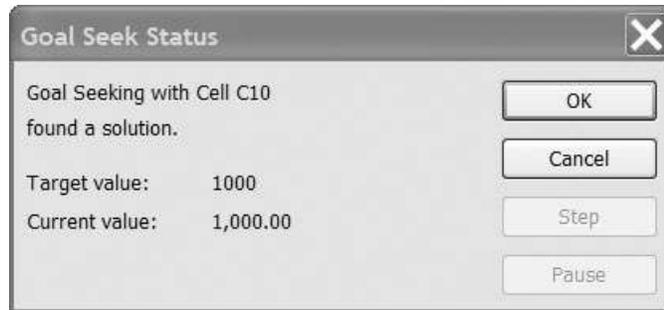
Example 15-15 may be solved using the spreadsheet from Sidebar 15-7 as shown in the following figure:

	A	B	C
1	Interest Rate	6.39%	
2			
3	Period	Amount	PV
4	0	0.00	0.00
5	1	-500.00	469.95
6	2	-600.00	530.05
7	3	0.00	0.00
8	4	0.00	0.00
9	5	0.00	0.00
10		Total	1,000.00

Once the data has been entered into the spreadsheet, the user must use the Goal Seek tool to solve for the interest rate. The Goal Seek dialogue box is shown in the following figure:



The interest rate is found by entering the cell containing the sum of the present values (cell C10) in the Set cell: textbox, entering the desired value of the present value in the To value: textbox (\$1,000 in this case), entering the cell containing the interest rate (cell B1) in the By changing cell: textbox, and clicking on the OK button. The Goal Seek tool then displays the Goal Seek Status dialogue box (shown next) and gives you the option to accept or reject the solution it has determined.



When the cash flows for the years contain both negative and positive cash flows, more than one solution for the interest rate may occur.

receipts and the cash disbursements at the end of year 2 at an interest rate of 6.394% compounded annually the cash receipts should be equivalent to the cash disbursements.

Example 15-16: Compare the value of the cash receipts to the cash disbursement from Example 15-15 at the end of year 2 at a periodic interest rate of 6.394% compounded annually.

Solution: First, find the future value of the cash receipts at the end of year 2 using Eq. (15-1):

$$F = \$1,000.00(1 + 0.0639)^2 = \$1,131.97$$

Next, find the future value of the first cash disbursement at the end of year 2 using Eq. (15-1). The cash disbursement occurs one year before the end of year 2; therefore, the number of periods is one. The calculations are as follows:

$$F_1 = \$500.00(1 + 0.06394)^1 = \$531.97$$

The second cash disbursement occurs at the end of the second year, which is the same point in time for which we are calculating the future value; therefore, the number of periods is zero and its future value is \$600.00. Summing these two future values we get the following:

$$F = \$531.97 + \$600.00 = \$1,131.97$$

At an interest rate of 6.394%, the cash receipts are equivalent to the cash disbursements.

INFLATION AND CONSTANT DOLLARS

While money is in the bank earning interest, inflation is eroding its purchasing power. During times of inflation, a dollar today will not purchase the same amount of goods that it would have purchased a year ago. If inflation were to be running 3% per year, it would take \$1.03 today to purchase the same amount of goods that \$1.00 would have purchased a year ago. Money's value is also determined by what it can purchase. Using f to represent the inflation rate, we could write the following equation:

$$\text{Cost}_n = \text{Cost}_0(1 + f)^n \quad (15-13)$$

As inflation increases costs, purchasing power decreases. Because purchasing power is inversely related to costs we may write Eq. (15-13) as follows:

$$\text{Future Purchasing Power} = \text{Today's Purchasing Power}/(1 + f)^n \quad (15-14)$$

Combining the effects of inflation with Eq. (15-1) we get the following:

$$F' = P(1 + i)^n/(1 + f)^n$$

where $(1 + i)^n$ is the increase in value due to interest, $(1 + f)^n$ is the decrease in purchasing power due to inflation, and F' is the future value in constant dollars. This formula not only produces equivalent cash flows at an interest rate of i but also cash flows in constant dollars or in dollars with the same purchasing power at an inflation rate of f .

To simplify the calculations, the periodic interest rate and inflation rate could be replaced with the constant dollar interest rate i' , which incorporates both the periodic interest rate and the inflation rate as follows:

$$(1 + i') = (1 + i)/(1 + f)$$

Solving for i' we get the following:

$$i' = (1 + i)/(1 + f) - 1 \quad (15-15)$$

The constant dollars interest rate (i') can be substituted for i into Eqs. (15-1) through (15-12) to get not only equivalent cash flows but also equivalent cash flows in constant dollars. The underlying assumption is that all cash flows are subject to the same inflation rate and the inflation rate is constant for all periods of time.

Example 15-17: How much money needs to be set aside today to purchase a new piece of equipment in five years? The money is expected to earn 8% interest compounded annually and the price of the equipment is expected to increase by 3% per year. The present cost of the equipment is \$10,000.

Solution: To solve this problem, we must first find the constant dollars interest rate for a periodic interest rate of 8% and an inflation rate of 3%. Substituting these values into Eq. (15-15) we get the following:

$$i' = (1 + 0.08)/(1 + 0.03) - 1 = 0.048544$$

Substituting i' into Eq. (15-3) and using five one-year periods we get the following:

$$P' = \$10,000/(1 + 0.048544)^5 = \$7,889.81$$

where P' is the present value in constant dollars.

In lieu of using the constant dollar inflation rate, we could have used Eq. (15-1) to find the future cost of the piece of equipment using the inflation rate in place of i as follows:

$$F' = \$10,000(1 + 0.03)^5 = \$11,592.74$$

We could then find the present value of the future cost of the piece of equipment by using Eq. (15-3):

$$P' = \$11,592.74/(1 + 0.08)^5 = \$7,889.82$$

During the five years that the \$7,889.82 was invested, it would have earned \$3,702.92 of interest, growing to \$11,592.74. During this same time, the price of the piece of equipment would have increased by \$1,592.74 to \$11,592.74.

CONCLUSION

Money's value is based on the size of the cash flows, the timing of the cash flows, and the interest rate. Before cash flows can be added, subtracted, or compared the cash flows must be converted to an equivalent cash flow or series of uniform cash flows at a specific point or points in time. Equivalent cash flows are cash flows that produce the same results at the specified interest rate. There are six basic formulas that may be used to convert cash flows of different time periods. To simplify calculations a cash flow may be split into two or more cash flows. The point or points in time at which the cash flows are compared is unimportant because when the cash flows are equivalent at a specified interest rate and point in time, they are equivalent at all points in time at that interest rate. Inflation decreases the purchasing power of money. The constant dollar inflation rate adjusts the interest rate to take into account inflation and can be substituted into any of the six conversion equations.

PROBLEMS

1. What is the future value, ten years from now, of \$1,000 invested today at a periodic interest rate of 12% compounded annually?

2. What is the future value, ten years from now, of \$1,000 invested today at a periodic interest rate of 1% compounded monthly?
3. What is the present value of \$1,000, received ten years from now, using a periodic interest rate of 12% compounded annually?
4. What is the present value of \$1,000, received four years from now, using a periodic interest rate of 7.5% compounded annually?
5. What is the future value, ten years from now, of ten \$1,000 cash flows using a periodic interest rate of 8% compounded annually? The cash flows are made at the end of each year.
6. What is the future value, five years from now, of sixty \$100 cash flows using a periodic interest rate of 0.7% compounded monthly? The cash flows are made at the end of each month.
7. What uniform series of cash flows is equivalent to a \$100,000 cash flow, ten years from now, if the uniform cash flows occur at the end of the year for the next ten years and the periodic interest rate is 11% compounded annually?
8. What uniform series of cash flows is equivalent to a \$100,000 cash flow, fifteen years from now, if the uniform cash flows occur at the end of the year for the next fifteen years and the periodic interest rate is 8.5% compounded annually?
9. What is the present value of five \$800 cash flows that occur at the end of each year for the next five years at a periodic interest rate of 8% compounded annually? The first cash flow occurs a year from now, the second cash flow occurs two years from now, . . . , and the fifth cash flow occurs five years from now.
10. What is the present value of twenty \$500 cash flows that occur at the end of each year for the next twenty years at a periodic interest rate of 7.5% compounded annually? The first cash flow occurs a year from now, the second cash flow occurs two years from now, . . . , and the twentieth cash flow occurs twenty years from now.
11. What uniform series of cash flows is equivalent to a \$15,000 cash flow occurring today if the uniform series of cash flows occur at the end of each year for the next five years and the periodic interest rate is 9% compounded annually?
12. What uniform series of cash flows is equivalent to a \$150,000 cash flow occurring today if the uniform series of cash flows occur at the end of each month for the next fifteen years and the periodic interest rate is 0.62% compounded annually?
13. Draw a cash flow diagram for the following cash flow:

YEAR	0	1	2	3	4
Receipts (\$)	0	1,000	2,000	2,000	1,000
Disbursements (\$)	4,750	0	0	0	0

14. Draw a cash flow diagram for the following cash flow:

YEAR	0	1	2	3	4
Receipts (\$)	0	5,000	0	5,000	5,000
Disbursements (\$)	7,500	0	5,000	0	0

15. Determine the future value at the end of August for the following cash flows using a periodic interest rate of 1% compounded monthly:

MONTH	AMOUNT (\$)
April	15,000
May	25,000
June	21,000
July	15,000

These cash flows occur at the end of the respective months.

16. Determine the future value at the end of June for the following cash flows using a periodic interest rate of 1% compounded monthly:

MONTH	AMOUNT (\$)
Dec.	15,000
Jan.	22,000
Feb.	28,000
March	35,000
April	30,000
May	15,000

These cash flows occur at the end of the respective months.

17. Determine the present value in year 0 of the following cash flows using a periodic interest rate of 9% compounded annually:

YEAR	1	2	3	4	5
Receipts (\$)	3,000	3,000	5,000	3,000	5,000

18. Determine the present value in year 0 of the following cash flows using a periodic interest rate of 11% compounded annually:

YEAR	1	2	3	4	5
Receipts (\$)	5,000	5,000	0	5,000	5,000
Disbursements (\$)	0	0	5,000	0	0

19. Determine the present value in year 0 of the following cash flows using a periodic interest rate of 5% compounded annually:

YEAR	5	6	7	8	9
Receipts (\$)	6,000	6,000	6,000	6,000	6,000

20. Determine the present value in year 0 of the following cash flows using a periodic interest rate of 12% compounded annually:

YEAR	11	12	13	14	15
Receipts (\$)	1,000	1,000	1,000	1,000	1,000

21. At what periodic interest rate is a \$1,000 cash disbursement occurring four years ago equivalent to a cash receipt of \$1,274.43 occurring today? The periodic interest rate is compounded annually.
22. At what periodic interest rate is a \$5,000 cash disbursement occurring today equivalent to a cash receipt of \$7,605.30 occurring five years from now? The periodic interest rate is compounded annually.
23. At what periodic interest rate is a \$2,000 cash receipt occurring at the beginning of year 1 equivalent to four annual \$600 cash disbursements? The first cash disbursement occurs at the end of year 1, the second occurs at the end of year 2, the third occurs at the end of year 3, and the fourth occurs at the end of year 4. The periodic interest rate is compounded annually.
24. At what periodic interest rate is a \$4,000 cash receipt occurring at the beginning of year 1 equivalent to ten annual \$750 cash disbursements? The first cash disbursement occurs at the end of year 1, the second occurs at the end of year 2, . . . , and the tenth occurs at the end of year 10.
25. What is the constant dollar interest rate for a periodic interest rate of 9% and an inflation rate of 4%?
26. What is the constant dollar interest rate for a periodic interest rate of 6% and an inflation rate of 3%?
27. How much money needs to be set aside today to purchase a new piece of equipment in five years? The money is expected to earn 5% interest compounded annually and the price of the equipment is expected to increase by 2% per year. The present cost of the equipment is \$100,000.
28. How much money needs to be set aside today to purchase a new piece of equipment in three years? The money is expected to earn 9% interest compounded annually and the price of the equipment is expected to increase by 3% per year. The present cost of the equipment is \$75,000.
29. Modify the spreadsheet in Sidebar 15-7 to calculate the future value instead of the present value. Expand the spreadsheet to cover 10 periods. Hint: The

periods must be in reverse numeric order. Solve Problem 15 and 16 using this modified spreadsheet.

30. Using the spreadsheet in Sidebar 15-8, determine at what periodic interest rate a \$2,000 cash receipt occurring at the beginning of year 1 is equivalent to four cash disbursements: a \$575 cash disbursement occurring at the end of year 1, a \$600 cash disbursement occurring at the end of year 2, a \$625 cash disbursement occurring at the end of year 3, and a \$650 cash disbursement occurring at the end of year 4. The periodic interest rate is compounded annually.
31. Using the spreadsheet in Sidebar 15-8, determine at what periodic interest rate a \$5,000 cash receipt occurring at the beginning of year 1 is equivalent to five annual cash disbursements. The first four annual cash disbursements are \$1,000 each and occur at the end of each of the first four years. The last cash disbursement is \$2,000 and occurs at the end of the fifth year. The periodic interest rate is compounded annually.